Universität
Stuttgart

Fachbereich
Mathematik

Parabolic Approximations of Diffusive-Dispersive Equations

Andrea Corli, Christian Rohde, Veronika Schleper

Preprint 2013/010

Universität
Stuttgart

Fachbereich
Mathematik

Parabolic Approximations of Diffusive-Dispersive Equations

Andrea Corli, Christian Rohde, Veronika Schleper

Preprint 2013/010

Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik.uni-stuttgart.de/preprints

ISSN **1613-8309**

Parabolic Approximations of Diffusive-Dispersive Equations

Andrea Corli* Christian Rohde† Veronika Schleper†

July 22, 2013

Abstract

We consider a lower-order approximation for a third-order diffusive-dispersive conservation law with nonlinear flux. It consists of a system of two second-order parabolic equations; a coupling parameter is also added.
If the flux has an inflection point it is well-known, on the one hand, that the diffusive-dispersive law admits traveling-wave solutions whose end states are also connected by undercompressive shock waves of the underlying hyperbolic conservation law. On the other hand, if the diffusive-dispersive regularization vanishes, the solutions of the corresponding initial-value problem converge to a weak solution of the hyperbolic conservation law. We show that both of these properties also hold for the lower-order approximation. Furthermore, when the coupling parameter tends to infinity, we prove that solutions of initial value problems for the approximation converge to a weak solution of the diffusive-dispersive law. The proofs rely on new a-priori energy estimates for higher-order derivatives and the technique of compensated compactness.

1 Introduction

We are interested in the following initial-value problem for a diffusive-dispersive conservation law:

$$\begin{aligned} u_t^\varepsilon + f(u^\varepsilon)_x &= \varepsilon u_{xx}^\varepsilon + \gamma\varepsilon^2 u_{xxx}^\varepsilon \quad \text{in } \Omega_T :\, = \mathbb{R}\times(0,T),\, T>0,\\ u^\varepsilon(.,0) &= u_0 \quad \text{in } \mathbb{R}. \end{aligned} \tag{1.1}$$

Here, $f:\mathbb{R}\to\mathbb{R}$ denotes a smooth flux with $f(0)=0$, $u_0:\mathbb{R}\to\mathbb{R}$ is the initial datum and ε, γ are positive real parameters.
The model (1.1) can be understood as a toy model to describe phase transition dynamics when viscous and capillary effects are taken into account. To see this, let us consider first the ideal case, i.e., the sharp interface limit $\varepsilon \to 0$ where the diffusive-dispersive equation in (1.1) reduces to the hyperbolic conservation law

$$u_t + f(u)_x = 0. \tag{1.2}$$

Assume that the flux f has an inflection point; this happens, for instance, in the modeling of phase transitions [13], saturation fronts [26] and precursors in thin film flows [3]. In such a case, shock-wave solutions to (1.2) that connect end states u_-, u_+ in different regions of convexity may be interpreted as phase boundaries; in particular, such waves can be undercompressive [11]. The crucial observation from [11, 13] is that these undercompressive shock waves possess a diffusive-dispersive profile, i.e., the equation (1.1) admits

*Department of Mathematics and Computer Science, University of Ferrara, I-44121 Ferrara, Italy.
†Institute for Applied Analysis and Numerical Simulation und Numerische Simulation, University of Stuttgart, D-70569 Stuttgart, Germany.

for some $\gamma > 0$ smooth traveling-wave solutions that connect the same end states u_-, u_+. Moreover, solutions of (1.1) converge for $\varepsilon \to 0$ to a weak solution of the corresponding initial-value problem for (1.2), see [11, 22].
Reverting to the dissipative equation in (1.1), it is obvious that its analytical treatment and even more its numerical simulation gets intricate due to the appearance of the third-order term. Therefore we introduce in the present paper *lower-order approximations* of (1.1) that keep the property of allowing *traveling-wave solutions* associated with under-compressive waves of (1.2) and reduce to (1.2) in the sharp-interface limit $\varepsilon \to 0$.
One possibility for lower-order approximations for (1.1) is the nonlocal equation

$$u_t^{\varepsilon,\alpha} + f(u^{\varepsilon,\alpha})_x = \varepsilon u_{xx}^{\varepsilon,\alpha} + \gamma\alpha \left(\mathcal{K}^{\varepsilon,\alpha} * u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\right)_x, \tag{1.3}$$

which depends on the *coupling parameter* $\alpha > 1$. In (1.3) the symbol "$*$" denotes spatial convolution and $\mathcal{K}^{\varepsilon,\alpha} : \mathbb{R} \to \mathbb{R}$ is a kernel function that satisfies

$$\int_{\mathbb{R}} \mathcal{K}^{\varepsilon,\alpha}(x)\, dx = 1, \quad \mathcal{K}^{\varepsilon,\alpha}(x) = \mathcal{K}^{\varepsilon,\alpha}(-x) \qquad (\alpha, \varepsilon > 0).$$

Under appropriate growth conditions on $\mathcal{K}^{\varepsilon,\alpha}$, the existence of traveling waves for (1.3) has been verified in [20]. For fixed $\varepsilon > 0$, formal asymptotics suggests that solutions $u^{\varepsilon,\alpha}$ of (1.3) converge for $\alpha \to \infty$ to solutions of (1.1). Though this has not been proven so far for arbitrary kernels $\mathcal{K}^{\varepsilon,\alpha}$ it holds true indeed for the choice

$$\mathcal{K}^{\varepsilon,\alpha}(x) = \frac{\sqrt{\alpha}}{2\varepsilon} e^{-\frac{\sqrt{\alpha}}{\varepsilon}|x|}, \tag{1.4}$$

see [6]. In this case (1.3) is equivalently rewritten in the local parabolic-elliptic form

$$\begin{cases} u_t^{\varepsilon,\alpha} + f(u^{\varepsilon,\alpha})_x &= \varepsilon u_{xx}^{\varepsilon,\alpha} - \alpha(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha})_x, \\ -\gamma\varepsilon^2 \lambda_{xx}^{\varepsilon,\alpha} &= \alpha(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha}), \end{cases} \quad \text{in } \Omega_T. \tag{1.5}$$

In other words, the approximation consists of a second-order equation for $u^{\varepsilon,\alpha} : \Omega_T \to \mathbb{R}$ coupled with a linear elliptic equation for an additional unknown $\lambda^{\varepsilon,\alpha} : \Omega_T \to \mathbb{R}$. The kernel (1.4) is then the Green function associated to $(1.5)_2$. We note that $(1.5)_2$ is exactly the screened Poisson equation, which is widely used in mathematical physics and visualization sciences (see for instance [9, 12]).
Instead of purely static equations as $(1.5)_2$, one may also think to select time dependent operators. The simplest choice in this context and indeed the main topic of this paper is the system

$$\begin{cases} u_t^{\varepsilon,\alpha} + f(u^{\varepsilon,\alpha})_x &= \varepsilon u_{xx}^{\varepsilon,\alpha} - \alpha(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha})_x, \\ \beta\lambda_t^{\varepsilon,\alpha} - \gamma\varepsilon^2 \lambda_{xx}^{\varepsilon,\alpha} &= \alpha(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha}), \end{cases} \quad \text{in } \Omega_T, \tag{1.6}$$

for $0 < \beta \leq 1$. We omitted in the functions $u^{\varepsilon,\alpha}$, $\lambda^{\varepsilon,\alpha}$ the dependence on β both for simplicity and because we shall see that β must be scaled with respect to α and ε in most results. The latter issue can be intuitively motivated as follows. By $(1.6)_2$ we can guess that $\lambda^{\varepsilon,\alpha}$ and $u^{\varepsilon,\alpha}$ converge to the same limit as $\alpha \to \infty$. If we plug $(1.6)_2$ into $(1.6)_1$, we see that a necessary condition to recover $(1.1)_1$ in the limit $\alpha \to \infty$ is that

$$\beta = \beta(\alpha) \to 0 \text{ for } \alpha \to \infty. \tag{1.7}$$

While (1.5) requires a single initial condition for $u^{\varepsilon,\alpha}$ we need two initial conditions for (1.6). We put

$$u^{\varepsilon,\alpha}(\cdot, 0) = \lambda^{\varepsilon,\alpha}(\cdot, 0) = u_0 \quad \text{ in } \mathbb{R}. \tag{1.8}$$

The case $\lambda^{\varepsilon,\alpha}(\cdot, 0) \neq u_0$, with $\lambda_0^{\varepsilon,\alpha} \to u_0$ for $\alpha \to \infty$ or $\varepsilon \to 0$ in some suitable norms, only leads to minor changes in the following.

As mentioned above, we are particularly interested in flux functions f that are neither concave nor convex. To avoid technicalities, we focus on two classes of flux functions: we assume that the flux function $f : \mathbb{R} \to \mathbb{R}$ has bounded first and second derivatives or it coincides with the simple non-convex flux

$$f(u) = u^3. \tag{1.9}$$

In Section 2 we prove the existence of traveling waves to (1.6) for the cubic flux (1.9). There we prove the first important result of this paper: under the assumption (1.7), traveling waves exist for particular values of γ. The proof relies on geometric singular perturbation theory [8].

The rest of the paper is concerned with fluxes having bounded derivatives. In Section 3 we prove some fundamental energy estimates for the initial-value problem (1.6)-(1.8); they will be used in Section 4 to prove the wellposedness of the initial-value problem for classical solutions.

The technical Section 5 presents refined a-priori estimates on smooth solutions of (1.6). In these energy-type estimates the dependence on both the sharp-interface parameter ε and, in particular, the coupling parameter α is carefully tracked. Such estimates provide us with the necessary compactness to study the limit $\alpha \to \infty$ in Section 6, where Theorem 6.1 is the second main result of this paper. It shows that a (sub)sequence of solutions to (1.6)-(1.8) converges to a weak solution of the diffusive-dispersive problem (1.1). Again, the scaling of β with respect to the coupling parameter α is crucial: we sharpen for fixed value of ε the asymptotic relation (1.7) to

$$\beta = \beta(\alpha) = \mathcal{O}(\alpha^{-1}) \text{ for } \alpha \to \infty. \tag{1.10}$$

We also obtain an explicit convergence rate with respect to the coupling parameter. Finally, the sharp interface limit $\varepsilon \to 0$ is analyzed in Section 7, now using for fixed α the scaling

$$\beta = \beta(\varepsilon) = \mathcal{O}(\varepsilon) \text{ for } \varepsilon \to 0. \tag{1.11}$$

The main result there is Theorem 7.1, which is proven by using the compensated compactness approach in the version presented in [16].

2 Undercompressive Shock Waves

In this section we prove that, for all $\varepsilon > 0$, the parabolic system (1.6) with f provided by (1.9) admits smooth traveling-wave solutions for α sufficiently large and β sufficiently small; moreover, such traveling waves converge almost everywhere for $\varepsilon \to 0$ to an *undercompressive* shock wave [15] of the homogeneous equation (1.2). Then, the parabolic system (1.6) can be seen as a singular perturbation of (1.1), for $\alpha \gg 1$ and $\beta \ll 1$. We refer to [13, 11] for traveling-wave profiles of both Lax and undercompressive shock waves for the equation (1.1). More precisely, if

$$U(x,t) = \begin{cases} u_- & \text{if } x - st < 0, \\ u_+ & \text{if } x - st > 0, \end{cases}$$

is a shock wave for (1.2) and the flux function is given by (1.9), then $s = u_-^2 + u_-u_+ + u_+^2$. Undercompressive shock waves satisfying the condition $f'(u_\pm) > s$ exist if, for instance,

$$-2u_- < u_+ < -\frac{u_-}{2} < 0. \tag{2.1}$$

Traveling solutions $U^\varepsilon(x,t) = u\left(\frac{x-st}{\varepsilon}\right)$ of (1.1) are solutions to

$$\begin{cases} u' &= z, & u(\pm\infty) = u_\pm, \\ \gamma z' &= -z - su + f(u) + c, & z(\pm\infty) = 0, \end{cases} \tag{2.2}$$

for $c := su_- - f(u_-) = su_+ - f(u_+)$. There are three rest points $(u, 0)$ for the flow in (2.2); they are $u_+ < u_0 = -(u_+ + u_-) < u_-$. If we denote $p(u) := su - f(u) - c$, then the eigenvalues at the rest points $(u, 0) = (u_\pm, 0)$ are $\lambda = [-1 \pm \sqrt{1 - 4\gamma p'(u)}]/(2\gamma)$ and $u_\pm$ are both saddles if and only if $p'(u_\pm) < 0$. This condition is equivalent to (2.1). We write system (2.2) as

$$\begin{cases} u' &= z, & u(\pm\infty) = u_\pm, \\ \gamma z' &= -z - p(u), & z(\pm\infty) = 0, \\ \gamma' &= 0, & \gamma(0) = l. \end{cases} \tag{2.3}$$

The sets $\mathcal{M}_0^\pm = \{(u_\pm, 0, l) \,|\, l \in \mathbb{R},\, l \neq 0\}$ are easily seen to be normally hyperbolic [14]. We rewrite (2.3) by introducing $w := z + p(u)$. We obtain

$$\begin{cases} u' &= w - p(u), & u(\pm\infty) = u_\pm, \\ w' &= -\dfrac{w}{\gamma} + p'(u)\left(w - p(u)\right), & w(\pm\infty) = 0, \\ \gamma' &= 0. \end{cases} \tag{2.4}$$

The transformed manifolds $M_0^\pm = \{(u_\pm, 0, l) \,|\, l \in \mathbb{R},\, l \neq 0\}$ of $\mathcal{M}_0^\pm$ are still normally hyperbolic. Inspired by [13] we proved in [6] the following result.

Theorem 2.1 *Consider the boundary-value problem* (2.4) *under assumptions* (1.9)*. If*

$$-u_- < u_+ < -u_-/2 < 0, \tag{2.5}$$

then there is a unique $\bar\gamma > 0$ such that, up to shifts, problem (2.4) *has a unique solution.*

Moreover, the intersection of the unstable manifold $W^u(u_-, \bar\gamma)$ from $(u_-, 0, \bar\gamma) \in M_0^-$ and the stable manifold $W^s(u_+, \bar\gamma)$ from $(u_+, 0, \bar\gamma) \in M_0^+$ is transverse with respect to the flow of (2.4)*.*

We pass now to (1.6). Since $\varepsilon > 0$ plays a minor role in this section and to avoid multiple upper indices, we write in the rest of this section u_α and λ_α instead of $u^{\varepsilon,\alpha}$ and $\lambda^{\varepsilon,\alpha}$. A traveling-wave solution to (1.6) with speed s is a solution of the form

$$\big(U_\alpha(x,t), L_\alpha(x,t)\big) = \left(u_\alpha\left(\frac{x - st}{\varepsilon}\right), \lambda_\alpha\left(\frac{x - st}{\varepsilon}\right)\right)$$

satisfying $(u_\alpha(\pm\infty), \lambda_\alpha(\pm\infty)) = (u_\pm, \lambda_\pm)$ and $u'_\alpha(\pm\infty) = \lambda'_\alpha(\pm\infty) = 0$. Then, denoting $\beta = \varepsilon b$ to have the correct scaling in ε, the functions $(u_\alpha, \lambda_\alpha)$ must solve the system

$$\begin{cases} u'_\alpha &= \alpha(u_\alpha - \lambda_\alpha) - p(u_\alpha), \\ -\gamma\lambda''_\alpha - sb\lambda'_\alpha &= \alpha(u_\alpha - \lambda_\alpha). \end{cases} \tag{2.6}$$

We need $\lambda_\pm = u_\pm$ in order that $(u_\pm, \lambda_\pm)$ are rest points of the flow in (2.6); in turn, the assumption $u'_\alpha(\pm\infty) = 0$ implies $c = -su_\pm + f(u_\pm)$. Then, system (2.6) is completed by the boundary conditions

$$u_\alpha(\pm\infty) = \lambda_\alpha(\pm\infty) = u_\pm\,, \quad \lambda'_\alpha(\pm\infty) = 0. \tag{2.7}$$

We make the change of variables

$$w_\alpha := \alpha(u_\alpha - \lambda_\alpha).$$

Equation $(2.6)_1$ now reads $u'_\alpha = w_\alpha - p_\alpha$; then, denoting $v_\alpha = w'_\alpha$, we compute $\lambda'_\alpha = w_\alpha - p_\alpha - \frac{v_\alpha}{\alpha}$. By $(2.6)_2$ we deduce

$$u''_\alpha \;=\; v_\alpha - p'_\alpha(w_\alpha - p_\alpha),$$

$$-\lambda''_\alpha \;=\; \frac{w_\alpha}{\gamma} + \frac{sb}{\gamma}\lambda'_\alpha \;=\; \frac{w_\alpha}{\gamma} + \frac{sb}{\gamma}\left(w_\alpha - p_\alpha\right) - \frac{sb}{\gamma\alpha}v_\alpha,$$

where we wrote $p_\alpha = p(u_\alpha)$ for short. We deduce

$$\begin{aligned} w''_\alpha &= \alpha(u''_\alpha - \lambda''_\alpha) \\ &= \alpha\left(v_\alpha - p'_\alpha(w_\alpha - p_\alpha) + \frac{w_\alpha}{\gamma} + \frac{sb}{\gamma}\left(w_\alpha - p_\alpha\right) - \frac{sb}{\gamma\alpha}v_\alpha\right) \\ &= \frac{\alpha}{\gamma}\left(w_\alpha + \left(sb - \gamma p'_\alpha\right)\left(w_\alpha - p_\alpha\right) + \left(\gamma - \frac{sb}{\alpha}\right)v_\alpha\right). \end{aligned}$$

We denote

$$G(u, w, \gamma, v, \alpha, b) = \frac{1}{\gamma}\left(w + \left(sb - \gamma p'(u)\right)\left(w - p(u)\right) + \left(\gamma - \frac{sb}{\alpha}\right)v\right)$$

and notice that until now the parameter b played no essential role; in particular. We deduce that (2.6)–(2.7) is equivalent to

$$\begin{cases} u'_\alpha = w_\alpha - p(u_\alpha), & u_\alpha(\pm\infty) = u_\pm, \\ w'_\alpha = v_\alpha, & w_\alpha(\pm\infty) = 0, \\ \gamma'_\alpha = 0, & \\ \frac{1}{\alpha}v'_\alpha = G(u_\alpha, w_\alpha, \gamma_\alpha, v_\alpha, \alpha, b), & v_\alpha(\pm\infty) = 0. \end{cases} \tag{2.8}$$

We remark that, differently from [6], the functional G depends not only on β (through b) but, in particular, on α, because of the presence of the term λ_t^α in $(1.6)_2$. However, for $\alpha \to \infty$ this dependence is not singular:

$$G(u, w, \gamma, v, \infty, b) = \frac{1}{\gamma}\left(w + \left(sb - \gamma p'(u)\right)\left(w - p(u)\right) + \gamma v\right).$$

In particular, $G(u, w, \gamma, v, \infty, b) = 0$ is equivalent to

$$v = -\frac{w}{\gamma} + \left(p'(u) - \frac{sb}{\gamma}\right)\left(w - p(u)\right).$$

Now, we compare (2.8) with (2.4); we see that (2.4) is the reduced system for $\alpha = \infty$ of (2.8) if and only if

$$\beta = \beta(\alpha) \to 0 \ \text{ as } \alpha \to \infty. \tag{2.9}$$

We notice that the requirement (2.9) introduces no singular behavior in the functional G defined above. Therefore, under the conditions (2.9), system (2.8), which is written with respect to a slow-time scale, falls into the framework of the geometric singular perturbation theory for α sufficiently large [8, 14]. We now state our final result.

Theorem 2.2 *Consider the boundary-value problem* (2.8) *with f given by* (1.9) *and assume* (2.5), (2.9).

Then, for $\alpha \gg 1$ there is a unique $\bar\gamma_\alpha > 0$ such that, up to shifts, problem (2.8) *has a unique solution, with $\gamma_\alpha = \bar\gamma_\alpha$. Moreover, we have $\bar\gamma_\infty = \bar\gamma$.*

Proof. We rely on the formulation of the geometric singular perturbation theory provided in [8, Theorem 9.1]. A simplified statement can be found in [10, Proposition 3.2], even if in that proposition G is independent from α; however, the result in [8, Theorem 9.1], which allows a regular dependence on the vanishing parameter, applies straightforwardly to the framework provided in [10].

We just need to check that the zero-set $G(u, w, \gamma, v, \infty, 0) = 0$ is indeed the graph $\mathcal{C}_0$ of a smooth function $h = h(u, w, \gamma)$ and that the condition $G_v(u, w, \gamma, v, \infty, 0) \neq 0$ is satisfied on $\mathcal{C}_0$. Both conditions are easily checked: since $\gamma > 0$, the implicit equation $G(u, w, \gamma, v, \infty, 0) = 0$ is uniquely solved as

$$v = -\frac{w}{\gamma} + \big(s - f'(u)\big)\,(w - p(u)) =: h(u, w, \gamma),$$

which defines a manifold $\mathcal{C}_0$ on which $G_v \equiv 1$. □

3 Energy estimates

In this section we prove some energy-type estimates for the case of flux functions with bounded derivatives. We start by making an assumption about the existence of solutions to (1.6); indeed, we shall prove in Section 4 that such solutions do exist by using the estimates of this section. We denote by $C_1^k(\Omega)$ the space of functions defined in $\Omega \subset \mathbb{R}^2$ having k continuous derivatives in x and one continuous derivative in t.

Assumption 3.1 *Let $L > 0$ be a constant such that $f \in C^3(\mathbb{R})$ satisfies $|f'(u)|+|f''(u)| \leq L$ and*

$$F(u) := \int_0^u f(w)\,\mathrm{d}w \geq 0 \qquad \textit{for all } u \in \mathbb{R}. \tag{3.1}$$

The role of condition (3.1) is to ensure the non-negativity of certain energy terms, see (3.11). Figure 1 illustrates a possible choice of $f(u)$ and the resulting function $F(u)$. In particular, since we assumed $f(0) = 0$, the function F satisfies

$$0 \leq F(u) \leq \frac{L}{2}\,u^2. \tag{3.2}$$

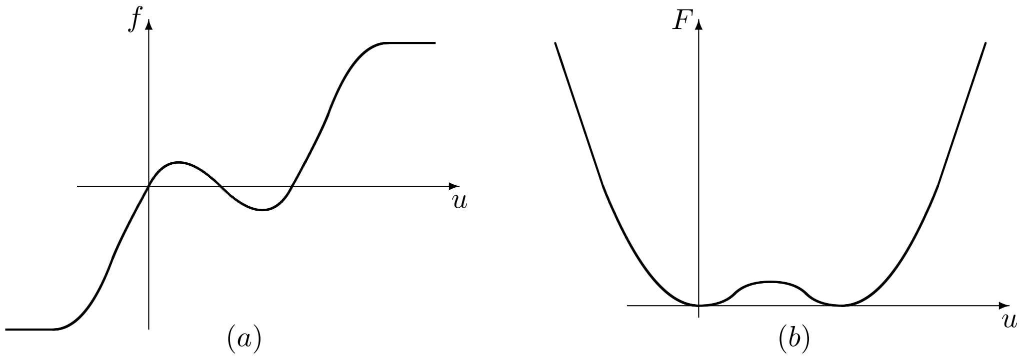

Figure 1: Example of a function f fulfilling Assumption 3.1, *(a)*, and its associated double-well function F, *(b)*.

We emphasize that Assumption 3.1 shall be fully exploited only from Section 5 on. As regards the current section, it only suffices that $f \in C^1(\mathbb{R})$ satisfies $|f'(u)| \leq L$ and (3.1).

Assumption 3.2 *Let $u_0 \in H^3(\mathbb{R})$ and $T > 0$ be fixed. We assume that for every $\varepsilon, \alpha, \beta > 0$ there exists a classical solution $(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha}) : \bar{\Omega}_T \to \mathbb{R}^2$ of* (1.6)-(1.8) *satisfying*

$$u^{\varepsilon,\alpha} \quad \in \quad C_1^2\Big((0,T] \times \mathbb{R}\Big) \cap L^\infty\left(0, T; H^3(\mathbb{R})\right). \tag{3.3}$$

We start by providing a representation formula of the function $\lambda^{\varepsilon,\alpha}$ in terms of $u^{\varepsilon,\alpha}$.

Lemma 3.3 *Let Assumptions 3.1 and 3.2 hold. Then the solution $\lambda^{\varepsilon,\alpha}$ satisfies for $(x,t)\in\bar\Omega_T$ the representation formula*

$$\lambda^{\varepsilon,\alpha}(x,t)=\int_0^t\int_{\mathbb{R}}K^{\varepsilon,\alpha}(x-y,t-s)u^{\varepsilon,\alpha}(y,s)\,\mathrm{d}y+\frac{\beta}{\alpha}\int_{\mathbb{R}}K^{\varepsilon,\alpha}(x-y,t)u_0(y)\,\mathrm{d}y\,, \tag{3.4}$$

where

$$K^{\varepsilon,\alpha}(x,t)=\frac{\alpha}{\beta}\sqrt{\frac{\beta}{4\pi\gamma\varepsilon^2 t}}e^{-\frac{\alpha}{\beta}t-\frac{\beta x^2}{4\gamma\varepsilon^2 t}}\qquad (t>0).$$

Furthermore, for $t\in(0,T]$ and $l=0,1,2,3$, we have the estimates

$$\|\lambda^{\varepsilon,\alpha}\|_{L^\infty\left(0,t;L^2(\mathbb{R})\right)}\ \le\ \|u^{\varepsilon,\alpha}\|_{L^\infty\left(0,t;L^2(\mathbb{R})\right)}+\|u_0\|_{L^2(\mathbb{R})}, \tag{3.5}$$

$$\|\partial_x^l\lambda^{\varepsilon,\alpha}\|_{L^2(\Omega_t)}\ \le\ \|\partial_x^l u^{\varepsilon,\alpha}\|_{L^2(\Omega_t)}+\sqrt{\frac{\beta}{2\alpha}}\,\|\partial_x^l u_0\|_{L^2(\mathbb{R})}. \tag{3.6}$$

Proof. Consider in $\bar\Omega_T$ the initial-value problem

$$\begin{cases}\lambda_t-\lambda_{xx}+\lambda=g,\\ \lambda(.,0)=u_0,\end{cases} \tag{3.7}$$

for $g\in L^\infty\left(0,T;L^2(\mathbb{R})\right)$ and $u_0\in L^2(\mathbb{R})$. We define the Bessel potential

$$B(x,t)=\begin{cases}\dfrac{1}{\sqrt{4\pi t}}\,e^{-t-\frac{x^2}{4t}} & \text{if } t>0,\\ 0 & \text{if } t\le 0.\end{cases}$$

Arguing as in [7, Examples 1 and 2, pages 186-188, Remark page 51], we see that any bounded solution of the initial value problem 3.7 satisfies

$$\begin{aligned}\lambda(x,t)\ &=\ \int_0^t\int_{\mathbb{R}}B(x-y,t-s)g(y,s)\,\mathrm{d}y\,\mathrm{d}s+\int_{\mathbb{R}}B(x-y,t)u_0(y)\,\mathrm{d}y\\ &=\ \int_0^t\big(B(t-s)*g(s)\big)(x)\,\mathrm{d}s+\big(B(t)*u_0\big)(x),\end{aligned}$$

where we denoted by "$*$" the convolution with respect to the x variable. Here and below we often use a functional notation and then drop the dependence on the variable x.

The uniqueness follows by [7, Theorem 7, §2.3]. To find the representation formula for $\lambda^{\varepsilon,\alpha}$ we divide the equation $(1.6)_2$ by α, denote $a=\beta/\alpha$, $b=\gamma\varepsilon^2/\alpha$, and then make the change of variables $t'=t/a$, $x'=x/\sqrt{b}$ to reduce to the equation in (3.7) with $u^{\varepsilon,\alpha}$ replacing g. Then, the kernel B becomes

$$K^{\varepsilon,\alpha}(x,t)=\begin{cases}\dfrac{\alpha}{\beta}\sqrt{\dfrac{\beta}{4\pi\gamma\varepsilon^2 t}}e^{-\frac{\alpha}{\beta}t-\frac{\beta x^2}{4\gamma\varepsilon^2 t}} & \text{if } t>0,\\ 0 & \text{if } t\le 0,\end{cases}$$

and the solution is given by

$$\lambda^{\varepsilon,\alpha}(x,t)=\int_0^t\big(K^{\varepsilon,\alpha}(t-s)*u^{\varepsilon,\alpha}(s)\big)(x)\,\mathrm{d}s+\frac{\beta}{\alpha}\big(K^{\varepsilon,\alpha}(t)*u_0\big)(x),\qquad \text{in } \bar\Omega_T. \tag{3.8}$$

This proves (3.4). To prove (3.5)–(3.6) we notice that for every $t>0$ we have

$$\int_{\mathbb{R}}K^{\varepsilon,\alpha}(x,t)\,\mathrm{d}x=\frac{\alpha}{\beta}\,e^{-\frac{\alpha}{\beta}t} \tag{3.9}$$

and then $\|K^{\varepsilon,\alpha}\|_{L^1(\mathbb{R}\times\mathbb{R}^+)} = 1$. We first apply Minkowski's inequality to (3.8) and then Minkowski integral inequality [1, Theorem 2.9]; we obtain

$$\begin{aligned}
\|\lambda^{\varepsilon,\alpha}(t)\|_{L^2(\mathbb{R})} &\leq \int_0^t \|K^{\varepsilon,\alpha}(t-s) * u^{\varepsilon,\alpha}(s)\|_{L^2(\mathbb{R})}\,\mathrm{d}s + \frac{\beta}{\alpha}\,\|K^{\varepsilon,\alpha}(t) * u_0\|_{L^2(\mathbb{R})} \\
&\leq \int_0^t \|K^{\varepsilon,\alpha}(t-s)\|_{L^1(\mathbb{R})}\|u^{\varepsilon,\alpha}(s)\|_{L^2(\mathbb{R})}\,\mathrm{d}s + \frac{\beta}{\alpha}\,\|K^{\varepsilon,\alpha}(t)\|_{L^1(\mathbb{R})}\|u_0\|_{L^2(\mathbb{R})}.
\end{aligned}$$

We fix any $\tau \in (0,T]$; by (3.9) we deduce

$$\begin{aligned}
\|\lambda^{\varepsilon,\alpha}\|_{L^\infty(0,\tau;L^2(\mathbb{R}))} &\leq \sup_{t\in[0,\tau]} \int_0^t \|K^{\varepsilon,\alpha}(t-s)\|_{L^1(\mathbb{R})}\|u^{\varepsilon,\alpha}(s)\|_{L^2(\mathbb{R})}\,\mathrm{d}s + \|u_0\|_{L^2(\mathbb{R})} \\
&\leq \left(1 - e^{-\frac{\alpha}{\beta}\tau}\right)\|u^{\varepsilon,\alpha}\|_{L^\infty(0,\tau;L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})},
\end{aligned}$$

whence (3.5). We now consider (3.6); it is clearly sufficient to prove this estimate in the case $l = 0$. By (3.8) it follows that

$$\|\lambda^{\varepsilon,\alpha}\|_{L^2(\Omega_t)} \leq \|U^{\varepsilon,\alpha}\|_{L^2(\Omega_t)} + \|U_0^{\varepsilon,\alpha}\|_{L^2(\Omega_t)},$$

where $U^{\varepsilon,\alpha}$ and $U_0^{\varepsilon,\alpha}$ denote the summands in (3.8). By Hölder inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\int_0^t\int_{\mathbb{R}} K^{\varepsilon,\alpha}(t-s,x-\xi)u^{\varepsilon,\alpha}(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}s\right)^2\mathrm{d}x \\
&\quad\leq \int_{\mathbb{R}}\left(\int_0^t\int_{\mathbb{R}} K^{\varepsilon,\alpha}(t-s,x-\xi)\,\mathrm{d}\xi\,\mathrm{d}s\right)\cdot\left(\int_0^t\int_{\mathbb{R}} K^{\varepsilon,\alpha}(t-s,x-\xi)|u^{\varepsilon,\alpha}(s,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}s\right)\mathrm{d}x \\
&\quad\leq \int_0^t \frac{\alpha}{\beta}\,e^{-\frac{\alpha}{\beta}(t-s)}\|u^{\varepsilon,\alpha}(s,.)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\,.
\end{aligned}$$

Now, we integrate in time to deduce

$$\begin{aligned}
\|U^{\varepsilon,\alpha}\|^2_{L^2(\Omega_\tau)} &\leq \int_0^\tau\int_0^t \frac{\alpha}{\beta}e^{-\frac{\alpha}{\beta}(t-s)}\|u^{\varepsilon,\alpha}(s,.)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\tau\int_s^\tau \frac{\alpha}{\beta}e^{-\frac{\alpha}{\beta}(t-s)}\|u^{\varepsilon,\alpha}(s,.)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t\,\mathrm{d}s \;\leq\; \|u^{\varepsilon,\alpha}\|^2_{L^2(\Omega_\tau)}.
\end{aligned}$$

The estimate of $\|U_0^{\varepsilon,\alpha}\|_{L^2(\Omega_\tau)}$ is proved by using Hölder inequality as above:

$$\int_0^\tau\int_{\mathbb{R}}\left(\int_{\mathbb{R}} K^{\varepsilon,\alpha}(t,x-\xi)u_0(\xi)\,d\xi\right)^2\mathrm{d}x\,\mathrm{d}t \leq \frac{\alpha}{2\beta}\left(1-e^{-\frac{2\alpha}{\beta}\tau}\right)\|u_0\|^2_{L^2(\mathbb{R})} \;\leq\; \frac{\alpha}{2\beta}\|u_0\|^2_{L^2(\mathbb{R})}.$$

Then

$$\|U_0^{\varepsilon,\alpha}\|_{L^2(\Omega_\tau)} \leq \sqrt{\frac{\beta}{2\alpha}}\|u_0\|_{L^2(\mathbb{R})}.$$

This proves (3.6). □

We point out that a much shorter proof of Lemma 3.3, which does not require a representation formula of $\lambda^{\varepsilon,\alpha}$ in term of $u^{\varepsilon,\alpha}$, can be obtained by “energy methods”. For instance, to obtain an estimate analogous to (3.6), $l = 0$, one multiplies $(1.6)_2$ by $\lambda^{\varepsilon,\alpha}$, integrates with respect to space and finally uses partial integration. However, this proof provides slightly worse constants and requires more regularity on $\lambda^{\varepsilon,\alpha}$.
An estimate of $\|\lambda_t^{\varepsilon,\alpha}\|_{L^2(\Omega_t)}$ could be deduced by exploiting directly equation $(1.6)_2$ jointly with (3.6); we omit it since we shall prove in a better estimate in Corollary 3.7.

A natural energy for (1.1) is the van der Waals' type energy

$$E^{\varepsilon}[u] = \int_{\mathbb{R}} \left(F(u) + \gamma \frac{\varepsilon^2}{2} u_x^2 \right) dx. \tag{3.10}$$

The energy (3.10) is dissipated by any smooth solution u^{ε} of (1.1) in the sense that $\frac{d}{dt}\left(E^{\varepsilon}[u^{\varepsilon}(.,t)]\right) \leq 0$, for $t \in (0,T)$. Another global ε-independent control on the evolution of the solution of (1.1) is given by the L^2-norm of u^{ε}.
The role of E^{ε} for the elliptic approximation (1.5) considered in [6] is taken by

$$\mathcal{E}^{\varepsilon,\alpha}[u,\lambda] = \int_{\mathbb{R}} \left(F(u) + \frac{\alpha}{2}(u-\lambda)^2 + \frac{\gamma\varepsilon^2}{2}\lambda_x^2 \right) dx.$$

We refer to [24, 25] for the variational analysis of $\mathcal{E}^{\varepsilon,\alpha}$ and to [4, 18, 19, 21] for related functionals. Again, the L^2-norm of solutions to (1.5) is uniformly bounded, now with respect to ε and α [6].
Now, we introduce a further generalization of the energy E^{ε} for the parabolic approximation (1.6). This is

$$E^{\varepsilon,\alpha}[u,\lambda] = \int_{\mathbb{R}} \left(F(u) + \frac{\alpha}{2}(u-\lambda)^2 + \frac{\varepsilon}{2}(\gamma\varepsilon + \beta)\lambda_x^2 \right) dx. \tag{3.11}$$

It formally collapses to $\mathcal{E}^{\varepsilon,\alpha}$ setting $\beta \equiv 0$. We will show that both the energy $E^{\varepsilon,\alpha}$ and the L^2-norm of the solution are bounded uniformly with respect to ε and $\alpha > 1$ (see (3.16)). Most notably, in the current parabolic case none of these two estimates can be derived independently as in [6, Lemmas 3.2 and 3.3] for the elliptic approximation (see Corollary 3.6 below).

Lemma 3.4 *Let Assumptions 3.1 and 3.2 hold. Then, for all $t \in [0,T]$ we have the estimate*

$$\begin{aligned} \int_{\mathbb{R}} (u^{\varepsilon,\alpha})^2 \,\mathrm{d}x + \varepsilon \int_0^t \int_{\mathbb{R}} (u_x^{\varepsilon,\alpha})^2 \,\mathrm{d}x\,\mathrm{d}t & \\ \leq \frac{2\beta^2}{\varepsilon} \int_0^t \int_{\mathbb{R}} (\lambda_t^{\varepsilon,\alpha})^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} (u_0)^2 \,\mathrm{d}x + \frac{\beta\varepsilon}{2\alpha} \int_{\mathbb{R}} (u_{0,x})^2 \,\mathrm{d}x\,. & \end{aligned} \tag{3.12}$$

Proof. We drop the upper indices and write for simplicity $(u,\lambda) = (u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})$. From the regularity assumption (3.3) and Morrey's estimate (see, e.g., [7, §5.6.2] or [5, Cor. VIII.8]), we deduce that $|\partial_x^l u(x,t)| \to 0$ as $|x| \to \infty$, for every $t \in (0,T)$ and $l = 0,1,2$. By multiplying $(1.6)_1$ by u and integrating with respect to x we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} u^2 \,\mathrm{d}x + \varepsilon \int_{\mathbb{R}} (u_x)^2 \,\mathrm{d}x = \alpha \int_{\mathbb{R}} (u-\lambda)\lambda_x \,\mathrm{d}x\,,$$

or, in the time-integrated form,

$$\frac{1}{2}\int_{\mathbb{R}} (u(\cdot,t))^2 \,\mathrm{d}x + \varepsilon \int_0^t \int_{\mathbb{R}} (u_x)^2 \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_{\mathbb{R}} u_0^2 \,\mathrm{d}x + \alpha \int_0^t \int_{\mathbb{R}} (u-\lambda)\lambda_x \,\mathrm{d}x\,\mathrm{d}t\,.$$

Using $(1.6)_2$, we obtain

$$\frac{1}{2}\int_{\mathbb{R}} (u(\cdot,t))^2 \,\mathrm{d}x + \varepsilon \int_0^t \int_{\mathbb{R}} (u_x)^2 \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_{\mathbb{R}} u_0^2 \,\mathrm{d}x + \beta \int_0^t \int_{\mathbb{R}} \lambda_x\lambda_t \,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.13}$$

Here, we exploited the fact that $\lambda_{xx}^{\varepsilon,\alpha} \to 0$ as $|x| \to \infty$ because of Lemma 3.3 and then argued as above. By the help of the basic inequality

$$ab \leq \rho a^2 + \frac{b^2}{4\rho}\,, \tag{3.14}$$

which holds for every $a, b, \rho > 0$ [7, (5) page 622] and estimate (3.6), we deduce

$$\beta \int_0^t \int_{\mathbb{R}} \lambda_x \lambda_t \,\mathrm{d}x\,\mathrm{d}t \le \rho \int_0^t \int_{\mathbb{R}} (\lambda_x)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\beta^2}{4\rho} \int_0^t \int_{\mathbb{R}} (\lambda_t)^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\le 2\rho \int_0^t \int_{\mathbb{R}} (u_x)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\beta^2}{4\rho} \int_0^t \int_{\mathbb{R}} (\lambda_t)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\beta\rho}{\alpha} \int_{\mathbb{R}} (u_{0,x})^2 \,\mathrm{d}x\,.$$

Choosing $\rho = \frac{\varepsilon}{4}$, by (3.13) we obtain (3.12). □

Due to the uncontrolled first term on the right-hand side of (3.12), Lemma 3.4 does not yet provide a uniform estimate with respect to ε of the L^2-norm of $u^{\varepsilon,\alpha}$. This is achieved by requiring that β is small with respect to ε, as in the following result where we consider the evolution of the energy $E^{\varepsilon,\alpha}$.

Lemma 3.5 *Let Assumptions 3.1 and 3.2 hold. Then, for all $t \in [0,T]$ we have the estimate*

$$\int_{\mathbb{R}} \left\{ F(u^{\varepsilon,\alpha}) + \frac{\alpha}{2}(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha})^2 + \frac{\varepsilon}{2}(\gamma\varepsilon + \beta)(\lambda_x^{\varepsilon,\alpha})^2 \right\} \mathrm{d}x +$$
$$+ \int_0^t \int_{\mathbb{R}} \left\{ \varepsilon \left((u_x^{\varepsilon,\alpha})^2 f'(u^{\varepsilon,\alpha}) + \alpha \left(u_x^{\varepsilon,\alpha} - \lambda_x^{\varepsilon,\alpha} \right)^2 + \gamma\varepsilon^2 (\lambda_{xx}^{\varepsilon,\alpha})^2 \right) + \beta (\lambda_t^{\varepsilon,\alpha})^2 \right\} \mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\mathbb{R}} \left\{ F(u_0) + \frac{\varepsilon}{2}(\gamma\varepsilon + \beta)(u_{0,x})^2 \right\} \mathrm{d}x\,. \tag{3.15}$$

Moreover, for $\beta < \frac{\varepsilon}{4L}$ we have the estimate

$$\begin{aligned} &E^{\varepsilon,\alpha}[u^{\varepsilon,\alpha}(.,t), \lambda^{\varepsilon,\alpha}(.,t)] - E^{\varepsilon,\alpha}[u_0, u_0] \\ &\le \; L\|u_0\|^2_{L^2(\mathbb{R})} + \frac{L\beta\varepsilon}{2\alpha}\|u_{0,x}\|^2_{L^2(\mathbb{R})} - \frac{\beta}{2}\left\|\lambda_t^{\varepsilon,\alpha}\right\|^2_{L^2(\Omega_t)} \\ &\quad -\varepsilon\left(\alpha\|u_x^{\varepsilon,\alpha} - \lambda_x^{\varepsilon,\alpha}\|^2_{L^2(\Omega_t)} + \gamma\varepsilon^2\|\lambda_{xx}^{\varepsilon,\alpha}\|^2_{L^2(\Omega_t)}\right). \end{aligned} \tag{3.16}$$

Proof. As in Lemma 3.4 we write $(u,\lambda) = (u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})$. We also recall the decay properties of $|\partial_x^i u(x,t)|$ and $|\partial_x^i \lambda(x,t)|$ for $|x| \to \infty$, for $i = 0,1,2$ and $t \in [0,T]$, mentioned in Lemma 3.4; they are used below to justify integration by parts. We multiply $(1.6)_1$ by $f(u) = F'(u)$; then we multiply again $(1.6)_1$ by $\alpha(u-\lambda)$ and $(1.6)_2$ by λ_t. Finally, we integrate with respect to x and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} F(u)\,\mathrm{d}x + \int_{\mathbb{R}} f(u)_x f(u)\,\mathrm{d}x = \varepsilon \int_{\mathbb{R}} u_{xx} f(u)\,\mathrm{d}x - \alpha\int_{\mathbb{R}} (u-\lambda)_x f(u)\,\mathrm{d}x\,, \tag{3.17}$$

$$\begin{aligned} \alpha\int_{\mathbb{R}} u_t(u-\lambda)\,\mathrm{d}x + \alpha\int_{\mathbb{R}} f(u)_x(u-\lambda)\,\mathrm{d}x =& \varepsilon\alpha\int_{\mathbb{R}} u_{xx}(u-\lambda)\,\mathrm{d}x \\ &- \alpha^2\int_{\mathbb{R}}(u-\lambda)_x(u-\lambda)\,\mathrm{d}x\,, \end{aligned} \tag{3.18}$$

$$\beta\int_{\mathbb{R}}(\lambda_t)^2\,\mathrm{d}x - \gamma\varepsilon^2\int_{\mathbb{R}}\lambda_{xx}\lambda_t\,\mathrm{d}x = \alpha\int_{\mathbb{R}}(u-\lambda)\lambda_t\,\mathrm{d}x\,. \tag{3.19}$$

We notice that

$$\begin{aligned} \int_{\mathbb{R}} u_{xx}(u-\lambda)\,\mathrm{d}x &= -\int_{\mathbb{R}}(u_x)^2\,\mathrm{d}x - \int_{\mathbb{R}} u\lambda_{xx}\,\mathrm{d}x \\ &= -\int_{\mathbb{R}}(u_x)^2\,\mathrm{d}x + \int_{\mathbb{R}}(\lambda_x)^2\,\mathrm{d}x + \frac{\gamma\varepsilon^2}{\alpha}\int_{\mathbb{R}}(\lambda_{xx})^2\,\mathrm{d}x - \frac{\beta}{\alpha}\int_{\mathbb{R}}\lambda_{xx}\lambda_t\,\mathrm{d}x\,. \end{aligned} \tag{3.20}$$

The second line above was obtained by plugging the expression of u deduced from $(1.6)_2$ into the first line. Summing up (3.17)–(3.19) and taking into account (3.20), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left\{F(u) + \frac{\alpha}{2}(u-\lambda)^2 + \frac{\varepsilon}{2}(\varepsilon\gamma - \beta)(\lambda_x)^2\right\}\mathrm{d}x$$

$$+\int_{\mathbb{R}}\left\{\varepsilon\,(u_x)^2 f'(u)+\varepsilon\alpha(u_x)^2+\beta(\lambda_t)^2\right\}\mathrm{d}x=\varepsilon\int_{\mathbb{R}}\left\{\alpha(\lambda_x)^2+\gamma\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x\,. \quad (3.21)$$

Moreover, by $(1.6)_2$, we observe that

$$\begin{aligned}-\gamma\varepsilon^2\int_{\mathbb{R}}(\lambda_{xx})^2\,\mathrm{d}x &= \gamma\varepsilon^2\int_{\mathbb{R}}\left\{2\lambda_x\lambda_{xxx}+(\lambda_{xx})^2\right\}\mathrm{d}x\\ &= \int_{\mathbb{R}}\left\{-2\alpha\lambda_x u_x+2\alpha(\lambda_x)^2+2\beta\lambda_{tx}\lambda_x+\gamma\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x\,.\end{aligned}$$

Then,

$$\int_{\mathbb{R}}\left\{\alpha(u_x)^2-\alpha(\lambda_x)^2-\gamma\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x=\int_{\mathbb{R}}\left\{\alpha\,(u_x-\lambda_x)^2+2\beta\lambda_{tx}\lambda_x+\gamma\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x\,.$$

Plugging this expression into (3.21) we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left\{F(u)+\frac{\alpha}{2}(u-\lambda)^2+\frac{\varepsilon}{2}(\gamma\varepsilon+\beta)(\lambda_x)^2\right\}\mathrm{d}x+\\ &\quad+\int_{\mathbb{R}}\left\{\varepsilon\left((u_x)^2 f'(u)+\alpha\,(u_x-\lambda_x)^2+\gamma\varepsilon^2(\lambda_{xx})^2\right)+\beta(\lambda_t)^2\right\}\mathrm{d}x=0\,. \quad (3.22)\end{aligned}$$

If we integrate with respect to t we obtain (3.15). To deduce (3.16) we simply use Lemma 3.4, the non-negativity of F and the assumption $\beta<\frac{\varepsilon}{4L}$. □

We observe that equality (3.15) cannot be fully exploited when deducing the inequality (3.16) since the sign of f' is not prescribed. Of course, if f is increasing and thus F is convex, the energy $E^{\varepsilon,\alpha}$ is decreasing. We have however the following result.

Corollary 3.6 *Let Assumptions 3.1 and 3.2 hold and assume $\beta<\frac{\varepsilon}{4L}$. Then, the following estimate holds:*

$$\|u^{\varepsilon,\alpha}(.,t)\|^2_{L^2(\mathbb{R})}+\varepsilon\|u^{\varepsilon,\alpha}_x\|^2_{L^2(\Omega_t)}\;\le\;2\|u_0\|^2_{L^2(\mathbb{R})}+\frac{\beta\varepsilon}{\alpha}\|u_{0,x}\|^2_{L^2(\mathbb{R})}+\frac{1}{L}E^{\varepsilon,\alpha}[u_0,u_0]. \quad (3.23)$$

Proof. Since the energy $E^{\varepsilon,\alpha}$ is non-negative, by (3.16) in Lemma 3.5 we deduce

$$\frac{\beta}{2}\left\|\lambda^{\varepsilon,\alpha}_t\right\|^2_{L^2(\Omega_t)}\le E^{\varepsilon,\alpha}[u_0,u_0]+L\|u_0\|^2_{L^2(\mathbb{R})}+\frac{L\beta\varepsilon}{2\alpha}\|u_{0,x}\|^2_{L^2(\mathbb{R})}.$$

We plug this estimate into (3.12) of Lemma 3.4 and obtain

$$\begin{aligned}&\|u^{\varepsilon,\alpha}(.,t)\|^2_{L^2(\mathbb{R})}+\varepsilon\|u^{\varepsilon,\alpha}_x\|^2_{L^2(\Omega_t)}\le\\ &\quad\left(1+\frac{4\beta L}{\varepsilon}\right)\|u_0\|^2_{L^2(\mathbb{R})}+\frac{\beta}{\alpha}\left(2L\beta+\frac{\varepsilon}{2}\right)\|u_{0,x}\|^2_{L^2(\mathbb{R})}+\frac{4\beta}{\varepsilon}E^{\varepsilon,\alpha}[u_0,u_0]. \quad (3.24)\end{aligned}$$

Then, estimate (3.23) follows because $\beta<\frac{\varepsilon}{4L}$. □

Note that in the case of $\beta=0$, (3.24) reduces to the estimate shown in the elliptic case in [6, Lemma 3.2]. In the next Corollary, we summarize the most important a-priori estimates for the subsequent analysis.

Corollary 3.7 *Let Assumptions 3.1 and 3.2 hold and consider the initial energy*

$$E^{\varepsilon,\alpha}[u_0,u_0]=\int_{\mathbb{R}}\left(F(u_0)+\frac{\varepsilon}{2}(\gamma\varepsilon+\beta)u_{0,x}^2\right).$$

Then, $E^{\varepsilon,\alpha}[u_0,u_0]$ *is uniformly bounded with respect to* $\varepsilon > 0$ *and* $0 < \beta < \frac{\varepsilon}{4L}$ *as*

$$E^{\varepsilon,\alpha}[u_0,u_0] \le \frac{L}{2}\|u_0\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}(\gamma\varepsilon + \beta)\|u_0\|^2_{H^1(\mathbb{R})}. \tag{3.25}$$

Moreover, there is a constant $C = C(L,\gamma) > 0$ *such that*

$$\begin{aligned} \max\Big\{\varepsilon\alpha\|u^{\varepsilon,\alpha}_x - \lambda^{\varepsilon,\alpha}_x\|^2_{L^2(\Omega_t)}, \gamma\varepsilon^3\|\lambda^{\varepsilon,\alpha}_{xx}\|^2_{L^2(\Omega_t)}, \beta\|\lambda^{\varepsilon,\alpha}_t\|^2_{L^2(\Omega_t)}\Big\} \\ \le \quad C\Big(\|u_0\|^2_{L^2(\mathbb{R})} + \varepsilon(\varepsilon+\beta)\|u_0\|^2_{H^1(\mathbb{R})}\Big), \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \max\Big\{\|u^{\varepsilon,\alpha}\|^2_{L^\infty(0,t;L^2(\mathbb{R}))}, \varepsilon\|u^{\varepsilon,\alpha}_x\|^2_{L^2(\Omega_t)}\Big\} \\ \le \quad C\left(\left(1+\frac{\beta}{\varepsilon}\right)\|u_0\|^2_{L^2(\mathbb{R})} + \varepsilon\beta\|u_0\|^2_{H^1(\mathbb{R})}\right). \end{aligned} \tag{3.27}$$

Proof. Estimate (3.25) simply follows by (3.2) while estimates (3.26) and (3.27) are consequence of (3.16) in Lemma 3.5 and Corollary 3.6, respectively. □

4 Wellposedness of Classical Solutions

In this section we prove that the initial-value problem (1.6)-(1.8) is well posed. By this we mean that, for any sufficiently smooth initial data u_0, problem (1.6)-(1.8) admits a unique classical solution that is globally defined in time.

Indeed, by slightly strengthening the assumptions on the initial data u_0 in Assumption 3.2, we show the existence of a solution $u^{\varepsilon,\alpha}$ as in (4.1) and therefore more regular of what was required in (3.3). The additional derivative in the variable x plays a crucial role in the following Section 5 and this is why we state Theorem 4.1 under this form. Clearly, the existence of a solution in $u^{\varepsilon,\alpha} \in C^2_1(\Omega_T) \cap L^\infty\left(0,T; H^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})\right)$ for initial data $u_0 \in H^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$ is a by-product of the proof.

We stress that the requirement of Assumption 3.1 in the next result is made only for sake of simplicity: the weaker condition that $f' \in C^3$ satisfies $|f'(u)| \le L$ is sufficient.

Theorem 4.1 (Global existence) *Let Assumption 3.1 hold and assume* $u_0 \in H^4(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R})$.
Then, for any $\varepsilon, \alpha > 0$, $0 < \beta < \frac{\varepsilon}{4L}$ *there is a unique classical solution* $(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})$ *of* (1.6)-(1.8) *defined in* $(0,+\infty)\times\mathbb{R}$. *Moreover, for any* $T > 0$ *we have*

$$u^{\varepsilon,\alpha} \quad \in \quad C^3_1\Big((0,T]\times\mathbb{R}\Big) \cap L^\infty\Big(0,T; H^4(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R})\Big). \tag{4.1}$$

Proof. Since ε, α and β are fixed in this proof, we drop the upper indices and write (u,λ) for $(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})$. We reformulate system (1.6) as

$$z_t - k(\varepsilon)z_{xx} + h(z)_x = g(z), \tag{4.2}$$

where $z = (u,\lambda)^T$, $z_0 = (u_0,u_0)^T$ and

$$k(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \frac{\gamma\varepsilon^2}{\beta} \end{pmatrix}, \qquad h(z) = \begin{pmatrix} f(u) + \alpha(u-\lambda) \\ 0 \end{pmatrix}, \qquad g(z) = \begin{pmatrix} 0 \\ \frac{\alpha}{\beta}(u-\lambda) \end{pmatrix}.$$

The norm in the space $\mathbb{R}^2_{u,\lambda}$ is defined as $|z| = \max\{|u|,|\lambda|\}$. The fundamental solution of the diagonal operator $z_t - k(\varepsilon)z_{xx}$ is $K = (K_1,K_2)$ with

$$K_i = \begin{cases} \frac{1}{\sqrt{4\pi d_i t}} e^{-\frac{x^2}{4d_i t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases}$$

for $d_1 = \varepsilon$ and $d_2 = \gamma\varepsilon^2/\beta$. To obtain (local in time) existence and uniqueness of solutions, we define the fixed-point mapping

$$\mathcal{L}z(t) = K(t) * z_0 - \int_0^t \partial_x K(t-s) * h(z(s))\,\mathrm{d}s + \int_0^t K(t-s) * g(z(s))\,\mathrm{d}s\,.$$

Since g is linear with respect to z and defining $C_1 = 2\frac{\alpha}{\beta}$, for $j = 0,\ldots,4$ we have

$$\|\partial_x^j g(z)(t)\|_{L^\infty(\mathbb{R})} \le C_1 \|\partial_x^j z(t)\|_{L^\infty(\mathbb{R})},$$
$$\|\partial_x^j g(z)(t)\|_{L^2(\mathbb{R})} \le C_1 \|\partial_x^j z(t)\|_{L^2(\mathbb{R})}.$$

We fix $T_0 > 0$ to be chosen sufficiently small later on. For $t \in [0, T_0]$ we have

$$\begin{aligned}\|\mathcal{L}z(t)\|_{L^\infty(\mathbb{R})} &\le \|u_0\|_{L^\infty(\mathbb{R})} + \left(C_1 T_0 + 2(L+\alpha)\int_0^t \|\partial_x K_1(t-s)\|_{L^1(\mathbb{R})}\,\mathrm{d}s\right)\|z\|_{L^\infty(\Omega_{T_0})}\\ &\le \|u_0\|_{L^\infty(\mathbb{R})} + \left(C_1 T_0 + C_2\sqrt{T_0}\right)\|z\|_{L^\infty(\Omega_{T_0})},\end{aligned}$$

for $C_2 = 4(L+\alpha)/\sqrt{\pi\varepsilon}$. This mapping is contracting for T_0 sufficiently small, yielding the existence and uniqueness of the solution in $L^\infty(\Omega_{T_0})$. Following the same steps as above, one can show that $z \in L^\infty(0, T_0; L^2(\mathbb{R}))$.

To prove that $z \in L^\infty(0, T_0; H^4(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R}))$, we argue as above and in Section 3. First, we have

$$\|K(t) * u_0\|^2_{L^2(\mathbb{R})} \le \|K_1(t) * u_0\|^2_{L^2(\mathbb{R})} + \|K_2(t) * u_0\|^2_{L^2(\mathbb{R})} \le 2\|u_0\|^2_{L^2(\mathbb{R})}.$$

Moreover

$$\begin{aligned}\left\|\int_0^t K_2(t-s) * \partial_x g(z(s))\,ds\right\|_{L^2(\mathbb{R})} &\le t \cdot \|\partial_x g(z(s))\|_{L^\infty(0,t;L^2(\mathbb{R}))}\\ &\le C_1 t \|z_x\|_{L^\infty(0,t;L^2(\mathbb{R}))}.\end{aligned}$$

At last,

$$\begin{aligned}\left\|\int_0^t \partial_x K_1(t-s) * \partial_x h(z(s))\,ds\right\|_{L^2(\mathbb{R})} &\le \frac{2\sqrt{t}}{\sqrt{\pi\varepsilon}}(L+2\alpha)\|z_x\|_{L^\infty(0,t;L^2(\mathbb{R}))}\\ &\le C_2\sqrt{t}\|z_x\|_{L^\infty(0,t;L^2(\mathbb{R}))}.\end{aligned}$$

As a consequence,

$$\|\partial_x \mathcal{L}z\|_{L^\infty(0,T_0;L^2(\mathbb{R}))} \le \sqrt{2}\|u_0\|_{H^1(\mathbb{R})} + \left(C_1 T_0 + C_2\sqrt{T_0}\right)\|\partial_x z\|_{L^\infty(0,T_0;L^2(\mathbb{R}))}.$$

Defining $l = C_1 T_0 + C_2\sqrt{T_0}$ and using a fixed point argument, we deduce

$$\|\partial_x z\|_{L^\infty(0,T_0;L^2(\mathbb{R}))} \le \frac{T_0}{1-l}\|u_0\|_{H^1(\mathbb{R})},$$

for T_0 sufficiently small to obtain $l < 1$. Now, we follow the same lines as [23, Lemma 6.2.4], with the above modifications to exploit the higher regularity of u_0. In particular, we use the L^∞ bound of u proved above to control the derivatives of f; here we need that $f \in C^3$. Then, we can deduce the following bounds, for $j = 1,\ldots,4$:

$$\|\partial_x^j z\|_{L^\infty(\Omega_{T_0})} \le C(\|u_0\|_{W^{4,\infty}(\mathbb{R})}, \|u_0\|_{H^4(\mathbb{R})})$$
$$\|\partial_x^j z\|_{L^\infty(0,T_0;L^2(\mathbb{R}))} \le C(\|u_0\|_{W^{4,\infty}(\mathbb{R})}, \|u_0\|_{H^4(\mathbb{R})}).$$

To extend the solution up to time T, note that for any $T > 0$ and $0 < t \le T$ we have by (3.5) and Corollary 3.6

$$\begin{aligned}\|z(.,t)\|_{L^2(\mathbb{R})} &\le \|u(.,t)\|_{L^2(\mathbb{R})} + \|\lambda(.,t)\|_{L^2(\mathbb{R})}\\ &\le C\|u_0\|_{H^1(\mathbb{R})}.\end{aligned}$$

The proof of [23, Theorem 6.2.7] is therefore still valid and yields the existence of classical solutions in $L^\infty(0,T;H^4(\mathbb{R})\cap W^{4,\infty}(\mathbb{R}))$ for all $T > 0$. Using Sobolev's embedding theorem for $H^4(\mathbb{R})$ and equation (4.2), we also get $z \in C_1^3([0,T];\mathbb{R})$ for all $T > 0$. □

5 Uniform boundedness of classical solutions

The main result of this technical section is Lemma 5.1, which shows that the L^2-norms of the derivatives of $u^{\varepsilon,\alpha}$ up to order 3 and the time derivative $u_t^{\varepsilon,\alpha}$ are uniformly bounded with respect to α. By Lemma 3.3 analogous bounds can be deduced for λ^α. Furthermore, a uniform L^∞-bound for $u^{\varepsilon,\alpha}$ is given in (5.3). We point out that the latter can be easily proven by an embedding argument and requiring only $u^{\varepsilon,\alpha} \in L^\infty(0,T;H^1(\mathbb{R}))$; however, this approach does not prove that the bound is independent of α. Below, the assumption that $u_0 \in H^4(\mathbb{R})$ and then the existence of solutions in $C_1^3(\Omega_T) \cap L^\infty(0,T;H^4(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R}))$ is needed to justify partial integrations in the proof. At last, notice the stronger scaling (1.10).

Lemma 5.1 (Uniform boundedness for higher-order norms) *Let* (1.10) *and the assumptions of Theorem 4.1 hold. Moreover, assume $\beta < \frac{\varepsilon}{4L}$.*
Then there exist three continuous, monotone increasing functions $\mathcal{C}_i(\varepsilon,\cdot) : [0,\infty) \to [0,\infty)$, $i \in \{1,\dots,3\}$, which depend on $\|u_0\|_{H^3(\mathbb{R})}$ but are independent of α and such that

$$\|u^{\varepsilon,\alpha}\|_{L^2(0,t;H^3(\mathbb{R}))} + \|u_t^{\varepsilon,\alpha}\|_{L^2(\Omega_t)} \le \mathcal{C}_1(\varepsilon,t) \qquad (t \in [0,\infty)), \tag{5.1}$$

$$\|u^{\varepsilon,\alpha}\|_{L^\infty(0,t;H^2(\mathbb{R}))} \le \mathcal{C}_2(\varepsilon,t) \qquad (t \in [0,\infty)), \tag{5.2}$$

$$\|u^{\varepsilon,\alpha}\|_{L^\infty(\Omega_t)} \le \mathcal{C}_3(\varepsilon,t) \qquad (t \in [0,\infty)). \tag{5.3}$$

Furthermore, there exists a continuous, monotone increasing function $\mathcal{C}_4(\varepsilon;\cdot) : [0,\infty) \to [0,\infty)$, depending on $\|u_0\|_{H^3(\mathbb{R})}$ but independent of α such that the following estimate holds for all $t \in [0,\infty)$:

$$\varepsilon\alpha\|u_x - \lambda_x\|^2_{L^2(0,t,H^2(\mathbb{R}))} + \beta\|\lambda_{xt}\|^2_{L^2(0,t,H^2(\mathbb{R}))} \le \mathcal{C}_4(\varepsilon,t). \tag{5.4}$$

Proof. First, estimate (5.3) follows by (5.2) because of the embedding of $H^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$.

So we are left to prove (5.1), (5.2) and (5.4). We use again the notation $(u,\lambda) = (u^{\varepsilon,\alpha},\lambda^{\varepsilon,\alpha})$. Remark that uniform estimates on $\|u\|_{L^2(0,T;H^1(\mathbb{R}))}$ and $\varepsilon\alpha\|u_x - \lambda_x\|_{L^2(\Omega_t)} + \beta\|\lambda_{xt}\|_{L^2(\Omega_t)}$ are already contained in Corollary 3.6 and Corollary 3.7. We divide the proof into four steps.

Step 1: estimate of $\|u_{xx}\|_{L^2(\Omega_t)}$. The main idea of the proof is to make use of the energy estimate in Lemma 3.5, which involves the energy functional $E^{\varepsilon,\alpha}$. We define $v = u_x$ and $\mu = \lambda_x$. Differentiating (1.6) with respect to x we see that v and μ must solve

$$\begin{cases} v_t + f(u)_{xx} = \varepsilon v_{xx} - \alpha(v-\mu)_x, \\ \beta\mu_t - \gamma\varepsilon^2\mu_{xx} = \alpha(v-\mu). \end{cases} \tag{5.5}$$

First, we multiply $(1.6)_2$ by μ_t and then integrate with respect to x; we find

$$\beta \int_{\mathbb{R}} \mu_t \lambda_t \, \mathrm{d}x - \gamma\varepsilon^2 \int_{\mathbb{R}} \mu_t \lambda_{xx} \, \mathrm{d}x = \alpha \int_{\mathbb{R}} \mu_t (u - \lambda) \, \mathrm{d}x \, .$$

The first summand in the left side vanishes and by partial integration we deduce that

$$\int_{\mathbb{R}} \mu_t \mu_x \, \mathrm{d}x = \frac{\alpha}{\gamma\varepsilon^2} \int_{\mathbb{R}} \lambda_t (u - \lambda)_x \, \mathrm{d}x \, . \tag{5.6}$$

Second, we multiply $(5.5)_1$ by v and then integrate with respect to x. It follows that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} v^2 \, \mathrm{d}x + \varepsilon \int_{\mathbb{R}} (v_x)^2 \, \mathrm{d}x &= - \int_{\mathbb{R}} f(u)_{xx} v \, \mathrm{d}x + \alpha \int_{\mathbb{R}} \mu_x v \, \mathrm{d}x \\ &\leq L \int_{\mathbb{R}} |v_x v| \, \mathrm{d}x + \alpha \int_{\mathbb{R}} \mu_x (v - \mu) \, \mathrm{d}x \\ &\leq L\rho_1 \int_{\mathbb{R}} (v_x)^2 \, \mathrm{d}x + \frac{L}{4\rho_1} \int_{\mathbb{R}} v^2 \, \mathrm{d}x + \beta \int_{\mathbb{R}} \mu_t \mu_x \, \mathrm{d}x \, , \end{aligned}$$

by (3.14). If we choose $\rho_1 = \frac{\varepsilon}{2L}$ and exploit identity (5.6) we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} v^2 \, \mathrm{d}x + \frac{\varepsilon}{2} \int_{\mathbb{R}} (v_x)^2 \, \mathrm{d}x \leq \frac{L^2}{2\varepsilon} \int_{\mathbb{R}} v^2 \, \mathrm{d}x + \frac{\beta\alpha}{\gamma\varepsilon^2} \int_{\mathbb{R}} \lambda_t (u - \lambda)_x \, \mathrm{d}x \, .$$

Now, it is time to apply 3.5 and its consequence Corollary 3.6. Integrating with respect to t and using once more (3.14) we deduce that for every $t \in [0, +\infty)$

$$\begin{aligned} &\frac{1}{2} \|v(t)\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2} \|v_x\|^2_{L^2(\Omega_t)} \\ &\leq \frac{1}{2} \|u_{0,x}\|^2_{L^2(\mathbb{R})} + \frac{L^2}{2\varepsilon} \|v\|^2_{L^2(\Omega_t)} + \frac{\beta}{\gamma\varepsilon^2} \|\lambda_t\|^2_{L^2(\Omega_t)} + \frac{\beta\alpha^2}{4\gamma\varepsilon^2} \|(u - \lambda)_x\|^2_{L^2(\Omega_t)} \\ &\leq \frac{1}{2} \|u_{0,x}\|^2_{L^2(\mathbb{R})} \\ &\quad + \left(\frac{L}{2\varepsilon^2} + \frac{2}{\gamma\varepsilon^2} + \frac{\beta\alpha}{4\gamma\varepsilon^3} \right) \left(2L \|u_0\|^2_{L^2(\mathbb{R})} + \frac{L\beta\varepsilon}{2\alpha} \|u_{0,x}\|^2_{L^2(\mathbb{R})} + E^{\varepsilon,\alpha}[u_0, u_0] \right) . \end{aligned} \tag{5.7}$$

Due to the scaling (1.10), the right side is uniformly bounded with respect to α.

Step 2: estimates on $\alpha \|v_x - \mu_x\|_{L^2(\Omega_t)}$ *and* $\beta \|\mu_t\|_{L^2(\Omega_t)}$. We multiply $(5.5)_1$ by v, then $(5.5)_1$ by $\alpha(v - \mu)$ and at last $(5.5)_2$ by μ_t. Then integration with respect to x yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} v^2 \, \mathrm{d}x + \varepsilon \int_{\mathbb{R}} (v_x)^2 \, \mathrm{d}x + \int_{\mathbb{R}} f(u)_{xx} v \, \mathrm{d}x = -\alpha \int_{\mathbb{R}} (v_x - \mu_x) v \, \mathrm{d}x \, , \tag{5.8}$$

$$\alpha \int_{\mathbb{R}} (v - \mu) v_t \, \mathrm{d}x + \alpha \int_{\mathbb{R}} f(u)_{xx} (v - \mu) \, \mathrm{d}x = \varepsilon\alpha \int_{\mathbb{R}} v_{xx} (v - \mu) \, \mathrm{d}x \, , \tag{5.9}$$

$$\beta \int_{\mathbb{R}} (\mu_t)^2 \, \mathrm{d}x - \gamma\varepsilon^2 \int_{\mathbb{R}} \mu_{xx} \mu_t \, \mathrm{d}x = \alpha \int_{\mathbb{R}} (v - \mu) \mu_t \, \mathrm{d}x \, . \tag{5.10}$$

About the right hand side of (5.9) we notice that, by $(5.5)_2$,

$$\begin{aligned} \alpha \int_{\mathbb{R}} v_{xx} (v - \mu) \, \mathrm{d}x &= -\alpha \int_{\mathbb{R}} (v_x)^2 \, \mathrm{d}x - \alpha \int_{\mathbb{R}} v \mu_{xx} \, \mathrm{d}x \\ &= -\alpha \int_{\mathbb{R}} (v_x)^2 \, \mathrm{d}x - \int_{\mathbb{R}} \mu_{xx} \left(\beta\mu_t - \gamma\varepsilon^2 \mu_{xx} + \alpha\mu \right) \mathrm{d}x \\ &= \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (\mu_x)^2 \, \mathrm{d}x - \alpha \int_{\mathbb{R}} (v_x)^2 \, \mathrm{d}x + \alpha \int_{\mathbb{R}} (\mu_x)^2 \, \mathrm{d}x + \gamma\varepsilon^2 \int_{\mathbb{R}} (\mu_{xx})^2 \, \mathrm{d}x \, . \end{aligned} \tag{5.11}$$

Now, we sum (5.8)–(5.10) and take into account (5.11) to deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(v^2+\alpha(v-\mu)^2+\left(\gamma\varepsilon^2-\varepsilon\beta\right)(\mu_x)^2\right)\mathrm{d}x\\
&\qquad+\varepsilon(1+\alpha)\int_{\mathbb{R}}(v_x)^2\,\mathrm{d}x+\beta\int_{\mathbb{R}}(\mu_t)^2\,\mathrm{d}x+\int_{\mathbb{R}}f(u)_{xx}\left(v+\alpha(v-\mu)\right)\mathrm{d}x\\
&\qquad=-\alpha\int_{\mathbb{R}}(v_x-\mu_x)v\,\mathrm{d}x+\varepsilon\left(\alpha\int_{\mathbb{R}}(\mu_x)^2\,\mathrm{d}x+\gamma\varepsilon^2\int_{\mathbb{R}}(\mu_{xx})^2\,\mathrm{d}x\right).
\end{aligned}\tag{5.12}$$

We differentiate $(5.5)_2$ with respect to x to deduce an expression for $\gamma\varepsilon^2\mu_{xxx}$ and obtain

$$\begin{aligned}
-\gamma\varepsilon^2\int_{\mathbb{R}}(\mu_{xx})^2\,\mathrm{d}x &= \gamma\varepsilon^2\int_{\mathbb{R}}\left(2\mu_x\mu_{xxx}+(\mu_{xx})^2\right)\mathrm{d}x\\
&=\beta\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\mu_x)^2\,\mathrm{d}x-2\alpha\int_{\mathbb{R}}v_x\mu_x\,\mathrm{d}x+2\alpha\int_{\mathbb{R}}(\mu_x)^2\,\mathrm{d}x+\gamma\varepsilon^2\int_{\mathbb{R}}(\mu_{xx})^2\,\mathrm{d}x\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}}\left((1+\alpha)(v_x)^2-\alpha(\mu_x)^2-\gamma\varepsilon^2(\mu_{xx})^2\right)\mathrm{d}x\\
&\qquad=\beta\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\mu_x)^2\,\mathrm{d}x+\int_{\mathbb{R}}(v_x)^2\,\mathrm{d}x+\alpha\int_{\mathbb{R}}(v_x-\mu_x)^2\,\mathrm{d}x+\gamma\varepsilon^2\int_{\mathbb{R}}(\mu_{xx})^2\,\mathrm{d}x\,.
\end{aligned}$$

We insert this identity into (5.12) and use $\partial_x^i\lambda_0=\partial_x^i u_0$ for all $i=0,\dots,3$ to obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left(v^2+\alpha(v-\mu)^2+\left(\gamma\varepsilon^2+\varepsilon\beta\right)(\mu_x)^2\right)\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{R}}\left(\varepsilon\left((v_x)^2+\alpha(v_x-\mu_x)^2+\gamma\varepsilon^2(\mu_{xx})^2\right)+\beta(\mu_t)^2\right)\mathrm{d}x\,\mathrm{d}s\\
&=\frac{1}{2}\int_{\mathbb{R}}\left(u_{0,x}^2+\left(\gamma\varepsilon^2+\varepsilon\beta\right)(u_{0,xx})^2\right)\mathrm{d}x+Q[u,v,\mu],
\end{aligned}\tag{5.13}$$

where

$$\begin{aligned}
Q[u,v,\mu]=&-\int_0^t\int_{\mathbb{R}}f(u)_{xx}v\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{R}}f(u)_{xx}\alpha(v-\mu)\,\mathrm{d}x\,\mathrm{d}s\\
&-\alpha\int_0^t\int_{\mathbb{R}}(v_x-\mu_x)\,v\,\mathrm{d}x\,\mathrm{d}s\\
=&\ I+II+III.
\end{aligned}\tag{5.14}$$

About I in (5.14), by (5.7), Corollary 3.6 and the bound $|f'|\le L$, we have, for $t\in[0,\infty)$,

$$\begin{aligned}
\left|\int_0^t\int_{\mathbb{R}}f(u)_{xx}v\,\mathrm{d}x\,\mathrm{d}s\right| &= \left|\int_0^t\int_{\mathbb{R}}f'(u)v_xv\,\mathrm{d}x\,\mathrm{d}s\right|\\
&\le 2L\left(\int_0^t\int_{\mathbb{R}}(v_x)^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}}v^2\,\mathrm{d}x\,\mathrm{d}s\right)\\
&\le \mathcal{C}_I(\varepsilon),
\end{aligned}\tag{5.15}$$

where $\mathcal{C}_I$ is a constant independent of α.
About II, by the bounds $|f'|,|f''|\le L$, $(5.5)_2$ and (3.14), we have

$$\left|\int_0^t\int_{\mathbb{R}}\alpha(v-\mu)f(u)_{xx}\,\mathrm{d}x\,\mathrm{d}s\right|=\left|\int_0^t\int_{\mathbb{R}}\alpha(v-\mu)\left(f'(u)v_x+f''(u)v^2\right)\mathrm{d}x\,\mathrm{d}s\right|$$

$$
\begin{aligned}
&= \left| \int_0^t \int_{\mathbb{R}} \Big(\beta\mu_t - \gamma\varepsilon^2\mu_{xx}\Big)\Big(f'(u)v_x + f''(u)v^2\Big)\,\mathrm{d}x\,\mathrm{d}s \right| \\
&\le L \int_0^t \int_{\mathbb{R}} \beta\,|\mu_t|\,(|v_x| + v^2) + \gamma\varepsilon^2\,|\mu_{xx}|\,(|v_x| + v^2)\,\mathrm{d}x\,\mathrm{d}s \\
&\le 2L^2\Big(\beta + \gamma\varepsilon^2\Big) \int_0^t \int_{\mathbb{R}} \Big((v_x)^2 + v^4\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{\beta}{4}\int_0^t \int_{\mathbb{R}} (\mu_t)^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\gamma\varepsilon^2}{4}\int_0^t \int_{\mathbb{R}} (\mu_{xx})^2\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (5.16)
\end{aligned}
$$

Concerning the summand $\|v\|^4_{L^4(\Omega_t)}$ on the right-hand side, we note that by [2, (13)] we have

$$
\|v\|^4_{L^4(\mathbb{R})} \le K_1\left(\|v_x\|^2_{L^2(\mathbb{R})} + \|v\|^6_{L^2(\mathbb{R})}\right). \qquad (5.17)
$$

Now, by (5.16) and using (5.17), Corollary 3.6, (5.7), we obtain

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \alpha(v-\mu) f(u)_{xx}\,\mathrm{d}x\,\mathrm{d}s \right| \\
&\qquad \le \mathcal{C}_{II}(\varepsilon)\cdot(1+t) + \frac{\beta}{4}\int_0^t \int_{\mathbb{R}} (\mu_t)^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\gamma\varepsilon^2}{4}\int_0^t \int_{\mathbb{R}} (\mu_{xx})^2\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (5.18)
\end{aligned}
$$

About *III* in (5.14), we notice that, by (3.14) and (5.7),

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \alpha\,(v_x - \mu_x)\,v\,\mathrm{d}x\,\mathrm{d}s \right| = \left| \int_0^t \int_{\mathbb{R}} (\beta\mu_t - \gamma\varepsilon^2\mu_{xx}) v_x\,\mathrm{d}x\,\mathrm{d}s \right| \\
&\le \Big(\beta + \gamma\varepsilon^2\Big)\int_0^t \int_{\mathbb{R}} (v_x)^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\beta}{4}\int_0^t \int_{\mathbb{R}} (\mu_t)^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\gamma\varepsilon^2}{4}\int_0^t \int_{\mathbb{R}} (\mu_{xx})^2\,\mathrm{d}x\,\mathrm{d}s \\
&\le \mathcal{C}_{III}(\varepsilon) + \frac{\beta}{4}\int_0^t \int_{\mathbb{R}} (\mu_t)^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\gamma\varepsilon^2}{4}\int_0^t \int_{\mathbb{R}} (\mu_{xx})^2\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (5.19)
\end{aligned}
$$

Inserting (5.15), (5.18) and (5.19) in (5.13), we obtain

$$
\varepsilon\alpha\|v_x - \mu_x\|^2_{L^2(\Omega_t)} + \frac{\beta}{2}\|\mu_t\|^2_{L^2(\Omega_t)} \le \tilde{\mathcal{C}}(\varepsilon, t). \qquad (5.20)
$$

for some function $\tilde{\mathcal{C}}$ affine linear in t.

Step 3: estimate of $\|u\|_{L^2(0,t;H^3(\mathbb{R}))}$. To derive this estimate, we need the assumption $u \in H^4$. We differentiate (5.5) with respect to x to obtain

$$
\begin{cases}
w_t + f(u)_{xxx} = \varepsilon w_{xx} - \alpha(w-\nu)_x, \\
\beta\nu_t - \gamma\varepsilon^2\nu_{xx} = \alpha(w-\nu),
\end{cases} \qquad (5.21)
$$

where we used the notation $w = v_x = u_{xx}$ and $\nu = \mu_x = \lambda_{xx}$. Analogously to the deduction of (5.6) in Step 1, we multiply $(5.5)_2$ by ν_t and integrate with respect to x and t to obtain

$$
\begin{aligned}
\beta\int_0^t \int_{\mathbb{R}} \nu_t\nu_x\,\mathrm{d}x\,\mathrm{d}s &= \frac{\beta\alpha}{\gamma\varepsilon^2}\int_0^t \int_{\mathbb{R}} \mu_t(v-\mu)_x\,\mathrm{d}x\,\mathrm{d}s \\
&\le \beta\int_0^t \int_{\mathbb{R}} (\mu_t)^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\beta\alpha^2}{4\gamma^2\varepsilon^4}\int_0^t \int_{R} (v_x - \mu_x)^2\,\mathrm{d}x\,\mathrm{d}s \\
&\le \left(2 + \frac{\beta\alpha}{4\gamma^2\varepsilon^3}\right)\tilde{\mathcal{C}}(\varepsilon, t) \qquad (5.22)
\end{aligned}
$$

by (5.20). Note that the scaling (1.10) ensures that $\beta\alpha = \mathcal{O}(1)$ and then the term $\frac{\beta\alpha}{4\gamma^2\varepsilon^3}$ does not depend on α in a critical way.

Multiply now $(5.21)_1$ by w to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} w^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}}(w_x)^2\,\mathrm{d}x &= -\int_{\mathbb{R}} f(u)_{xxx}w\,\mathrm{d}x + \alpha\int_{\mathbb{R}}(w-\nu)\nu_x\,\mathrm{d}x \\ &\le \int_{\mathbb{R}}\Big(f''(u)v^2 + f'(u)v_x\Big)w_x\,\mathrm{d}x + \beta\int_{\mathbb{R}}\nu_t\nu_x\,\mathrm{d}x \\ &\le L\int_{\mathbb{R}}(|v^2|+|v_x|)|w_x|\,\mathrm{d}x + \beta\int_{\mathbb{R}}\nu_t\nu_x\,\mathrm{d}x \\ &\le \frac{L^2}{\varepsilon}\left(\|v\|^4_{L^4(\mathbb{R})} + \|v_x\|^2_{L^2(\mathbb{R})}\right) + \frac{\varepsilon}{2}\int_{\mathbb{R}}(w_x)^2\,\mathrm{d}x + \beta\int_{\mathbb{R}}\nu_t\nu_x\,\mathrm{d}x\,, \end{aligned} \tag{5.23}$$

where we used the bounds on f', f'' and (3.14). Integrating in time and using (5.17), (5.7), Corollary 3.6 and (5.22) in (5.23), we obtain

$$\frac{1}{2}\|w(t)\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\|w_x\|^2_{L^2(\Omega_t)} \le \bar{\mathcal{C}}(\varepsilon,t), \tag{5.24}$$

for some function $\bar{\mathcal{C}}$ affine linear in t.

Analogously to the derivation of (5.20) in Step 2, one can also show the estimate

$$\varepsilon\alpha\|w_x - \nu_x\|_{L^2(\Omega_t)} + \frac{\beta}{2}\|\nu_t\|_{L^2(\Omega_t)} \le \hat{\mathcal{C}}(\varepsilon,t),$$

where $\hat{\mathcal{C}}$ is affine linear in t.

Step 4: estimate of $\|u_t\|_{L^2(\Omega_t)}$. Squaring $(1.6)_1$ and integrating with respect to x, we obtain

$$\begin{aligned}\int_{\mathbb{R}}(u_t)^2\,\mathrm{d}x \quad = \quad & \varepsilon^2\int_{\mathbb{R}}(u_{xx})^2\,\mathrm{d}x + \int_{\mathbb{R}}(f'(u))^2(u_x)^2\,\mathrm{d}x + \alpha^2\int_{\mathbb{R}}(u_x-\lambda_x)^2\,\mathrm{d}x \\ & -2\varepsilon\int_{\mathbb{R}} f'(u)u_xu_{xx}\,\mathrm{d}x + 2\alpha\int_{\mathbb{R}} f'(u)u_x(u_x-\lambda_x)\,\mathrm{d}x \\ & -2\alpha\int_{\mathbb{R}}\varepsilon u_{xx}(u_x-\lambda_x)\,\mathrm{d}x\,. \end{aligned}$$

This leads to

$$\begin{aligned}\|u_t\|^2_{L^2(\Omega_t)} \quad \le \quad & \varepsilon^2\|u_{xx}\|^2_{L^2(\Omega_t)} + \|f'(u)u_x\|^2_{L^2(\Omega_t)} + \|\alpha(u-\lambda)_x\|^2_{L^2(\Omega_t)} \\ & + 2\varepsilon\|f'(u)u_x\|_{L^2(\Omega_t)}\|u_{xx}\|_{L^2(\Omega_t)} \\ & + 2\|f'(u)u_x\|_{L^2(\Omega_T)}\|\alpha(u-\lambda)_x\|_{L^2(\Omega_t)} \\ & + 2\varepsilon\|u_{xx}\|_{L^2(\Omega_t)}\|\alpha(u-\lambda)_x\|_{L^2(\Omega_t)}. \end{aligned}$$

We need to prove that the term $\|\alpha(u-\lambda)_x\|^2_{L^2(\Omega_t)}$ is uniformly bounded with respect to α since this then also holds for the term $\|\alpha(u-\lambda)_x\|_{L^2(\Omega_t)}$. By $(1.6)_2$, we have

$$\|\alpha(u-\lambda)_x\|^2_{L^2(\Omega_t)} \le 2\|\beta\lambda_{tx}\|^2_{L^2(\Omega_t)} + 2\|\gamma\varepsilon^2\lambda_{xxx}\|^2_{L^2(\Omega_t)}.$$

The first summand on the right-hand side is $2\|\beta\mu_t\|^2_{L^2(\Omega_t)}$, which is bounded uniformly with respect to α by (5.20). Using Lemma 3.3, the second summand is bounded by

$$4\gamma^2\varepsilon^4\left(\|u_{xxx}\|^2_{L^2(\Omega_t)} + \frac{\beta}{2\alpha}\|u_{0,xxx}\|^2_{L^2(\mathbb{R})}\right),$$

which in turn is bounded uniformly with respect to α by (5.24). The uniform boundedness of $\|u_t\|_{L^2(\Omega_t)}$ now follows from (5.7) and Corollary 3.7. □

6 Asymptotics for the Coupling Limit $\alpha \to \infty$

In this section we analyze the limit $\alpha \to \infty$ of the system (1.6), for ε fixed, and prove that solutions of (1.6) weakly converge to a solution of the diffusive-dispersive conservation law (1.1). Moreover, we provide an estimate of the convergence rate. For simplicity, we denote the family of classical solutions to (1.6), (1.8) by $\{(u^\alpha, \lambda^\alpha)\}$, dropping the superscript index ε.

Theorem 6.1 *Let* (1.10) *and the assumptions of Theorem 4.1 hold. Furthermore, let $\varepsilon > 0$, $0 < \beta < \frac{\varepsilon}{4L}$ and denote by $\{(u^\alpha, \lambda^\alpha)\}_{\alpha>0}$ the family of unique classical solutions to* (1.6), (1.8), *as provided by Theorem 4.1. At last, fix any $T > 0$.*
Then there exists a subsequence of $\{(u^\alpha, \lambda^\alpha)\}_{\alpha>0}$, still denoted by $\{(u^\alpha, \lambda^\alpha)\}_{\alpha>0}$, and a function $u \in H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$ such that:

(i) we have

$$u^\alpha \rightharpoonup u \text{ in } H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R})) \text{ for } \alpha \to \infty,$$
$$\lambda^\alpha \rightharpoonup u \text{ in } H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R})) \text{ for } \alpha \to \infty;$$

(ii) there exists a positive constant $\mathcal{C}(\varepsilon, L, T)$, depending on $\|u_0\|_{H^3(\mathbb{R})}$ but not on α such that

$$\|u^\alpha - u\|_{L^2(0,T;H^1(\mathbb{R}))} + \|u^\alpha - u\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq \mathcal{C}(\varepsilon, L, T) \cdot \alpha^{-\frac{1}{2}}; \tag{6.1}$$

(iii) u is a distributional solution of the initial value problem (1.1), (1.8), *i.e.,*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t + f(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0 \varphi(0,.)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}} -\varepsilon u \varphi_{xx} + \gamma\varepsilon^2 u \varphi_{xxx} \,\mathrm{d}x\,\mathrm{d}t\,, \tag{6.2}$$

for all $\varphi \in C_0^\infty([0,T),\mathbb{R})$.

Proof.

(i) Due to Lemma 5.1, we know that the sequence $\{u^\alpha\}_{\alpha>0}$ is bounded uniformly with respect to α in the reflexive space $H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$. We can thus deduce the existence of a function $u \in H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$ and of a subsequence (still denoted by $\{u^\alpha\}_{\alpha>0}$) such that $u^\alpha \rightharpoonup u$ in $H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$, as $\alpha \to \infty$. This also implies $\partial_t u^\alpha \rightharpoonup \partial_t u$ and $\partial_x^j u^\alpha \rightharpoonup \partial_x^j u$ in $L^2(\Omega_T)$, for $j = 0, \ldots, 3$.
Due to Lemma 3.3, also the sequence $\{\lambda^\alpha\}_{\alpha>0}$ is bounded and, analogously as above, we can deduce the existence of a function $\lambda \in H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$ and of a subsequence such that $\lambda^\alpha \rightharpoonup \lambda$ in $H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$.
To prove that $\lambda = u$, we observe that by $(1.6)_2$ we have, for all $y \in H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \lambda^\alpha y \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}} (\lambda^\alpha - u^\alpha) y \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} u^\alpha y \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}} \lambda_t^\alpha \left(\frac{\beta}{\alpha} y\right) \mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}} \gamma\varepsilon^2 \lambda_{xx}^\alpha \left(\frac{1}{\alpha} y\right) \mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} u^\alpha y \,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

By the scaling (1.10), the weak convergence of λ_t^α, λ_{xx}^α, u^α and the strong convergence of $\frac{\beta}{\alpha} y$, $\frac{1}{\alpha} y$, we thus obtain $\lambda^\alpha \rightharpoonup u$ in $H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$ by the uniqueness of the weak limit.

(ii) First, we note that the sequence $\{\sqrt{\beta}\,\lambda_{tx}^\alpha\}_{\alpha>0}$ is bounded uniformly with respect to α because of Lemma 5.1. By the weak convergence of λ^α and its derivatives we deduce

$$\int_0^T \int_{\mathbb{R}} \alpha(u_x^\alpha - \lambda_x^\alpha) y \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}} \sqrt{\beta}\lambda_{tx}^\alpha (\sqrt{\beta} y) \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}} \gamma\varepsilon^2 \lambda_{xxx}^\alpha y \,\mathrm{d}x\,\mathrm{d}t$$

$$\rightharpoonup -\int_0^T \int_{\mathbb{R}} \gamma\varepsilon^2 u_{xxx} y \,\mathrm{d}x\,\mathrm{d}t\,. \tag{6.3}$$

for all $y \in H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$. By (1.8) we have $u^\alpha(0,.) = u_0$, which of course converges strongly to u_0. Due to the continuity in time of u^α and u, we have $u(0,.) = u_0$ in $L^2(\mathbb{R})$. From the weak convergence of $(u^\alpha)_{\alpha>0}$ and (6.3), we deduce that u fulfills

$$\int_0^T \int_{\mathbb{R}} \Big(u_t + f(u)_x - \varepsilon u_{xx} - \gamma\varepsilon^2 u_{xxx}\Big) y \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{6.4}$$

for all $y \in H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$. We define $y^\alpha := u^\alpha - u$. Since $y^\alpha \in H^1(\Omega_T) \cap L^2(0,T;H^3(\mathbb{R}))$, we can use it as a test function both in (6.4) and in the analogous expression for u^α. Subtracting these two identities we find, by (3.14),

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} (y^\alpha(T))^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} (y^\alpha(0))^2 \,\mathrm{d}x + \varepsilon \int_0^T \int_{\mathbb{R}} (y^\alpha_x)^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{\mathbb{R}} (f(u^\alpha) - f(u)) y^\alpha_x \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}} \Big(\alpha(u^\alpha_x - \lambda^\alpha_x) + \gamma\varepsilon^2 u_{xxx}\Big) y^\alpha \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{\mathbb{R}} (f(u^\alpha) - f(u)) y^\alpha_x \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}} \Big(\beta\lambda^\alpha_{tx} + \gamma\varepsilon^2(\lambda^\alpha_{xxx} - u_{xxx})\Big) y^\alpha \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \frac{L^2}{4\rho_1}\int_0^T \int_{\mathbb{R}} (y^\alpha)^2 \,\mathrm{d}x\,\mathrm{d}t + \rho_1 \int_0^T \int_{\mathbb{R}} (y^\alpha_x)^2 \,\mathrm{d}x\,\mathrm{d}t + 2\rho_2 \int_0^T \int_{\mathbb{R}} (y^\alpha)^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \frac{\gamma^2\varepsilon^4}{4\rho_2}\int_0^T \int_{\mathbb{R}} (\lambda^\alpha_{xxx} - u^\alpha_{xxx})^2 \,\mathrm{d}x\,\mathrm{d}t - \frac{\gamma^2\varepsilon^4}{4\rho_2}\int_0^T \int_{\mathbb{R}} y^\alpha_{xxx} y^\alpha \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Now, we choose $\rho_1 = \frac{\varepsilon}{2}$ and $\rho_2 = 1/2$. By Lemma 5.1 and noting that $y^\alpha(0) = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} (y^\alpha(T))^2 \,\mathrm{d}x + \frac{\varepsilon}{2}\int_0^T \int_{\mathbb{R}} (y^\alpha_x)^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \left(\frac{L^2}{2\varepsilon} + 1\right) \int_0^T \int_{\mathbb{R}} (y^\alpha)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\beta^2}{2}\int_0^T \int_{\mathbb{R}} (\lambda^\alpha_{tx})^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \frac{\gamma^2\varepsilon^4}{2}\int_0^T \int_{\mathbb{R}} (\lambda^\alpha_{xxx} - u^\alpha_{xxx})^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \left(\frac{L^2}{2\varepsilon} + 1\right) \int_0^T \int_{\mathbb{R}} (y^\alpha)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\left(\beta + \frac{\gamma^2\varepsilon^3}{\alpha}\right) \mathcal{C}_4(\varepsilon,T).
\end{aligned} \tag{6.5}$$

Therefore the following estimate holds true:

$$\|y^\alpha(T)\|^2_{L^2(\mathbb{R})} \le \left(\beta + \frac{2\gamma^2\varepsilon^3}{\alpha}\right) \mathcal{C}_4(\varepsilon,T) + \left(\frac{L^2}{\varepsilon} + 2\right) \int_0^T \|y^\alpha(t)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}t\,.$$

Since $\mathcal{C}_4(\varepsilon,T)$ is a constant, by Gronwall's inequality we obtain

$$\|y^\alpha(T)\|^2_{L^2(\mathbb{R})} \le \frac{1}{2}\left(\beta + \frac{\gamma^2\varepsilon^3}{\alpha}\right) \mathcal{C}_4(\varepsilon,T) e^{\left(\frac{L^2}{\varepsilon}+2\right)T}. \tag{6.6}$$

Combining (6.5), (6.6) and (1.10) yields (6.1).

(iii) By the weak convergence of λ^α and the scaling (1.10) we deduce $\beta\lambda^\alpha \rightharpoonup 0$ in $L^2(\Omega_T)$. Any classical solution u^α of (1.6), (1.8) fulfills

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} u^\alpha \varphi_t + f(u^\alpha)\varphi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0 \varphi \,\mathrm{d}x \\ = \int_0^T \int_{\mathbb{R}} \varepsilon u^\alpha \varphi_{xx} + \gamma\varepsilon^2 \lambda^\alpha \varphi_{xxx} + \beta\lambda^\alpha \varphi_{tx} \,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned}$$

for all $\varphi \in C_0^\infty([0,T];\mathbb{R})$. Since $C_0^\infty([0,T],\mathbb{R}) \subset L^2(\Omega_T)$, we obtain by weak convergence that

$$\int_0^T \int_{\mathbb{R}} u\varphi_t + f(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0\varphi \,\mathrm{d}x = \int_0^T \int_{\mathbb{R}} \varepsilon u \varphi_{xx} + \gamma\varepsilon^2 u \varphi_{xxx} \,\mathrm{d}x\,\mathrm{d}t\,.$$

for all $\varphi \in C_0^\infty([0,T);\mathbb{R})$, which proves (6.2).

□

7 Asymptotics for the Sharp-Interface Limit $\varepsilon \to 0$

In the previous section we studied the limit $\alpha \to \infty$ of the system (1.6) for ε fixed. Here we consider the sharp-interface limit $\varepsilon \to 0$ for fixed values of $\alpha > 0$.

For any initial datum $u_0 \in L^2(\mathbb{R})$ there is a family $u_0^\varepsilon \in H^4(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R})$ of smooth approximations of u_0 satisfying

$$\begin{aligned}\lim_{\varepsilon\to 0} \|u_0 - u_0^\varepsilon\|_{L^2(\mathbb{R})} &= 0\,, \\ \|u_0^\varepsilon\|_{L^2(\mathbb{R})} + \varepsilon\|u_0^\varepsilon\|_{H^1(\mathbb{R})} &\leq K_0 \text{ for every } \varepsilon > 0,\end{aligned} \tag{7.1}$$

where $K_0 = K_0(\|u_0\|_{L^2(\mathbb{R})}) > 0$ is a constant independent of ε. From Theorem 4.1 we know that there is a family of classical solutions to (1.6) with initial datum u_0^ε. To keep notation short, we denote this family by $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$, dropping now the index α.

We prove below that if the coefficient β is suitably scaled with respect to ε, then the solutions $\{u^\varepsilon\}$ of (1.6) with initial datum u_0^ε converge for $\varepsilon \to 0$ to a weak solution u of the homogeneous equation (1.2); moreover, $u(.,0) = u_0$ in a weak sense. More precisely, we have the following result.

Theorem 7.1 *Assume that $f \in C^3(\mathbb{R})$ satisfies $|f'| < L$ for some constant $L > 0$ and, moreover, that*

$$\operatorname{meas}\{u \in \mathbb{R} \colon f''(u) = 0\} = 0. \tag{7.2}$$

Let $u_0 \in L^2(\mathbb{R})$ and consider $u_0^\varepsilon \in H^4(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R}))$ satisfying (7.1). We choose $\beta = \varepsilon\mathcal{O}(\alpha^{-1})$ be given such that

$$\beta \leq \frac{\varepsilon}{4L}. \tag{7.3}$$

Then, for a family $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$ of classical solutions of (1.6) with initial datum u_0^ε, there is a subsequence of $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$ still denoted by $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$, and a function $u \in L^p(\Omega_T)$, $1 \leq p < 2$, such that

$$u^\varepsilon,\ \lambda^\varepsilon \to u \text{ in } L^p_{loc}(\Omega_T) \quad (1 \leq p < 2)\ .$$

Moreover, u is a weak solution to the initial value problem (1.2) with datum $u_0 \in L^2(\mathbb{R})$, i.e.,

$$\int_0^T \int_{\mathbb{R}} u\varphi_t + f(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0 \varphi(.,0)\,\mathrm{d}x = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R} \times [0,T))$.

From Section 2 we know that the limit solution u can contain undercompressive waves; then, it may fail to be a Kružkov solution.

The proof of Theorem 7.1 relies on the compensated compactness theory [17] in the L^p-framework [16, 22]. We recall that an *entropy pair* (η, q) for (1.2) is a pair of functions of class $C^2(\mathbb{R})$ satisfying

$$\eta'(w)f'(w) = q'(w)$$

for every $w \in \mathbb{R}$. In the following, we consider entropies satisfying the condition

$$|\eta'(w)| + |\eta''(w)| \leq C_\eta, \tag{7.4}$$

for every $w \in \mathbb{R}$. The following compactness lemma will lead to the proof of Theorem 7.1. Its proof is similar to that of an analogous result in [20]; see also [6]. We denote by $\mathcal{M}(Q)$ the set of Radon measures on some open set Q.

Lemma 7.2 *Let the assumptions of Theorem 7.1 be valid and consider the family of classical solutions $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$ of (1.6) with initial datum u_0^ε defined above.*
Then, for every open bounded set $Q \subset \Omega_T$ there exist a compact set $\mathcal{K} \subset W^{-1,2}(Q)$ and a bounded set $\mathcal{B} \subset \mathcal{M}(Q)$ such that

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \in \mathcal{K} + \mathcal{B}\,,$$

for every entropy pair (η, q) satisfying (7.4).

Proof. We multiply (1.6) by $\eta'(u^\varepsilon)$ and obtain

$$\begin{aligned}
\eta(u^\varepsilon)_t + q(u^\varepsilon)_x &= \varepsilon\eta(u^\varepsilon)_{xx} - \varepsilon\eta''(u^\varepsilon)(u_x^\varepsilon)^2 - \alpha\left(\eta'(u^\varepsilon)(u^\varepsilon - \lambda^\varepsilon)\right)_x + \alpha\eta''(u^\varepsilon)u_x^\varepsilon(u^\varepsilon - \lambda^\varepsilon) \\
&= A_1^\varepsilon + A_2^\varepsilon + A_3^\varepsilon + A_4^\varepsilon.
\end{aligned}$$

In the sequel we use the notation $\langle\cdot,\cdot\rangle$ both for the duality of $H^{-1}(Q)$ and $H_0^1(Q)$ as well as for the one between $\mathcal{M}(Q)$ and $\mathcal{C}^0(Q)$. We will prove that $A_1^\varepsilon, A_3^\varepsilon \in \mathcal{K}$ and $A_2^\varepsilon, A_4^\varepsilon \in \mathcal{B}$. We notice that the estimates in Corollary 3.7 hold because of (7.3).

Let $\varphi \in H_0^1(Q)$ and consider

$$|\langle A_1^\varepsilon, \varphi\rangle| \leq \varepsilon \int_Q |\eta'(u^\varepsilon)u_x^\varepsilon\varphi_x|\,\mathrm{d}x\,\mathrm{d}t \leq C_\eta\varepsilon\|u_x^\varepsilon\|_{L^2(Q)}\|\varphi_x\|_{L^2(Q)}.$$

By (3.27) in Corollary 3.7 we obtain

$$|\langle A_1^\varepsilon, \varphi\rangle| \leq C_\eta\sqrt{C}\sqrt{\varepsilon}\Big((1+\beta\varepsilon^{-1})\|u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon\beta\|u_0^\varepsilon\|^2_{H^1(\mathbb{R})}\Big)^{1/2}\|\varphi\|_{H^1(Q)},$$

and then, by (7.3) and (7.1),

$$|\langle A_1^\varepsilon, \varphi\rangle| \leq C_\eta\sqrt{C}\sqrt{\varepsilon}\Big(\Big(1+\frac{1}{4L}\Big)\|u_0\|^2_{L^2(\mathbb{R})} + \frac{K_0^2}{4L}\Big)^{1/2}\|\varphi\|_{H^1(Q)} \xrightarrow{\varepsilon\to 0} 0.$$

We turn to A_3^ε and observe from equation $(1.6)_2$ that

$$\begin{aligned}
|\langle A_3^\varepsilon, \varphi\rangle| &\leq \alpha\int_Q |\ \eta'(u^\varepsilon)(u^\varepsilon - \lambda^\varepsilon)\varphi_x\ |\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C_\eta\int_Q |\beta\lambda_t^\varepsilon - \gamma\varepsilon^2\lambda_{xx}^\varepsilon|\,|\varphi_x|\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C_\eta\left(\beta\|\lambda_t^\varepsilon\|_{L^2(Q)} + \gamma\varepsilon^2\|\lambda_{xx}^\varepsilon\|_{L^2(Q)}\right)\|\varphi\|_{H^1(Q)}.
\end{aligned}$$

With Corollary 3.7 this leads to

$$|\langle A_3^\varepsilon, \varphi\rangle| \leq C_\eta(\sqrt{\beta} + \sqrt{\gamma\varepsilon})\sqrt{C}\Big(\|u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon(\varepsilon+\beta)\|u_0^\varepsilon\|^2_{H^1(\mathbb{R})}\Big)^{1/2}\|\varphi\|_{H^1(Q)}.$$

Condition (7.3) and (7.1) ensure then $|\langle A_3^\varepsilon, \varphi\rangle| \to 0$ for $\varepsilon \to 0$.

It remains to analyze A_2^ε and A_4^ε. In view of (3.27) in Corollary 3.7 it is straightforward to verify for any $\psi \in \mathcal{C}^0(Q)$

$$|\langle A_2^\varepsilon, \psi\rangle| \le C_\eta C\Big((1+\beta\varepsilon^{-1})\|u_0^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon\beta\|u_0^\varepsilon\|^2_{H^1(\mathbb{R})}\Big)\|\psi\|_{C^0(Q)}.$$

Once again, condition (7.3) and (7.1) show that $|\langle A_2^\varepsilon, \psi\rangle|$ is uniformly bounded with respect to ε. For the remaining term A_4^ε we have, for any $\psi \in \mathcal{C}^0(Q)$,

$$\begin{aligned} |\langle A_4^\varepsilon, \psi\rangle| &\le \alpha \int_Q |\eta''(u^\varepsilon)u_x^\varepsilon(u^\varepsilon - \lambda^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t \\ &\le C_\eta \int_Q |\beta\lambda_t^\varepsilon + \gamma\varepsilon^2\lambda_{xx}^\varepsilon|\, |u_x^\varepsilon|\, |\psi| \,\mathrm{d}x\,\mathrm{d}t \\ &\le C_\eta \left(\beta\|\lambda_t^\varepsilon\|_{L^2(Q)} + \gamma\varepsilon^2\|\lambda_{xx}^\varepsilon\|_{L^2(Q)}\right)\|u_x^\varepsilon\|_{L^2(Q)}\|\psi\|_{C^0(Q)}. \end{aligned}$$

Estimates (3.26) and (3.27) in Corollary 3.7 apply to bound $|\langle A_4^\varepsilon, \psi\rangle|$ uniformly with respect to ε. □

With this compactness result we can finally prove Theorem 7.1.

Proof of Theorem 7.1. The family of norms $\|u^\varepsilon\|_{L^2(\Omega_T)}$ is uniformly bounded, because of Corollary 3.6 and (7.1). By Lemma 7.2 and the results in [16, 17] we deduce that $u^\varepsilon \to u$ in $L^p_{loc}(\Omega_T)$, for $1 \le p < 2$. In particular, the Lipschitz bound on the flux and condition (7.2) are necessary to apply the results in [16].

In order to prove that the limit function u solves (1.2), consider any $\varphi \in C_0^\infty(\mathbb{R}\times[0,T))$; then

$$\begin{aligned} &\int_{\Omega_T} \big(u^\varepsilon\varphi_t + f(u^\varepsilon)\varphi_x\big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0^\varepsilon\varphi(.,0)\,\mathrm{d}x \\ &= \; -\varepsilon\int_{\Omega_T} u^\varepsilon\varphi_{xx}\,\mathrm{d}x\,\mathrm{d}t - \alpha\int_{\Omega_T}(u^\varepsilon - \lambda^\varepsilon)\varphi_x\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

We showed above that the sequence $\{u^\varepsilon\}$ converges in $L^p_{loc}(\Omega_T)$ for $1 \le p < 2$; therefore, and using (7.1), the left side of the identity above converges to

$$\int_{\Omega_T}\big(u\varphi_t + f(u)\varphi_x\big)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} u_0\varphi(.,0)\,\mathrm{d}x\ .$$

By the same reason, the first term on the right side vanishes in the limit. At last, by $(1.6)_2$, the second term equals

$$-\int_{\Omega_T}\Big(\beta\lambda_t^\varepsilon - \gamma\varepsilon^2\lambda_{xx}^\varepsilon\Big)\varphi_x\,\mathrm{d}x\,\mathrm{d}t\,.$$

This term vanishes in the limit $\varepsilon \to 0$ due to (7.3) and the estimate (3.26) in Corollary 3.7.

The convergence $\lambda^\varepsilon \to u$ is an immediate consequence of the parabolic equation $(1.6)_2$ for λ^ε and (3.26). □

References

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] M. Agueh. Sharp Gagliardo-Nirenberg inequalities via p-Laplacian type equations. *NoDEA Nonlinear Differential Equations Appl.*, 15(4-5):457–472, 2008.

[3] A. L. Bertozzi, A. Münch, and M. Shearer. Undercompressive shocks in thin film flows. *Phys. D*, 134(4):431–464, 1999.

[4] D. Brandon, T. Lin, and R. C. Rogers. Phase transitions and hysteresis in nonlocal and order-parameter models. *Meccanica*, 30(5):541–565, 1995.

[5] H. Brezis. *Analyse fonctionnelle.* Masson, Paris, 1983.

[6] A. Corli and C. Rohde. Singular limits for a parabolic-elliptic regularization of scalar conservation laws. *J. Differential Equations*, 253(5):1399–1421, 2012.

[7] L. C. Evans. *Partial differential equations.* American Mathematical Society, Providence, RI, second edition, 2010.

[8] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, 31(1):53–98, 1979.

[9] A. Fetter and J. Walecka. *Theoretical Mechanics of Particles and Continua.* Courier, Dover, 2003.

[10] H. Freistühler and P. Szmolyan. Existence and bifurcation of viscous profiles for all intermediate magnetohydrodynamic shock waves. *SIAM J. Math. Anal.*, 26(1):112–128, 1995.

[11] B. T. Hayes and P. G. LeFloch. Non-classical shocks and kinetic relations: scalar conservation laws. *Arch. Rational Mech. Anal.*, 139(1):1–56, 1997.

[12] B. Horn. Height and gradient from shading. *Inter. J. Computer Vision*, 4(1):59–78, 1990.

[13] D. Jacobs, B. McKinney, and M. Shearer. Travelling wave solutions of the modified Korteweg-de Vries-Burgers equation. *J. Differential Equations*, 116(2):448–467, 1995.

[14] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical systems*, volume 1609 of *Lecture Notes in Math.*, pages 44–118. Springer, Berlin, 1995.

[15] P. G. LeFloch. *Hyperbolic systems of conservation laws.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. The theory of classical and nonclassical shock waves.

[16] Y. G. Lu. Convergence of solutions to nonlinear dispersive equations without convexity conditions. *Appl. Anal.*, 31(4):239–246, 1989.

[17] F. Murat. L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. *J. Math. Pures Appl. (9)*, 60(3):309–322, 1981.

[18] X. Ren and L. Truskinovsky. Finite scale microstructures in nonlocal elasticity. *J. Elasticity*, 59(1-3):319–355, 2000.

[19] R. C. Rogers and L. Truskinovsky. Discretization and hysteresis. *Physica B*, 233:370–375, 1997.

[20] C. Rohde. Scalar conservation laws with mixed local and nonlocal diffusion-dispersion terms. *SIAM J. Math. Anal.*, 37(1):103–129, 2005.

[21] C. Rohde. A local and low-order Navier-Stokes-Korteweg system. In *Nonlinear partial differential equations and hyperbolic wave phenomena*, volume 526 of *Contemp. Math.*, pages 315–337. Amer. Math. Soc., Providence, RI, 2010.

[22] M. E. Schonbek. Convergence of solutions to nonlinear dispersive equations. *Comm. Partial Differential Equations*, 7(8):959–1000, 1982.

[23] D. Serre. *Systems of conservation laws. 1.* Cambridge University Press, Cambridge, 1999. Hyperbolicity, entropies, shock waves, Translated from the 1996 French original by I. N. Sneddon.

[24] M. Solci. A variational model for phase separation. *Ph.D. Thesis, University of Pavia*, 2002.

[25] M. Solci and E. Vitali. Variational models for phase separation. *Interfaces Free Bound.*, 5(1):27–46, 2003.

[26] C. J. van Duijn, L. A. Peletier, and I. S. Pop. A new class of entropy solutions of the Buckley-Leverett equation. *SIAM J. Math. Anal.*, 39(2):507–536, 2007.

Andrea Corli
Department of Mathematics and Computer Science,
University of Ferrara,
I-44121 Ferrara, Italy
E-Mail: `crl@unife.it`

Christian Rohde
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: `Christian.Rohde@mathematik.uni-stuttgart.de`
WWW: `http://www.mathematik.uni-stuttgart.de/fak8/ians/lehrstuhl/LstAngMath/mitarbeiter/rohde/`

Veronika Schleper
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: `Veronika.Schleper@mathematik.uni-stuttgart.de`
WWW: `http://www.mathematik.uni-stuttgart.de/fak8/ians/lehrstuhl/LstAngMath/mitarbeiter/schleper`

Erschienene Preprints ab Nummer 2007/2007-001

Komplette Liste: `http://www.mathematik.uni-stuttgart.de/preprints`

2013-010 *Corli, A.; Rohde, C.; Schleper, V.:* Parabolic Approximations of Diffusive-Dispersive Equations

2013-009 *Nava-Yazdani, E.; Polthier, K.:* De Casteljau's Algorithm on Manifolds

2013-008 *Bächle, A.; Margolis, L.:* Rational conjugacy of torsion units in integral group rings of non-solvable groups

2013-007 *Knarr, N.; Stroppel, M.J.:* Heisenberg groups over composition algebras

2013-006 *Knarr, N.; Stroppel, M.J.:* Heisenberg groups, semifields, and translation planes

2013-005 *Eck, C.; Kutter, M.; Sändig, A.-M.; Rohde, C.:* A Two Scale Model for Liquid Phase Epitaxy with Elasticity: An Iterative Procedure

2013-004 *Griesemer, M.; Wellig, D.:* The Strong-Coupling Polaron in Electromagnetic Fields

2013-003 *Kabil, B.; Rohde, C.:* The Influence of Surface Tension and Configurational Forces on the Stability of Liquid-Vapor Interfaces

2013-002 *Devroye, L.; Ferrario, P.G.; Györfi, L.; Walk, H.:* Strong universal consistent estimate of the minimum mean squared error

2013-001 *Kohls, K.; Rösch, A.; Siebert, K.G.:* A Posteriori Error Analysis of Optimal Control Problems with Control Constraints

2012-018 *Kimmerle, W.; Konovalov, A.:* On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II

2012-017 *Stroppel, B.; Stroppel, M.:* Desargues, Doily, Dualities, and Exceptional Isomorphisms

2012-016 *Moroianu, A.; Pilca, M.; Semmelmann, U.:* Homogeneous almost quaternion-Hermitian manifolds

2012-015 *Steinke, G.F.; Stroppel, M.J.:* Simple groups acting two-transitively on the set of generators of a finite elation Laguerre plane

2012-014 *Steinke, G.F.; Stroppel, M.J.:* Finite elation Laguerre planes admitting a two-transitive group on their set of generators

2012-013 *Diaz Ramos, J.C.; Dominguez Vázquez, M.; Kollross, A.:* Polar actions on complex hyperbolic spaces

2012-012 *Moroianu; A.; Semmelmann, U.:* Weakly complex homogeneous spaces

2012-011 *Moroianu; A.; Semmelmann, U.:* Invariant four-forms and symmetric pairs

2012-010 *Hamilton, M.J.D.:* The closure of the symplectic cone of elliptic surfaces

2012-009 *Hamilton, M.J.D.:* Iterated fibre sums of algebraic Lefschetz fibrations

2012-008 *Hamilton, M.J.D.:* The minimal genus problem for elliptic surfaces

2012-007 *Ferrario, P.:* Partitioning estimation of local variance based on nearest neighbors under censoring

2012-006 *Stroppel, M.:* Buttons, Holes and Loops of String: Lacing the Doily

2012-005 *Hantsch, F.:* Existence of Minimizers in Restricted Hartree-Fock Theory

2012-004 *Grundhöfer, T.; Stroppel, M.; Van Maldeghem, H.:* Unitals admitting all translations

2012-003 *Hamilton, M.J.D.:* Representing homology classes by symplectic surfaces

2012-002 *Hamilton, M.J.D.:* On certain exotic 4-manifolds of Akhmedov and Park

2012-001 *Jentsch, T.:* Parallel submanifolds of the real 2-Grassmannian

2011-028 *Spreer, J.:* Combinatorial 3-manifolds with cyclic automorphism group

2011-027 *Griesemer, M.; Hantsch, F.; Wellig, D.:* On the Magnetic Pekar Functional and the Existence of Bipolarons

2011-026 *Müller, S.:* Bootstrapping for Bandwidth Selection in Functional Data Regression

2011-025 *Felber, T.; Jones, D.; Kohler, M.; Walk, H.:* Weakly universally consistent static forecasting of stationary and ergodic time series via local averaging and least squares estimates

2011-024 *Jones, D.; Kohler, M.; Walk, H.:* Weakly universally consistent forecasting of stationary and ergodic time series

2011-023 *Györfi, L.; Walk, H.:* Strongly consistent nonparametric tests of conditional independence

2011-022 *Ferrario, P.G.; Walk, H.:* Nonparametric partitioning estimation of residual and local variance based on first and second nearest neighbors

2011-021 *Eberts, M.; Steinwart, I.:* Optimal regression rates for SVMs using Gaussian kernels

2011-020 *Frank, R.L.; Geisinger, L.:* Refined Semiclassical Asymptotics for Fractional Powers of the Laplace Operator

2011-019 *Frank, R.L.; Geisinger, L.:* Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain

2011-018 *Hänel, A.; Schulz, C.; Wirth, J.:* Embedded eigenvalues for the elastic strip with cracks

2011-017 *Wirth, J.:* Thermo-elasticity for anisotropic media in higher dimensions

2011-016 *Höllig, K.; Hörner, J.:* Programming Multigrid Methods with B-Splines

2011-015 *Ferrario, P.:* Nonparametric Local Averaging Estimation of the Local Variance Function

2011-014 *Müller, S.; Dippon, J.:* k-NN Kernel Estimate for Nonparametric Functional Regression in Time Series Analysis

2011-013 *Knarr, N.; Stroppel, M.:* Unitals over composition algebras

2011-012 *Knarr, N.; Stroppel, M.:* Baer involutions and polarities in Moufang planes of characteristic two

2011-011 *Knarr, N.; Stroppel, M.:* Polarities and planar collineations of Moufang planes

2011-010 *Jentsch, T.; Moroianu, A.; Semmelmann, U.:* Extrinsic hyperspheres in manifolds with special holonomy

2011-009 *Wirth, J.:* Asymptotic Behaviour of Solutions to Hyperbolic Partial Differential Equations

2011-008 *Stroppel, M.:* Orthogonal polar spaces and unitals

2011-007 *Nagl, M.:* Charakterisierung der Symmetrischen Gruppen durch ihre komplexe Gruppenalgebra

2011-006 *Solanes, G.; Teufel, E.:* Horo-tightness and total (absolute) curvatures in hyperbolic spaces

2011-005 *Ginoux, N.; Semmelmann, U.:* Imaginary Kählerian Killing spinors I

2011-004 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part II — Gain-Scheduled Control

2011-003 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part I — Robust Control

2011-002 *Alexandrov, B.; Semmelmann, U.:* Deformations of nearly parallel G_2-structures

2011-001 *Geisinger, L.; Weidl, T.:* Sharp spectral estimates in domains of infinite volume

2010-018 *Kimmerle, W.; Konovalov, A.:* On integral-like units of modular group rings

2010-017 *Gauduchon, P.; Moroianu, A.; Semmelmann, U.:* Almost complex structures on quaternion-Kähler manifolds and inner symmetric spaces

2010-016 *Moroianu, A.; Semmelmann,U.:* Clifford structures on Riemannian manifolds

2010-015 *Grafarend, E.W.; Kühnel, W.:* A minimal atlas for the rotation group $SO(3)$

2010-014 *Weidl, T.:* Semiclassical Spectral Bounds and Beyond

2010-013 *Stroppel, M.:* Early explicit examples of non-desarguesian plane geometries

2010-012 *Effenberger, F.:* Stacked polytopes and tight triangulations of manifolds

2010-011 *Györfi, L.; Walk, H.:* Empirical portfolio selection strategies with proportional transaction costs

2010-010 *Kohler, M.; Krzyżak, A.; Walk, H.:* Estimation of the essential supremum of a regression function

2010-009 *Geisinger, L.; Laptev, A.; Weidl, T.:* Geometrical Versions of improved Berezin-Li-Yau Inequalities

2010-008 *Poppitz, S.; Stroppel, M.:* Polarities of Schellhammer Planes

2010-007 *Grundhöfer, T.; Krinn, B.; Stroppel, M.:* Non-existence of isomorphisms between certain unitals

2010-006 *Höllig, K.; Hörner, J.; Hoffacker, A.:* Finite Element Analysis with B-Splines: Weighted and Isogeometric Methods

2010-005 *Kaltenbacher, B.; Walk, H.:* On convergence of local averaging regression function estimates for the regularization of inverse problems

2010-004 *Kühnel, W.; Solanes, G.:* Tight surfaces with boundary

2010-003 *Kohler, M; Walk, H.:* On optimal exercising of American options in discrete time for stationary and ergodic data

2010-002 *Gulde, M.; Stroppel, M.:* Stabilizers of Subspaces under Similitudes of the Klein Quadric, and Automorphisms of Heisenberg Algebras

2010-001 *Leitner, F.:* Examples of almost Einstein structures on products and in cohomogeneity one

2009-008 *Griesemer, M.; Zenk, H.:* On the atomic photoeffect in non-relativistic QED

2009-007 *Griesemer, M.; Moeller, J.S.:* Bounds on the minimal energy of translation invariant n-polaron systems

2009-006 *Demirel, S.; Harrell II, E.M.:* On semiclassical and universal inequalities for eigenvalues of quantum graphs

2009-005 *Bächle, A, Kimmerle, W.:* Torsion subgroups in integral group rings of finite groups

2009-004 *Geisinger, L.; Weidl, T.:* Universal bounds for traces of the Dirichlet Laplace operator

2009-003 *Walk, H.:* Strong laws of large numbers and nonparametric estimation

2009-002 *Leitner, F.:* The collapsing sphere product of Poincaré-Einstein spaces

2009-001 *Brehm, U.; Kühnel, W.:* Lattice triangulations of E^3 and of the 3-torus

2008-006 *Kohler, M.; Krzyżak, A.; Walk, H.:* Upper bounds for Bermudan options on Markovian data using nonparametric regression and a reduced number of nested Monte Carlo steps

2008-005 *Kaltenbacher, B.; Schöpfer, F.; Schuster, T.:* Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems

2008-004 *Leitner, F.:* Conformally closed Poincaré-Einstein metrics with intersecting scale singularities

2008-003 *Effenberger, F.; Kühnel, W.:* Hamiltonian submanifolds of regular polytope

2008-002 *Hertweck, M.; Höfert, C.R.; Kimmerle, W.:* Finite groups of units and their composition factors in the integral group rings of the groups $PSL(2,q)$

2008-001 *Kovarik, H.; Vugalter, S.; Weidl, T.:* Two dimensional Berezin-Li-Yau inequalities with a correction term

2007-006 *Weidl, T.:* Improved Berezin-Li-Yau inequalities with a remainder term

2007-005 *Frank, R.L.; Loss, M.; Weidl, T.:* Polya's conjecture in the presence of a constant magnetic field

2007-004 *Ekholm, T.; Frank, R.L.; Kovarik, H.:* Eigenvalue estimates for Schrödinger operators on metric trees

2007-003 *Lesky, P.H.; Racke, R.:* Elastic and electro-magnetic waves in infinite waveguides

2007-002 *Teufel, E.:* Spherical transforms and Radon transforms in Moebius geometry

2007-001 *Meister, A.:* Deconvolution from Fourier-oscillating error densities under decay and smoothness restrictions