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A relative entropy approach to convergence of a low order approximation to a nonlinear elasticity model with viscosity and capillarity

Jan Giesselmann*

Abstract

In this work we study the dynamics of an elastic bar undergoing phase transitions. It is modeled by two regularizations of the equations of nonlinear elastodynamics with a non-convex energy. We estimate the difference between solutions to the two regularizations if in one of them a coupling parameter α is sent to infinity. This estimate is based on an adaptation of the relative entropy framework using the regularizing terms in order to compensate for the non-convexity of the energy density.

1 Introduction

This paper is concerned with two models describing longitudinal or shearing motions of an elastic bar undergoing phase transitions between a *low strain* and *high strain* phase. The models are based on the (isothermal) equations of elastodynamics which, in the multi-phase case, form a system of hyperbolic/elliptic conservation laws. It is well-known that for such systems standard entropy conditions are insufficient to guarantee uniqueness of weak solutions. There are two approaches to overcome this obstacle: One is to impose so called kinetic relations at discontinuities, [1, 21, e.g.]. The other approach is to require solutions to be limits of solutions of regularized equations. We are interested in the second approach and study two such regularizations. One of these systems is well-accepted in the literature while the other one offers computational advantages. We estimate the difference between solutions of the two regularized models. In particular, we will see that the regularizations compensate for the non-convex energy density, in that they make the models well-posed (which is well-known and can be seen as the main reason for their introduction) and in that they allow us to use the relative entropy framework to derive estimates for the difference between solutions. To be more precise, let us introduce the two models: We consider the following third order model including viscous and capillary effects:

$$\begin{aligned} w_t - v_x &= 0 \\ v_t - (W'(w) - \gamma w_{xx})_x &= \mu v_{xx}, \end{aligned} \tag{1.1}$$

where w is the deformation gradient, v is the velocity, $\gamma, \mu > 0$ are the capillarity and the viscosity parameter and $W = W(w)$ is a (possibly) non-convex energy density. This model was studied

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in [2, 18, 26, 25, e.g.]. We will assume that $W \in C^3(\mathbb{R}, [0,\infty))$ and that there exists a possibly negative constant $C \in \mathbb{R}$ such that

$$W''(w) \geq C \quad \forall w \in \mathbb{R}. \tag{1.2}$$

It is important to note that we do not make any assumptions on convexity of W. The case we have in mind is that W is a double-well potential. Recently the following family of lower order approximations parametrized by $\alpha > 0$ (whose solutions will be marked by $\tilde{\cdot}$) was suggested in [13]:

$$\begin{aligned} \tilde{w}_t^\alpha - \tilde{v}_x^\alpha &= 0 \\ \tilde{v}_t^\alpha - (W'(\tilde{w}^\alpha) + \gamma\alpha\tilde{w}^\alpha)_x &= \mu\tilde{v}_{xx}^\alpha - \gamma\alpha\tilde{c}_x^\alpha \\ \alpha(\tilde{c}^\alpha - \tilde{w}^\alpha) &= \tilde{c}_{xx}^\alpha. \end{aligned} \tag{1.3}$$

Here $\tilde{w}^\alpha$ is the deformation gradient, $\tilde{v}^\alpha$ is the velocity, $\gamma, \mu > 0$ are as above and $\tilde{c}^\alpha$ is an auxiliary variable without any immediate physical interpretation. For later use, let us note that the second and third line of (1.3) can be combined to obtain

$$\tilde{v}_t^\alpha - (W'(\tilde{w}^\alpha) - \gamma\tilde{c}_{xx}^\alpha)_x = \mu\tilde{v}_{xx}^\alpha. \tag{1.4}$$

The minimization problem associated to the energy of (1.3), see (2.5), was studied in [7, 27]. In [13] several advantages of (1.3) over (1.1) are stated. While arguments are provided indicating that (1.3) is a meaningful physical model in itself the authors of [13] argue that their main reasons for introducing (1.3) are numerical in nature. In particular, provided α is sufficiently large, the first two equations in (1.3) form a strictly hyperbolic system of balance laws for $\tilde{w}^\alpha$ and $\tilde{v}^\alpha$ such that the whole wealth of schemes developed for such problems can be employed to solve $(1.3)_{1,2}$, while $(1.3)_3$ can easily be solved by any elliptic solver. Therefore, it is expected that numerical schemes for (1.3), see [23], are much more robust and efficient than those developed for (1.1). The construction of numerical methods for (1.1) is a delicate issue [8]. We refer to [8] and [6, 11, 16] (dealing with numerical methods for the Navier-Stokes-Korteweg system) for ideas to overcome the problems introduced by the hyperbolic-elliptic structure of the first order part of the equations.

As a matter of justifying (1.3) it was shown in [13] that for $\alpha \to \infty$ a sub-sequence of the solutions of (1.3) converges weakly in L^2 to a solution of (1.1). This is the starting point of this study. We aim at making this convergence more explicit. In particular, we will prove an estimate for the difference between strong solutions (w, v) of (1.1) and $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ of (1.3) if the initial data for both models are the same and sufficiently regular, see Theorem 3.12. We show that the convergence of $\|w - \tilde{c}^\alpha\|_{H^1} + \|v - \tilde{v}^\alpha\|_{L^2}$ is of order $\alpha^{-1/4}$. However, it must be noted that the error constant is proportional to $\exp(T/\mu\gamma)$ if $[0, T]$ is the time interval under consideration.

Our estimate is based on an adaptation of the relative entropy framework, going back to [9, 12], to higher order models making no assumptions on the convexity of the energy density W. In recent years the relative entropy technique was frequently used for the study of hyperbolic conservation laws and related systems. For a general overview of the development in the last decades we refer to the references in [10, Section 5.7]. More recent works employing relative entropy arguments include [5, 14, 15, 19, 20, 22, e.g.]. In these cases the energy densities are at least quasi-convex or poly-convex. Estimates obtained using the relative entropy framework usually involve the use of Gronwall's inequality and, therefore, an exponential dependence on time. The dependence of the estimate on γ, μ in our case is due to the fact that the estimate heavily relies on the convexity of the energy functional and, here, the local part of the energy density $W(w)$ is non-convex.

In fact, a main novelty of the work at hand is that there exist higher order (regularizing) mechanisms which may compensate for non-convex energy densities, in that they make stability estimates based on the relative entropy possible even for entropies which are not quasi-convex.

This is also true in several space dimensions. In order to show this, we consider generalizations of (1.3) and (1.1) which still allow for additional balance laws, which might be seen as energy balances. However, in both multi-dimensional models neither the capillary nor the viscous terms are materially frame indifferent [3, 4]. Therefore, the physical content of the multi-dimensional equations studied here is at least dubious.

Still, from a mathematical viewpoint it turns out that the results from the one dimensional situation can easily be generalized, provided the multi-dimensional equations admit sufficiently smooth solutions. The proper physical generalization of the models (1.1) and (1.3) to several space dimensions is a matter of ongoing research beyond the scope of this work.

The remainder of this work is structured as follows: In the forthcoming Section 2.1 we state the energy balance laws associated with the systems under consideration. In particular, this enables us to determine energies and energy fluxes which is a valuable prerequisite for employing the relative entropy arguments later. Section 2.2 is devoted to the well-posedness analysis of (1.1) and (1.3). In Section 3.1 we describe a generalized relative entropy approach and in Section 3.2 we establish estimates related to the elliptic operator $(1.3)_3$. These results are combined with Gronwall's Lemma in order to estimate the difference between the solutions of (1.3) and (1.1) in Section 3.3. Section 4 is devoted to a generalization of these results to several space dimensions.

2 Thermodynamics and well-posedness

2.1 Thermodynamical structure

In this section we recall the energy inequalities satisfied by (1.3) and (1.1). In doing so we identify entropies and entropy fluxes. Moreover, the energy dissipation equation (2.3) will be crucial in establishing the existence of global strong solutions to (1.1). In our analysis we will use the classical Lebesgue L^p, Sobolev $W^{k,p}$ and Sobolev (Bochner) spaces $L^p(0,T;H^k(\cdot))$, for $p \geq 1$ and $k \in \mathbb{N}$, where H^k refers to $W^{k,2}$.

We will consider both systems on S^1 by which we denote the unit interval with its endpoints glued together. We choose functions $\bar{w} : S^1 \to \mathbb{R}$ and $\bar{v} : S^1 \to \mathbb{R}$ and complement (1.1), (1.3) with identical initial data

$$w(0,\cdot) = \bar{w}, \quad v(0,\cdot) = \bar{v} \quad \text{in } S^1, \tag{2.1}$$

$$\tilde{w}^\alpha(0,\cdot) = \bar{w}, \quad \tilde{v}^\alpha(0,\cdot) = \bar{v} \quad \text{in } S^1. \tag{2.2}$$

Both systems are thermodynamically consistent in that they satisfy energy balance equations and are materially frame indifferent.

Lemma 2.1 (Energy balance for (1.1)**)** *Let T, γ, $\mu > 0$ be given and let*

$$(w,v) \in \big(C^0([0,T],H^3(S^1)) \cap C^1((0,T),H^1(S^1))\big) \times \big(C^0([0,T],H^2(S^1)) \cap C^1((0,T),L^2(S^1))\big)$$

be a strong solution of (1.1)*. Then, the following energy balance law holds in* $(0,T) \times S^1$ *:*

$$\big(W(w) + \frac{\gamma}{2}(w_x)^2 + \frac{1}{2}v^2\big)_t - \big(vW'(w) - \gamma v w_{xx} + \gamma v_x w_x + \mu v_x v\big)_x + \mu(v_x)^2 = 0. \tag{2.3}$$

Proof :
Equation (2.3) is obtained by multiplying $(1.1)_1$ by $W'(w) - \gamma w_{xx}$, multiplying $(1.1)_2$ by v and adding both equations. ■

Lemma 2.2 (Energy balance for (1.3)**)** *Let T, γ, $\mu > 0$ and let $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ with*

$$\begin{aligned} \tilde{w}^\alpha &\in W^{1,\infty}((0,T), L^2(S^1)) \cap C([0,T], H^1(S^1)) \\ \tilde{v}^\alpha &\in H^1(0,T; L^2(S^1)) \cap L^\infty(0,T; H^1(S^1)) \cap L^2(0,T; H^2(S^1)) \\ \tilde{c}^\alpha &\in C^1((0,T), H^1(S^1)) \cap C([0,T], H^3(S^1)), \end{aligned} \tag{2.4}$$

be a strong solution of (1.3)*. Then, the following energy balance is satisfied in* $(0,T) \times S^1$:

$$\begin{aligned} &\big(W(\tilde{w}^\alpha) + \frac{\gamma\alpha}{2}(\tilde{w}^\alpha - \tilde{c}^\alpha)^2 + \frac{\gamma}{2}(\tilde{c}_x^\alpha)^2 + \frac{1}{2}(\tilde{v}^\alpha)^2\big)_t \\ &\qquad - \big(\tilde{v}^\alpha W'(\tilde{w}^\alpha) + \gamma\alpha\tilde{v}^\alpha(\tilde{w}^\alpha - \tilde{c}^\alpha) + \tilde{c}_t^\alpha \tilde{c}_x^\alpha + \mu \tilde{v}_x^\alpha \tilde{v}^\alpha\big)_x + \mu(\tilde{v}_x^\alpha)^2 = 0. \end{aligned} \tag{2.5}$$

Proof :
To obtain (2.5) we multiply $(1.3)_1$ by $W'(\tilde{w}^\alpha) + \alpha\gamma(\tilde{w}^\alpha - \tilde{c}^\alpha)$ and $(1.3)_2$ by $\tilde{v}^\alpha$ and, further, use

$$\alpha(\tilde{w}^\alpha - \tilde{c}^\alpha)\tilde{w}_t^\alpha = \alpha(\tilde{w}^\alpha - \tilde{c}^\alpha)(\tilde{w}^\alpha - \tilde{c}^\alpha)_t + \tilde{c}_x^\alpha \tilde{c}_{tx}^\alpha - (\tilde{c}_t^\alpha \tilde{c}_x^\alpha)_x. \tag{2.6}$$

■

2.2 Well-posedness of the models

From now on we will use the notation $\sigma = W'$. Let us start with well-posedness of (1.1). In [2, 17] weak solutions of (1.1) were investigated in case of natural boundary conditions. We consider periodic boundary conditions and are interested in strong solutions. Still, our analysis goes along the same lines as the one in [2, Sec. 6]. Using $w = u_x$ the system (1.1) can be rewritten as

$$\begin{aligned} u_t - v &= 0 \\ v_t - \sigma(u_x)_x &= \mu v_{xx} - \gamma u_{xxxx} \end{aligned} \quad \text{in } (0,\infty) \times S^1, \tag{2.7}$$

with initial conditions

$$u(0,\cdot) = u_0, \quad v(0,\cdot) = v_0 \ \text{ in } S^1. \tag{2.8}$$

We denote

$$H_m^k(S^1) := \big\{ f \in H^k(S^1) \,:\, \int_{S^1} f \,\mathrm{d}\,x = 0 \big\}.$$

Theorem 2.3 (Well-posedness of (2.7)**)** *Let initial data $u_0 \in H_m^4(S^1)$, $v_0 \in H_m^2(S^1)$ and $T > 0$ be given. Then, the problem* (2.7),(2.8) *has a unique strong solution*

$$(u,v) \in \big(C^0([0,T], H_m^4(S^1)) \cap C^1((0,T), H_m^2(S^1))\big) \times \big(C^0([0,T], H_m^2(S^1)) \cap C^1((0,T), L_m^2(S^1))\big).$$

Corollary 2.4 (Well-posedness of (1.1)**)** *Let initial data $\bar{w} \in H_m^3(S^1)$, $\bar{v} \in H_m^2(S^1)$ and $T > 0$ be given. Then, the problem* (1.1),(2.1) *has a unique strong solution*

$$(w,v) \in \big(C^0([0,T], H_m^3(S^1)) \cap C^1((0,T), H_m^1(S^1))\big) \times \big(C^0([0,T], H_m^2(S^1)) \cap C^1((0,T), L_m^2(S^1))\big).$$

Proof of Theorem 2.3:
Let us note that the periodicity of u_0 implies $(u_0)_x \in H^3_m(S^1)$. In abstract form (2.7) can be written as

$$y_t = Ay + f(y) \tag{2.9}$$

with

$$y = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \mathrm{Id} \\ -\gamma\partial_{xxxx} & \mu\partial_{xx} \end{pmatrix}, \quad f(y) = \begin{pmatrix} 0 \\ \sigma(u_x)_x \end{pmatrix}. \tag{2.10}$$

Let us define the spaces

$$X := H^2_m(S^1), \quad Y := X \times L^2(S^1). \tag{2.11}$$

For every $f \in X$ it holds that $f_x \in H^1_m(S^1)$ such that, by Poincaré's inequality,

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_Y := \int_{S^1} \gamma(u_1)_{xx}(u_2)_{xx} + v_1 v_2 \,\mathrm{d}\,x, \quad \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_Y^2 := \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_Y \tag{2.12}$$

define a scalar product and a norm on Y. The operator A is densely defined on Y with

$$D(A) = \big(H^4(S^1) \cap X\big) \times H^2(S^1). \tag{2.13}$$

The operator A induces a C^0 semi-group on Y which can be seen analogously to the arguments in [2] using $\{\sin(2n\pi\cdot), \cos(2n\pi\cdot) : n \in \mathbb{N}\}$ as a basis of X. Note that for all $t \geq 0$ it holds

$$\int_{S^1} u(t,\cdot)\,\mathrm{d}\,x = 0, \quad \int_{S^1} u(t,\cdot)_x \,\mathrm{d}\,x = 0, \quad \int_{S^1} v(t,\cdot)\,\mathrm{d}\,x = 0$$

due to our assumptions on the initial data and the fact that u_x, v satisfy conservation laws. The semi-group induced by A is, in fact, contractive as any solution (u,v) of

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = A \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.14}$$

satisfies

$$\begin{aligned} \frac{d}{dt} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_Y^2 &= 2 \int_{S^1} \gamma u_{xx} u_{xxt} + v v_t \,\mathrm{d}\,x \\ &= 2 \int_{S^1} \gamma u_{xx} u_{xxt} - \gamma u_t u_{xxxx} + \mu v v_{xx} \,\mathrm{d}\,x \leq -2 \int_{S^1} \mu (v_x)^2 \,\mathrm{d}\,x \leq 0. \end{aligned} \tag{2.15}$$

Moreover, the map $f : Y \to Y$ is locally Lipschitz continuous as $X \subset H^2(S^1)$ is continuously embedded in $C^1(S^1)$ and, therefore,

$$\begin{aligned} \left\| f\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right) - f\left(\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right) \right\|_Y^2 &= \int_{S^1} (\sigma((u_1)_x)_x - \sigma((u_2)_x)_x)^2 \,\mathrm{d}\,x \\ &\leq C_1 \|u_1 - u_2\|^2_{H^2(S^1)} + C_2 \|u_2\|^2_{H^2(S^1)} \|u_1 - u_2\|^2_{C^1(S^1)} \leq C \left\| \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right\|_Y^2, \end{aligned} \tag{2.16}$$

where

$$C_1 := \max\left\{ |\sigma'(w)|^2 \,:\, |w| \leq \|u_1\|_{H^2(S^1)} \right\}, \; C_2 := \max\left\{ |\sigma''(w)|^2 \,:\, |w| \leq \|u_1\|_{H^2(S^1)} + \|u_2\|_{H^2(S^1)} \right\}$$

and C is a generic constant depending on $C_1, C_2, \|u_2\|^2_{H^2}, \gamma$, and the constant from the embedding $X \to C^1(S^1)$. We have also used that the Poincaré constant on S^1 is smaller than 1. Invoking [24, Theorem 5.8] we infer that there exists a maximal time of existence $T_m \in (0, \infty]$ and a unique strong solution (u, v) of (2.7), (2.8) with

$$\begin{aligned} u &\in C^0([0, T_m), H^4_m(S^1)) \cap C^1((0, T_m), H^2_m(S^1)), \\ v &\in C^0([0, T_m), H^2_m(S^1)) \cap C^1((0, T_m), L^2_m(S^1)). \end{aligned} \tag{2.17}$$

In case $T_m < \infty$ the result [24, Theorem 5.8] implies

$$\|(u(t,\cdot), v(t,\cdot))^T\|_Y \to \infty \quad \text{for } t \nearrow T_{\max}.$$

We know from (2.3) that strong solutions of (2.7) satisfy

$$\frac{d}{dt} \int_{S^1} W(u_x) + \frac{\gamma}{2}(u_{xx})^2 + \frac{1}{2} v^2 \, \mathrm{d}\, x \leq 0. \tag{2.18}$$

As W is bounded from below (2.18) implies that $\|(u(t,\cdot), v(t,\cdot))^T\|_Y$ is uniformly bounded in time and therefore $T_{\max} = +\infty$.

■

Proof of Corollary 2.4:
As $\bar{w} \in H^3_m(S^1)$ every primitive is in $H^4(S^1)$, and there is exactly one primitive $u_0 \in H^4_m(S^1)$ of $\bar{w}$. Let (u, v) denote the solution of (2.7), (2.8) with initial data $(u_0, \bar{v})$. Then, (u_x, v) is the unique solution of (1.1), (2.1). ■

The well-posedness of (1.3) is investigated in [13] in a similar way. There the system

$$\begin{aligned} \tilde{u}^\alpha_t - \tilde{v}^\alpha &= 0 \\ \tilde{v}^\alpha_t - \sigma(\tilde{v}^\alpha_x)_x &= \tilde{v}^\alpha_{xx} - \alpha\gamma(\tilde{c}^\alpha - \tilde{u}^\alpha_x)_x \\ \frac{1}{\alpha}\tilde{c}^\alpha_{xx} &= \tilde{c}^\alpha - \tilde{u}^\alpha_x \end{aligned} \tag{2.19}$$

equipped with the following initial and boundary conditions

$$\begin{aligned} &\tilde{u}^\alpha(t,0) = \tilde{u}^\alpha(t,1) = 0, \quad \tilde{v}^\alpha(t,0) = \tilde{v}^\alpha(t,1) = 0, \quad \tilde{c}^\alpha_x(t,0) = \tilde{c}^\alpha_x(t,1) = 0, \quad \forall t \in (0,T), \\ &\tilde{u}^\alpha(0,x) = u_0(x), \ \tilde{u}^\alpha_t(0,x) = v_0(x) \quad \forall x \in (0,1), \end{aligned} \tag{2.20}$$

is considered. The following result on existence of strong solutions is shown in [13] with $I := [0, 1]$:

Proposition 2.5 (Well-posedness of (2.19)**)** *Let $\alpha, \gamma > 0$ be fixed and suppose that*

$$u_0 \in H^2(I) \cap H_0(I) \quad \text{and} \quad v_0 \in H^2(I) \cap H^1_0(I).$$

Then, given $T > 0$, there is a unique solution $(\tilde{u}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ of (2.19), (2.20) *with*

$$\begin{aligned} \tilde{u}^\alpha_t = \tilde{v}^\alpha &\in H^1(0,T; L^2(I)) \cap L^\infty(0,T; H^1_0(I)) \cap L^2(0,T; H^2(I)) \\ \tilde{u}^\alpha &\in C^1((0,T), L^2(I)) \cap C([0,T], H^2(I) \cap H^1_0(I)) \\ \tilde{c}^\alpha &\in C^1((0,T), H^1(I)) \cap C([0,T], H^3(I)). \end{aligned} \tag{2.21}$$

In particular, $\tilde{u}^\alpha \in W^{1,\infty}(0,T;H^1(I))$. In addition, there exists a constant $C = C(T)$ independent of α such that

$$\|\tilde{u}^\alpha_{xx}(t,\cdot)\|_{L^2(I)} \le C(T) \quad \forall\, t \in [0,T]. \tag{2.22}$$

Remark 2.6 (Boundary conditions) *The analogous result to Proposition 2.5 with $H^2(I)$ and $H^1_0(I)$ replaced by $H^2(S^1)$ and $H^1_m(S^1)$ can be shown in the same way. Note that the result does not require the choice of W from [13] but works for the more general class considered here. In particular, assumption* (1.2) *is required to derive* (2.22). *Similarly, the viscous term $\tilde{v}^\alpha_{xx}$ may be scaled by some $\mu > 0$ without changing the result.*

With respect to system (1.3) this implies

Proposition 2.7 (Well-posedness of (1.3)) *Let $\alpha, \gamma, \mu > 0$ be fixed and suppose that*

$$\bar{w} \in H^1_m(S^1) \quad \text{and} \quad \bar{v} \in H^2(S^1) \cap H^1_m(S^1).$$

Then, given $T > 0$, there is a unique solution $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ of (1.3), (2.2) *with*

$$\begin{aligned} \tilde{v}^\alpha &\in H^1(0,T;L^2_m(S^1)) \cap L^\infty(0,T;H^1_m(S^1)) \cap L^2(0,T;H^2_m(S^1)) \\ \tilde{w}^\alpha &\in W^{1,\infty}(0,T;L^2_m(S^1)) \cap C([0,T],H^1_m(S^1)) \\ \tilde{c}^\alpha &\in C^1((0,T),H^1(S^1)) \cap C([0,T],H^3(S^1)). \end{aligned} \tag{2.23}$$

In addition, there exists a constant $C = C(T)$ independent of α such that

$$\|\tilde{w}^\alpha_x(t,\cdot)\|_{L^2(S^1)} \le C(T) \quad \forall\, t \in [0,T].$$

3 Estimates in the one dimensional case

3.1 A relative entropy argument

This section is devoted to an adaptation of the relative entropy framework to the situation of a non-convex energy and higher order problems. It should be noted that the higher order mechanisms compensate for the lack of convexity of W. For the general setup of the relative entropy framework and the way it may be used to prove stability for hyperbolic balance laws, we refer to [10, Chapter 5]. As in this isothermal problem energy and entropy coincide we will use the terms relative entropy and relative energy interchangeably.

We start our considerations with a relative entropy inequality still involving W. The main issue here is the correct choice of the parts of the relative entropy and relative entropy flux corresponding to the third order terms.

Proposition 3.1 (Relative entropy equality) *For $T > 0$, let (w, v) be a strong solution of* (1.1),(2.1) *in the sense of Corollary 2.4 and let $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ be a solution of* (1.3),(2.2) *in the sense of Proposition 2.7. Then, the following relative entropy equation is satisfied*

$$\begin{aligned} \frac{d}{dt} \int_{S^1} & W(\tilde{w}^\alpha) + \frac{\alpha\gamma}{2}(\tilde{w}^\alpha - \tilde{c}^\alpha)^2 + \frac{\gamma}{2}(\tilde{c}^\alpha_x)^2 + \frac{1}{2}(\tilde{v}^\alpha)^2 - W(w) - \frac{\gamma}{2}(w_x)^2 - \frac{1}{2}v^2 \\ & \qquad - W'(w)(\tilde{w}^\alpha - w) - \gamma w_x(\tilde{c}^\alpha - w)_x - v(\tilde{v}^\alpha - v)\,\mathrm{d}\,x \\ = \int_{S^1} & -\mu(\tilde{v}^\alpha_x - v_x)^2 + v_x\big(W'(\tilde{w}^\alpha) - W'(w) - W''(w)(\tilde{w}^\alpha - w)\big) - \gamma w_{xx}(\tilde{w}^\alpha_t - \tilde{c}^\alpha_t)\,\mathrm{d}\,x. \end{aligned} \tag{3.1}$$

Proof :
We start with a direct computation following the general relative entropy framework:

$$
\begin{aligned}
A :=& \int_{S^1} \partial_t \Big(W(\tilde w^\alpha) + \frac{\alpha\gamma}{2}(\tilde w^\alpha - \tilde c^\alpha)^2 + \frac{\gamma}{2}(\tilde c^\alpha_x)^2 + \frac{1}{2}(\tilde v^\alpha)^2 - W(w) - \frac{\gamma}{2}(w_x)^2 - \frac{1}{2}v^2 \\
&\qquad - W'(w)(\tilde w^\alpha - w) - \gamma w_x(\tilde c^\alpha - w)_x - v(\tilde v^\alpha - v)\Big) \\
&+ \partial_x \Big(- W'(\tilde w^\alpha)\tilde v^\alpha + \gamma \tilde c^\alpha_{xx}\tilde v^\alpha + W'(w)v - \gamma w_{xx}v + W'(w)(\tilde v^\alpha - v) - \gamma w_{xx}(\tilde v^\alpha - v) \\
&\qquad + vW'(\tilde w^\alpha) - \gamma \tilde c^\alpha_{xx} v - W'(w)v + \gamma w_{xx} v\Big) \,\mathrm{d}\, x \\
=& \int_{S^1} W'(\tilde w^\alpha)\tilde w^\alpha_t + \alpha\gamma(\tilde w^\alpha - \tilde c^\alpha)(\tilde w^\alpha_t - \tilde c^\alpha_t) + \gamma \tilde c^\alpha_x \tilde c^\alpha_{xt} + \tilde v^\alpha \tilde v^\alpha_t - W'(w)w_t \\
&\qquad - \gamma w_x w_{xt} - v v_t - W''(w)w_t(\tilde w^\alpha - w) - W'(w)\tilde w^\alpha_t + W'(w)w_t \\
&\qquad - \gamma w_{xt}\tilde c^\alpha_x + \gamma w_x w_{xt} - \gamma w_x \tilde c^\alpha_{xt} + \gamma w_x w_{xt} - v_t \tilde v^\alpha + v_t v - v\tilde v^\alpha_t + v v_t \\
&\qquad - W'(\tilde w^\alpha)\tilde v^\alpha_x - \tilde v^\alpha W'(\tilde w^\alpha)_x + \gamma \tilde v^\alpha \tilde c^\alpha_{xxx} + \gamma \tilde c^\alpha_{xx}\tilde v^\alpha_x \\
&\qquad + \tilde v^\alpha W'(w)_x - vW'(w)_x + W'(w)\tilde v^\alpha_x - W'(w)v_x \\
&\qquad - \gamma \tilde v^\alpha w_{xxx} + \gamma v w_{xxx} - \gamma w_{xx}\tilde v^\alpha_x + \gamma w_{xx} v_x \\
&\qquad + v_x W'(\tilde w^\alpha) + vW'(\tilde w^\alpha)_x - \gamma v \tilde c^\alpha_{xxx} - \gamma \tilde c^\alpha_{xx} v_x \,\mathrm{d}\, x.
\end{aligned} \tag{3.2}
$$

In this equation several terms cancel out. We use the evolution equations (1.1), (1.3) and

$$
-w_x \tilde c^\alpha_{xt} = -w_x \tilde v^\alpha_{xx} + (w_x \tilde v^\alpha_x - w_x \tilde c^\alpha_t)_x - w_{xx}(\tilde w^\alpha_t - \tilde c^\alpha_t) \tag{3.3}
$$

to get rid of the time derivatives and obtain

$$
\begin{aligned}
A =& \int_{S^1} W'(\tilde w^\alpha)\tilde v^\alpha_x - \gamma \tilde c^\alpha_{xx}\tilde v^\alpha_x + (\tilde c^\alpha_t \tilde c^\alpha_x)_x + \tilde v^\alpha W'(\tilde w^\alpha)_x - \gamma \tilde v^\alpha \tilde c^\alpha_{xxx} + \mu \tilde v^\alpha \tilde v^\alpha_{xx} \\
&\qquad - W''(w)v_x(\tilde w^\alpha - w) - W'(w)\tilde v^\alpha_x - \gamma v_{xx}\tilde c^\alpha_x - \gamma w_x \tilde v^\alpha_{xx} + \gamma(w_x \tilde v^\alpha_x - w_x \tilde c^\alpha_t)_x \\
&\qquad - \gamma w_{xx}(\tilde w^\alpha_t - \tilde c^\alpha_t) + \gamma w_x v_{xx} - \tilde v^\alpha W'(w)_x + \gamma \tilde v^\alpha w_{xxx} - \mu \tilde v^\alpha v_{xx} \\
&\qquad + vW'(w)_x - \gamma v w_{xxx} + \mu v v_{xx} - vW'(\tilde w^\alpha)_x + \gamma v \tilde c^\alpha_{xxx} - \mu v \tilde v^\alpha_{xx} \\
&\qquad - W'(\tilde w^\alpha)\tilde v^\alpha_x - \tilde v^\alpha W'(\tilde w^\alpha)_x + \gamma \tilde v^\alpha \tilde c^\alpha_{xxx} + \gamma \tilde c^\alpha_{xx}\tilde v^\alpha_x \\
&\qquad + \tilde v^\alpha W'(w)_x - vW'(w)_x + W'(w)\tilde v^\alpha_x - W'(w)v_x \\
&\qquad - \gamma \tilde v^\alpha w_{xxx} + \gamma v w_{xxx} - \gamma w_{xx}\tilde v^\alpha_x + \gamma w_{xx} v_x \\
&\qquad + v_x W'(\tilde w^\alpha) + vW'(\tilde w^\alpha)_x - \gamma v \tilde c^\alpha_{xxx} - \gamma v_x \tilde c^\alpha_{xx} \,\mathrm{d}\, x \\
=& \int_{S^1} \mu(\tilde v^\alpha - v)(\tilde v^\alpha - v)_{xx} + \gamma\big(\tilde c^\alpha_t \tilde c^\alpha_x - v_x \tilde c^\alpha_x + v_x w_x - \tilde v^\alpha_x w_x + w_x \tilde v^\alpha_x - w_x \tilde c^\alpha_t\big)_x \\
&\qquad + v_x\big(W'(\tilde w^\alpha) - W'(w) - W''(w)(\tilde w^\alpha - w)\big) - \gamma w_{xx}(\tilde w^\alpha_t - \tilde c^\alpha_t) \,\mathrm{d}\, x.
\end{aligned} \tag{3.4}
$$

The assertion of the proposition follows from (3.2),(3.4) upon using Gauss' Theorem and the boundary conditions.

■

By rearranging the velocity and gradient terms in the relative energy in (3.1) we obtain:

Corollary 3.2 (Reformulated relative entropy equality) *Under the assumptions of Proposition 3.1 the following relative energy equation is satisfied*

$$\frac{d}{dt}\int_{S^1} W(\tilde{w}^\alpha) - W(w) - W'(w)(\tilde{w}^\alpha - w) + \frac{\alpha\gamma}{2}(\tilde{w}^\alpha - \tilde{c}^\alpha)^2 + \frac{\gamma}{2}(\tilde{c}^\alpha_x - w_x)^2 + \frac{1}{2}(\tilde{v}^\alpha - v)^2 \,\mathrm{d}\,x$$
$$= \int_{S^1} -\mu(\tilde{v}^\alpha_x - v_x)^2 + v_x\big(W'(\tilde{w}^\alpha) - W'(w) - W''(w)(\tilde{w}^\alpha - w)\big) - \gamma w_{xx}(\tilde{w}^\alpha_t - \tilde{c}^\alpha_t)\,\mathrm{d}\,x. \quad (3.5)$$

If W were a convex function the only remaining problem preventing us to use (3.5) and Gronwall's inequality to show convergence of solutions for $\alpha \to \infty$ would be to estimate the integral of $\gamma w_{xx}(\tilde{w}^\alpha_t - \tilde{c}^\alpha_t)$. In fact, this is a major step in our analysis, see Lemma 3.6. However, we have the additional difficulty that the gradient terms in the energy functional do not make the energy functional globally convex. Thus, we have to deal with the (non-convex) W-terms on the left hand side of (3.1). Our next step is to remove these terms.

Corollary 3.3 (Reduced relative entropy inequality) *Provided the assumptions of Proposition 3.1 are satisfied, then*

$$\frac{d}{dt}\int_{S^1} \frac{\alpha\gamma}{2}(\tilde{w}^\alpha - \tilde{c}^\alpha)^2 + \frac{\gamma}{2}(\tilde{c}^\alpha_x - w_x)^2 + \frac{1}{2}(\tilde{v}^\alpha - v)^2 \,\mathrm{d}\,x$$
$$\leq \int_{S^1} \frac{1}{4\mu}\big(W'(\tilde{w}^\alpha) - W'(w)\big)^2 - \gamma w_{xx}(\tilde{w}^\alpha_t - \tilde{c}^\alpha_t)\,\mathrm{d}\,x. \quad (3.6)$$

Proof :
A straightforward calculation using the evolution equations (1.1), (1.3) shows

$$\begin{aligned}
&\partial_t\Big(W(\tilde{w}^\alpha) - W(w) - W'(w)(\tilde{w}^\alpha - w)\Big) - v_x\Big(W'(\tilde{w}^\alpha) - W'(w) - W''(w)(\tilde{w}^\alpha - w)\Big)\\
&= W'(\tilde{w}^\alpha)\tilde{v}^\alpha_x - W'(w)v_x - W''(w)v_x(\tilde{w}^\alpha - w) - W'(w)\tilde{v}^\alpha_x + W'(w)v_x\\
&\quad - v_x W'(\tilde{w}^\alpha) + v_x W'(w) + v_x W''(w)(\tilde{w}^\alpha - w)\\
&= \big(W'(\tilde{w}^\alpha) - W'(w)\big)\big(\tilde{v}^\alpha_x - v_x\big).
\end{aligned} \quad (3.7)$$

Inserting (3.7) into (3.5) implies

$$\frac{d}{dt}\int_{S^1} \frac{\alpha\gamma}{2}(\tilde{w}^\alpha - \tilde{c}^\alpha)^2 + \frac{\gamma}{2}(\tilde{c}^\alpha_x - w_x)^2 + \frac{1}{2}(\tilde{v}^\alpha - v)^2 \,\mathrm{d}\,x$$
$$= \int_{S^1} -\mu(\tilde{v}^\alpha_x - v_x)^2 - \big(W'(\tilde{w}^\alpha) - W'(w)\big)\big(\tilde{v}^\alpha_x - v_x\big) - \gamma w_{xx}(\tilde{w}^\alpha_t - \tilde{c}^\alpha_t)\,\mathrm{d}\,x. \quad (3.8)$$

The assertion of the corollary follows from (3.8) and Young's inequality. ■

3.2 Estimates for the elliptic operator

Let us denote the solution operator to $\mathrm{Id} - \frac{1}{\alpha}\partial_{xx}$ on S^1 by G_α, i.e., for $f \in L^2(S^1)$ we have

$$G_\alpha[f] - \frac{1}{\alpha}G_\alpha[f]_{xx} = f \quad (3.9)$$

at least weakly. Thus,

$$\tilde{c}^\alpha = G_\alpha[\tilde{w}^\alpha] \tag{3.10}$$

by definition.

Lemma 3.4 (Regularity of elliptic approximation) *For any $f \in L^2(S^1)$ the following inequality holds*

$$\|G_\alpha[f]\|_{L^2(S^1)} \leq \|f\|_{L^2(S^1)}. \tag{3.11}$$

Moreover, for any $k \in \mathbb{N}$ the fact that G_α is the solution operator to a linear elliptic equation with constant coefficients implies

$$G_\alpha[f] \in H^{k+2}(S^1) \quad \text{for } f \in H^k(S^1)$$

and

$$G_\alpha[f_x] = (G_\alpha[f])_x \tag{3.12}$$

for all $f \in H^1(S^1)$.

Proof :
Upon testing (3.9) with $G_\alpha[f]$ and using the periodic boundary conditions we obtain

$$\|G_\alpha[f]\|^2_{L^2(S^1)} + \frac{1}{\alpha}|G_\alpha[f]|^2_{H^1(S^1)} = \int_{S^1} f G_\alpha[f] \, \mathrm{d}\, x \leq \|G_\alpha[f]\|_{L^2(S^1)} \|f\|_{L^2(S^1)}, \tag{3.13}$$

which implies (3.11). The other assertions of the Lemma follow as ∂_x commutes with $\mathrm{Id} - \frac{1}{\alpha}\partial_{xx}$. ■

Our next step is to investigate the approximation properties of the elliptic operator in $(1.3)_3$.

Lemma 3.5 (Elliptic approximation estimate) *Let $f \in H^1(S^1)$, then*

$$\|f - G_\alpha[f]\|^2_{L^2(S^1)} \leq \frac{2}{\alpha}|f|^2_{H^1(S^1)}.$$

In case $f \in H^2(S^1)$ the following (stronger) estimate is satisfied:

$$\|f - G_\alpha[f]\|^2_{L^2(S^1)} \leq \frac{1}{\alpha^2}|f|^2_{H^2(S^1)}.$$

Proof :
From testing (3.9) with $f - G_\alpha[f]$ we infer

$$\begin{aligned} \|f - G_\alpha[f]\|^2_{L^2(S^1)} &= \frac{-1}{\alpha}\int_{S^1}(f - G_\alpha[f])\partial_{xx} G_\alpha[f] \, \mathrm{d}\, x = \frac{1}{\alpha}\int_{S^1}(f - G_\alpha[f])_x G_\alpha[f]_x \, \mathrm{d}\, x \\ &\leq \frac{1}{\alpha}\|f_x - G_\alpha[f_x]\|_{L^2(S^1)}\|G_\alpha[f_x]\|_{L^2(S^1)} \leq \frac{2}{\alpha}|f|^2_{H^1(S^1)} \end{aligned} \tag{3.14}$$

upon applying (3.11). This shows the first assertion of the Lemma. For $f \in H^2(S^1)$ we compute

$$\|f - G_\alpha[f]\|^2_{L^2(S^1)} = \frac{1}{\alpha^2}\int_{S^1}(\partial_{xx} G_\alpha[f])^2 \, \mathrm{d}\, x = \frac{1}{\alpha^2}\int_{S^1}(G_\alpha[f_{xx}])^2 \, \mathrm{d}\, x \leq \frac{1}{\alpha^2}|f|^2_{H^2(S^1)}. \tag{3.15}$$

■

3.3 Difference estimates

Lemma 3.6 (Estimate on time derivatives) *Under the assumptions of Proposition 3.1 the following inequality is satisfied for all $t \in (0,T)$:*

$$\Big| \int_{S^1} w_{xx}(\tilde{c}^\alpha_t - \tilde{w}^\alpha_t)\,\mathrm{d}\,x \Big| \le \sqrt{\frac{2}{\alpha}} |w|_{H^3(S^1)} |\tilde{v}^\alpha|_{H^1(S^1)}. \tag{3.16}$$

Proof :
A straightforward calculation, based on $\tilde{c}^\alpha_t = G_\alpha[\tilde{w}^\alpha_t]$, gives

$$\int_{S^1} w_{xx}(\tilde{c}^\alpha_t - \tilde{w}^\alpha_t)\,\mathrm{d}\,x = \int_{S^1} w_{xx}(G_\alpha[\tilde{v}^\alpha_x] - \tilde{v}^\alpha_x)\,\mathrm{d}\,x = -\int_{S^1} w_{xxx}(G_\alpha[\tilde{v}^\alpha] - \tilde{v}^\alpha)\,\mathrm{d}\,x. \tag{3.17}$$

Using Lemma 3.5 we obtain

$$\Big| \int_{S^1} w_{xx}(\tilde{c}^\alpha_t - \tilde{w}^\alpha_t)\,\mathrm{d}\,x \Big| \le |w|_{H^3(S^1)} \sqrt{\frac{2}{\alpha}} |\tilde{v}^\alpha|_{H^1(S^1)}.$$

■

Remark 3.7 (Boundary conditions) *If we considered natural boundary conditions, as in [13], instead of periodic ones, there would be additional (non-vanishing) boundary terms in* (3.16)*. It is not clear whether it is possible to estimate them properly.*

Proposition 3.8 (Reduced relative entropy growth estimate) *Let $\bar{w} \in H^3_m(S^1)$, $\bar{v} \in H^2_m(S^1)$ and $T, \mu, \gamma > 0$ be given. Then, it exists a constant $C > 0$ such that for α large enough the strong solution (w, v) of* (1.1),(2.1) *and the strong solution $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ of* (1.3),(2.2) *satisfy the following estimate for all $t \in (0,T)$:*

$$\frac{d}{dt}\Big(\frac{\alpha\gamma}{2} \|\tilde{w}^\alpha - \tilde{c}^\alpha\|^2_{L^2(S^1)} + \frac{\gamma}{2} |\tilde{c}^\alpha - w|^2_{H^1(S^1)} + \frac{1}{2} \|\tilde{v}^\alpha - v\|^2_{L^2(S^1)} \Big)$$
$$\le \frac{C}{\gamma\mu} \Big(\frac{\alpha\gamma}{2} \|\tilde{w}^\alpha - \tilde{c}^\alpha\|^2_{L^2(S^1)} + \frac{\gamma}{2} |\tilde{c}^\alpha - w|^2_{H^1(S^1)} \Big) + \gamma |w|_{H^3(S^1)} \sqrt{\frac{2}{\alpha}} |\tilde{v}^\alpha|_{H^1(S^1)}. \tag{3.18}$$

Proof :
The existence of (w, v) and $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ follows from Corollary 2.4 and Proposition 2.7. As $\tilde{w}^\alpha(t, \cdot) \in H^1_m(S^1)$ for all $\alpha > 0$ and all $t \in [0,T]$ the bound (independent of α) on $\sup_{t\in[0,T]} |\tilde{w}^\alpha|_{H^1(S^1)}$ asserted in Proposition 2.7 implies that $\|\tilde{w}^\alpha(t, \cdot)\|_{H^1(S^1)}$ is bounded independent of α. Due to the continuous embedding of $H^1(S^1)$ into $C^0(S^1)$ this implies

$$w_{\max} := \max \big\{ \|w\|_{L^\infty([0,T]\times S^1)}, \sup_{\alpha>0} \|\tilde{w}^\alpha\|_{L^\infty([0,T]\times S^1)} \big\} < \infty, \text{ and } \bar{W} := \max_{|w|\le w_{\max}} |W''(w)| < \infty. \tag{3.19}$$

Combining (3.6) and Lemma 3.6 we find

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{\alpha\gamma}{2}\|\tilde{w}^\alpha-\tilde{c}^\alpha\|^2_{L^2(S^1)}+\frac{\gamma}{2}|\tilde{c}^\alpha-w|^2_{H^1(S^1)}+\frac{1}{2}\|\tilde{v}^\alpha-v\|^2_{L^2(S^1)}\Big)\\
&\qquad\le\frac{\bar{W}^2}{2\mu}\Big(\|\tilde{w}^\alpha-\tilde{c}^\alpha\|^2_{L^2(S^1)}+\|\tilde{c}^\alpha-w\|^2_{L^2(S^1)}\Big)+\gamma|w|_{H^3(S^1)}\sqrt{\frac{2}{\alpha}}|\tilde{v}^\alpha|_{H^1(S^1)}\\
&\quad\le\max\Big\{\frac{1}{\alpha},1\Big\}\frac{\bar{W}^2}{\mu\gamma}\Big(\frac{\alpha\gamma}{2}\|\tilde{w}^\alpha-\tilde{c}^\alpha\|^2_{L^2(S^1)}+\frac{\gamma}{2}|\tilde{c}^\alpha-w|^2_{H^1(S^1)}\Big)+\gamma|w|_{H^3(S^1)}\sqrt{\frac{2}{\alpha}}|\tilde{v}^\alpha|_{H^1(S^1)},
\end{aligned}
\tag{3.20}
$$

because Poincaré's inequality is applicable to $\tilde{c}^\alpha-w$ as

$$
\int_{S^1}\tilde{c}^\alpha(t,\cdot)-w(t,\cdot)\,\mathrm{d}\,x=\int_{S^1}\tilde{w}^\alpha(t,\cdot)-w(t,\cdot)\,\mathrm{d}\,x=\int_{S^1}\tilde{w}^\alpha(0,\cdot)-w(0,\cdot)\,\mathrm{d}\,x=0.
$$

Equation (3.20) proves the assertion of the proposition.

■

Remark 3.9 (Dependency of $\bar{W}$.) *While the constant $\bar{W}$ defined in* (3.19) *is independent of α, it might very well depend on $\bar{w}$, $\bar{v}$, γ and μ as $w_{\max}$ might depend on those data.*

Remark 3.10 (Higher order estimate) *Analogous to the derivation of Proposition 3.8, but using a modification of Lemma 3.6 which relies on the second assertion of Lemma 3.5 instead of the first one, we can show that*

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{\alpha\gamma}{2}\|\tilde{w}^\alpha-\tilde{c}^\alpha\|^2_{L^2(S^1)}+\frac{\gamma}{2}|\tilde{c}^\alpha-w|^2_{H^1(S^1)}+\frac{1}{2}\|\tilde{v}^\alpha-v\|^2_{L^2(S^1)}\Big)\\
&\qquad\le\frac{C}{\gamma\mu}\Big(\frac{\alpha\gamma}{2}\|\tilde{w}^\alpha-\tilde{c}^\alpha\|^2_{L^2(S^1)}+\frac{\gamma}{2}|\tilde{c}^\alpha-w|^2_{H^1(S^1)}\Big)+\gamma|w|_{H^3(S^1)}\frac{1}{\alpha}|\tilde{v}^\alpha|_{H^2(S^1)},
\end{aligned}
\tag{3.21}
$$

for some $C>0$ independent of α, holds under the assumptions of Proposition 3.8.

Let us define the following reduced relative entropy (without the W-terms)

$$
\eta^\alpha(t):=\frac{\alpha\gamma}{2}\|\tilde{w}^\alpha(t,\cdot)-\tilde{c}^\alpha(t,\cdot)\|^2_{L^2(S^1)}+\frac{\gamma}{2}|\tilde{c}^\alpha(t,\cdot)-w(t,\cdot)|^2_{H^1(S^1)}+\frac{1}{2}\|\tilde{v}^\alpha(t,\cdot)-v(t,\cdot)\|^2_{L^2(S^1)}. \tag{3.22}
$$

Using this notation we can write the assertion of Proposition 3.8 as

$$
(\eta^\alpha)'(t)\le\frac{C}{\gamma\mu}\eta^\alpha(t)+\gamma\sqrt{\frac{2}{\alpha}}|w(t,\cdot)|_{H^3(S^1)}|\tilde{v}^\alpha(t,\cdot)|_{H^1(S^1)}. \tag{3.23}
$$

In order to derive a bound for η^α via Gronwall's inequality we need to study $\eta^\alpha(0)$.

Proposition 3.11 (Estimate on initial relative entropy) *Provided the assumptions of Proposition 3.8 are fulfilled, then η^α defined in* (3.22) *satisfies*

$$
\eta^\alpha(0)\le\frac{\gamma}{\alpha}\|\bar{w}\|^2_{H^3(S^1)} \tag{3.24}
$$

and $|\tilde{v}^\alpha(t,\cdot)|_{L^2(0,T;H^1_s(S^1))}$ is bounded uniformly for all $\alpha\ge1$, where $|\cdot|_{L^2(0,T;H^1_s(S^1))}$ indicates that we consider the H^1-semi-norm in space.

Proof :
As $\tilde{v}^\alpha(0,\cdot) = v(0,\cdot)$ and $\tilde{w}^\alpha(0,\cdot) = w(0,\cdot)$ we have

$$\eta^\alpha(0) = \frac{\alpha\gamma}{2}\|\tilde{w}^\alpha(0,\cdot) - \tilde{c}^\alpha(0,\cdot)\|^2_{L^2(S^1)} + \frac{\gamma}{2}|\tilde{c}^\alpha(0,\cdot) - \tilde{w}^\alpha(0,\cdot)|^2_{H^1(S^1)}. \tag{3.25}$$

The second assertion of Lemma 3.6 implies

$$\frac{\gamma}{2}|\tilde{c}^\alpha(0,\cdot) - \tilde{w}^\alpha(0,\cdot)|^2_{H^1(S^1)} = \frac{\gamma}{2}|G_\alpha[\bar{w}] - \bar{w}|^2_{H^1(S^1)} \le \frac{\gamma}{2\alpha^2}|\bar{w}|^2_{H^3(S^1)} \tag{3.26}$$

and

$$\frac{\gamma\alpha}{2}\|\tilde{c}^\alpha(0,\cdot) - \tilde{w}^\alpha(0,\cdot)\|^2_{L^2(S^1)} = \frac{\gamma\alpha}{2}\|G_\alpha[\bar{w}] - \bar{w}\|^2_{L^2(S^1)} \le \frac{\gamma}{2\alpha}|\bar{w}|^2_{H^2(S^1)}. \tag{3.27}$$

Combining (3.26) and (3.27) proves (3.24). Integrating (2.5) in space implies

$$\frac{d}{dt}\int_{S^1} W(\tilde{w}^\alpha) + \frac{\gamma\alpha}{2}(\tilde{w}^\alpha - \tilde{c}^\alpha)^2 + \frac{\gamma}{2}(\tilde{c}^\alpha_x)^2 + \frac{1}{2}(\tilde{v}^\alpha)^2 \,\mathrm{d}\,x = -\mu\int_{S^1}(\tilde{v}^\alpha_x)^2 \,\mathrm{d}\,x$$

such that, because the energy density is non-negative,

$$\begin{aligned}|\tilde{v}^\alpha(t,\cdot)|_{L^2(0,T;H^1_s(S^1))} &\le \int_{S^1} W(\bar{w}) + \frac{\gamma\alpha}{2}(\bar{w} - G_\alpha[\bar{w}])^2 + \frac{\gamma}{2}(G_\alpha[\bar{w}]_x)^2 + \frac{1}{2}\bar{v}^2 \,\mathrm{d}\,x \\ &\le \int_{S^1} W(\bar{w}) + \frac{\gamma}{2}(\bar{w}_x)^2 + \frac{1}{2}(\bar{v})^2 \,\mathrm{d}\,x + \eta^\alpha(0).\end{aligned} \tag{3.28}$$

■

Combining our preparatory results we are now in position to prove our main result:

Theorem 3.12 (Model convergence) *Let $\bar{w} \in H^3_m(S^1)$, $\bar{v} \in H^2_m(S^1)$ and μ, γ, $T > 0$ be given. Then, it exists a constant $\bar{W}$ so that for α large enough the strong solution (w, v) of (1.1),(2.1) and the strong solution $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ of (1.3),(2.2) fulfill the following estimate for all $t \in (0, T)$:*

$$\begin{aligned}&\frac{\alpha\gamma}{2}\|\tilde{w}^\alpha(t,\cdot) - \tilde{c}^\alpha(t,\cdot)\|^2_{L^2(S^1)} + \frac{\gamma}{2}|\tilde{c}^\alpha(t,\cdot) - w(t,\cdot)|^2_{H^1(S^1)} + \frac{1}{2}\|\tilde{v}^\alpha(t,\cdot) - v(t,\cdot)\|^2_{L^2(S^1)} \\ &\qquad \le \frac{\gamma}{\alpha}\|\bar{w}\|^2_{H^3(S^1)}e^{Kt} + \gamma\sqrt{\frac{2}{\alpha}}e^{Kt}\Big(\|w\|^2_{L^2(0,T;H^3(S^1))} + \frac{E_0}{\mu} + \frac{\gamma}{\alpha\mu}\|\bar{w}\|^2_{H^3(S^1)}\Big)\end{aligned} \tag{3.29}$$

with

$$E_0 := \int_{S^1} W(\bar{w}) + \frac{\gamma}{2}|\bar{w}_x|^2 + \frac{1}{2}\bar{v}^2 \,\mathrm{d}\,x \quad \text{and} \quad K := \frac{\bar{W}^2}{\gamma\mu}.$$

Proof :
Applying Gronwall's inequality to (3.23) we find

$$\eta^\alpha(t) \le \eta^\alpha(0)e^{Kt} + \gamma\sqrt{\frac{2}{\alpha}}\int_0^t e^{K(s-t)}|w(s,\cdot)|_{H^3(S^1)}|\tilde{v}^\alpha(s,\cdot)|_{H^1(S^1)} \,\mathrm{d}\,s. \tag{3.30}$$

Using Proposition 3.11 and Young's inequality in (3.30) we find

$$\eta^\alpha(t) \le \frac{\gamma}{\alpha}\|\bar{w}\|^2_{H^3(S^1)}e^{Kt} + \gamma\sqrt{\frac{2}{\alpha}}e^{Kt}\Big(\|w\|^2_{L^2(0,T;H^3(S^1))} + |\tilde{v}^\alpha|^2_{L^2(0,T;H^1_s(S^1))}\Big). \tag{3.31}$$

This completes the proof as

$$\mu|\tilde{v}^\alpha|^2_{L^2(0,T;H^1_s(S^1))} \leq E_0 + \eta^\alpha(0) \leq E_0 + \frac{\gamma}{\alpha}\|\bar{w}\|^2_{H^3(S^1)}$$

by Proposition 3.11.

■

Remark 3.13 (Parameter and time dependence) *Theorem 3.12 implies that for given $\gamma, \mu > 0$ it holds $\eta^\alpha(t) = \mathcal{O}(\alpha^{-1/2})$ locally uniform in time. It must be noted that the constant in this estimate depends strongly on γ and μ and for γ, μ very small the error might be quite large. In particular, the strong dependence on γ was to be expected as it scales the part of the energy which is convex in w.*

Remark 3.14 (Initial data) *Note that we do not need to impose identical initial data for* (1.1) *and* (1.3) *but we simply did so for simplicity. For Theorem 3.12 to hold, it is sufficient that the initial data are sufficiently regular and such that Proposition 3.11 is valid.*

Remark 3.15 (Different convergence rates) *Theorem 3.12 guarantees strong convergence of solutions provided the initial data are sufficiently smooth. However, in most numerical examples higher orders of convergence are observed, see [23]. We expect that in those cases some additional terms are uniformly bounded in α, while it is not clear how to uniformly bound these terms in general. As an indication in this direction we will give an estimate below, which shows how the convergence is accelerated in case $\|\tilde{v}^\alpha\|^2_{L^2(0,T;H^2(S^1))}$ is uniformly bounded*

Theorem 3.16 (Model convergence II) *Let $\bar{w} \in H^3_m(S^1)$, $\bar{v} \in H^2_m(S^1)$ and $T, \gamma, \mu > 0$ be given. Let (w, v) be the strong solution of* (1.1),(2.1) *and let $(\tilde{w}^\alpha, \tilde{v}^\alpha, \tilde{c}^\alpha)$ denote the strong solution of* (1.3),(2.2). *In case there exist constants $C, \alpha_0 > 0$ such that*

$$\|\tilde{v}^\alpha\|^2_{L^2(0,T;H^2(S^1))} \leq C \quad \forall \alpha > \alpha_0,$$

there exists a constant $\bar{W}$ so that for α large enough and $K := \frac{\bar{W}}{\gamma\mu}$ the following estimate is satisfied:

$$\frac{\alpha\gamma}{2}\|\tilde{w}^\alpha(t,\cdot) - \tilde{c}^\alpha(t,\cdot)\|^2_{L^2(S^1)} + \frac{\gamma}{2}|\tilde{c}^\alpha(t,\cdot) - w(t,\cdot)|^2_{H^1(S^1)} + \frac{1}{2}\|\tilde{v}^\alpha(t,\cdot) - v(t,\cdot)\|^2_{L^2(S^1)} \\ \leq \frac{\gamma}{\alpha}\|\bar{w}\|^2_{H^2(S^1)}e^{Kt} + \frac{\gamma}{\alpha}e^{Kt}\Big(\|w\|^2_{L^2(0,T;H^3(S^1))} + C\Big). \quad (3.32)$$

This can be proven analogous to Theorem 3.12. The only difference is that Proposition 3.8 has to be replaced by Remark 3.10.

4 The multidimensional case

4.1 Layout of the problem

In this section we consider the multi-dimensional equations of non-linear elastodynamics supplemented by a rate type and a second gradient type term. Those terms are multidimensional versions of the viscosity and capillarity terms in (1.1). However, the multi-dimensional terms are not materially frame indifferent, cf. [4, e.g.], and therefore do not allow for any physical interpretation.

We will denote the strain by $\mathbf{F} = (F_{ij})$ and the velocity by $\mathbf{v} = (v_i)$; space derivatives are abbreviated by ∂_i. Whenever one index appears twice it is understood that it is summed. For given $\gamma, \mu > 0$ we consider the following third order model

$$\begin{aligned} \partial_t F_{ij} - \partial_j v_i = 0 \quad \text{for } i, j = 1, \dots, d \\ \partial_t v_i - \partial_j \big(\frac{\partial W}{\partial F_{ij}}(\mathbf{F}) - \gamma \Delta F_{ij}\big) = \mu \Delta v_i \quad \text{for } i = 1, \dots, d \end{aligned} \tag{4.1}$$

subject to the involution $\partial_k F_{ij} = \partial_j F_{ik}$ for $i, j, k = 1, \dots, d$. In (4.1) we assume that $W \in C^3(M_+^{d\times d}, \mathbb{R})$, where $M_+^{d\times d}$ is the space of $d \times d$ matrices with positive determinant. For the first order part of (4.1) to be materially frame indifferent W needs to satisfy certain conditions, see [3, 10, e.g.] which rule out convexity. Instead of investigating the effects of certain assumptions of W, e.g. poly-convexity or quasi-convexity, which are compatible with material frame indifference, our analysis will rely on the higher order regularization mechanisms. We only impose that there exists a constant $\bar{W}$ such that

$$\Big(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\Big)^2 \leq \bar{W} |\tilde{\mathbf{F}}^\alpha - \mathbf{F}|^2 \quad \forall\, \mathbf{F}, \tilde{\mathbf{F}}^\alpha \in M_+^{d\times d} \tag{4.2}$$

and we will see that in this case the higher order regularizations will indeed compensate for the non-convexity. We are well aware that Assumption (4.2) is problematic for realistic energy densities as $W(\mathbf{F})$ will usually be singular for $\det(\mathbf{F}) \to 0$. However, assumptions like (4.2), i.e. boundedness of the second derivative, are standard for relative entropy estimates.

We will compare solutions of (4.1) to those of a family of lower order approximations (parametrized by $\alpha > 0$)

$$\begin{aligned} \partial_t \tilde{F}^\alpha_{ij} - \partial_j \tilde{v}^\alpha_i = 0 \quad \text{for } i, j = 1, \dots, d \\ \partial_t \tilde{v}^\alpha_i - \partial_j \big(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) + \gamma\alpha \tilde{F}^\alpha_{ij}\big) = \mu \Delta \tilde{v}^\alpha_i - \gamma\alpha \partial_j \tilde{C}^\alpha_{ij} \quad \text{for } i = 1, \dots, d \\ \alpha(\tilde{C}^\alpha_{ij} - \tilde{F}^\alpha_{ij}) = \Delta \tilde{C}^\alpha_{ij} \quad \text{for } i, j = 1, \dots, d \end{aligned} \tag{4.3}$$

subject to the involution $\partial_k \tilde{F}_{ij} = \partial_j \tilde{F}_{ik}$ for $i, j, k = 1, \dots, d$. Note that the second and third line of (4.3) can be combined to obtain

$$\partial_t \tilde{v}^\alpha_i - \partial_j \big(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) - \gamma \Delta \tilde{C}^\alpha_{ij}\big) = \mu \Delta \tilde{v}^\alpha_i. \tag{4.4}$$

In addition, equations $(4.3)_{1,2}$ form a hyperbolic system of balance laws for $(\mathbf{F}, \mathbf{v})$ provided α is large enough.

Both systems satisfy an additional balance law, which, in case of periodic boundary conditions, gives rise to a Lyapunov function, upon integration in space. Strong solutions of the third order system (4.1) satisfy

$$\begin{aligned} &\partial_t\big(W(\mathbf{F}) + \frac{\gamma}{2}|\nabla \mathbf{F}|^2 + \frac{1}{2}|\mathbf{v}|^2\big) \\ &\qquad - \partial_j\Big(v_i \frac{\partial W}{\partial F_{ij}}(\mathbf{F}) - \gamma v_i \Delta F_{ij} + \gamma \partial_t F_{ik} \partial_j F_{ik} + \mu v_i \partial_j v_i\Big) + \mu |D\mathbf{v}|^2 = 0 \end{aligned} \tag{4.5}$$

where $D\mathbf{v}$ denotes the Jacobian of $\mathbf{v}$. Equation (4.5) is obtained by multiplying $(4.1)_1$ by $\frac{\partial W}{\partial F_{ij}}(\mathbf{F}) - \gamma\Delta F_{ij}$ and $(4.1)_2$ by v_i and summing. Strong solutions of the lower order system (4.3) fulfill

$$\partial_t(W(\tilde{\mathbf{F}}^\alpha) + \frac{\gamma\alpha}{2}|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha|^2 + \frac{\gamma}{2}|\nabla\tilde{\mathbf{C}}^\alpha|^2 + \frac{1}{2}|\tilde{\mathbf{v}}^\alpha|^2) \\ - \partial_j\Big(\tilde{v}_i^\alpha \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) - \gamma\alpha\tilde{v}_i^\alpha(\tilde{F}_{ij}^\alpha - \tilde{C}_{ij}^\alpha) + \gamma(\partial_t\tilde{C}_{ik})\partial_j\tilde{C}_{ik} + \mu\tilde{v}_i^\alpha\partial_j\tilde{v}_i^\alpha\Big) + \mu|D\tilde{\mathbf{v}}^\alpha|^2 = 0. \quad (4.6)$$

To obtain (4.6) we multiply $(4.3)_1$ by $\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) + \alpha\gamma(\tilde{F}_{ij}^\alpha - \tilde{C}_{ij}^\alpha)$ and $(4.3)_2$ by $\tilde{v}_i^\alpha$ and, further, use

$$\alpha(\tilde{F}_{ij}^\alpha - \tilde{C}_{ij}^\alpha)\partial_t\tilde{C}_{ij}^\alpha = -\partial_k(\partial_t\tilde{C}_{ij}^\alpha\partial_k\tilde{C}_{ij}^\alpha) + \frac{1}{2}\partial_t\big((\partial_k\tilde{C}_{ij}^\alpha)^2\big). \quad (4.7)$$

In the sequel we will consider both systems on the flat d-dimensional torus $\mathbb{T}^d$, i.e., the cube $[0,1]^d$ with periodic boundary conditions. We choose functions $\bar{\mathbf{F}} \in H^3(\mathbb{T}^d, M_+^{d\times d})$ and $\bar{\mathbf{v}} \in H^2(\mathbb{T}^d, \mathbb{R}^d)$ and complement (4.1), (4.3) with the following initial data

$$\tilde{\mathbf{F}}^\alpha(0,\cdot) = \mathbf{F}(0,\cdot) = \bar{\mathbf{F}}, \quad \tilde{\mathbf{v}}^\alpha(0,\cdot) = \mathbf{v}(0,\cdot) = \bar{\mathbf{v}}, \quad \text{in } \mathbb{T}^d. \quad (4.8)$$

Definition 4.1 (Strong solution of (4.1)**)** *We call a tuple* $(\mathbf{F}, \mathbf{v})$ *a strong solution of* (4.1) *if*

$$\begin{aligned} &\mathbf{F} \in C^0([0,T], H^3(\mathbb{T}^d, M_+^{d\times d})) \cap C^1((0,T), H^1(\mathbb{T}^d, M_+^{d\times d})) \\ &\mathbf{v} \in C^0([0,T], H^2(\mathbb{T}^d, \mathbb{R}^d)) \cap C^1((0,T), L^2(\mathbb{T}^d, \mathbb{R}^d)) \end{aligned} \quad (4.9)$$

and if it satisfies (4.1) *in a point-wise sense almost everywhere.*

Definition 4.2 (Strong solution of (4.3)**)** *We call a tuple* $(\tilde{\mathbf{F}}^\alpha, \tilde{\mathbf{v}}^\alpha, \tilde{\mathbf{C}}^\alpha)$ *a strong solution of* (4.3) *if*

$$\begin{aligned} &\tilde{\mathbf{F}}^\alpha \in W^{1,\infty}(0,T; L^2(\mathbb{T}^d, M_+^{d\times d})) \cap C([0,T], H^1(\mathbb{T}^d, M_+^{d\times d})) \\ &\tilde{\mathbf{v}}^\alpha \in H^1(0,T; L^2(\mathbb{T}^d, \mathbb{R}^d)) \cap L^\infty(0,T; H^1(\mathbb{T}^d, \mathbb{R}^d)) \cap L^2(0,T; H^2(\mathbb{T}^d, \mathbb{R}^d)) \\ &\tilde{\mathbf{C}}^\alpha \in C^1((0,T), H^1(\mathbb{T}^d, M_+^{d\times d})) \cap C([0,T], H^3(\mathbb{T}^d, M_+^{d\times d})) \end{aligned} \quad (4.10)$$

and if it satisfies (4.3) *in a point-wise sense almost everywhere.*

We will not provide any well-posedness analysis in the multi-dimensional case. A main difficulty of such an analysis would be to exclude $\det(\tilde{\mathbf{F}}^\alpha) \to 0$ at some point in finite time. We will show results under the assumption that strong solutions exist.

4.2 Relative entropy

Proposition 4.3 (Relative entropy equality) *Let* T, γ, $\mu > 0$ *be given. Let* $(\mathbf{F}, \mathbf{v})$ *be a strong solution of* (4.1),(4.8) *and let* $(\tilde{\mathbf{F}}^\alpha, \tilde{\mathbf{v}}^\alpha, \tilde{\mathbf{C}}^\alpha)$ *be a strong solution of* (4.3),(4.8)*. Then, the following* *relative energy estimate is satisfied:*

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^d} W(\tilde{\mathbf{F}}^\alpha) &+ \frac{\alpha\gamma}{2}|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha|^2 + \frac{\gamma}{2}|\nabla\tilde{\mathbf{C}}^\alpha|^2 + \frac{1}{2}|\tilde{\mathbf{v}}^\alpha|^2 - W(\mathbf{F}) - \frac{\gamma}{2}|\nabla\mathbf{F}|^2 - \frac{1}{2}|\mathbf{v}|^2 \\
&- \frac{\partial W}{\partial F_{ij}}(\mathbf{F})(\tilde{F}^\alpha_{ij} - F_{ij}) - \gamma\partial_k F_{ij}\partial_k(\tilde{C}^\alpha_{ij} - F_{ij}) - v_i(\tilde{v}^\alpha_i - v_i)\,\mathrm{d}\,\mathbf{x} \\
&= \int_{\mathbb{T}^d} -\mu|D\tilde{\mathbf{v}}^\alpha - D\mathbf{v}|^2 + \partial_j v_i\Big[\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) - \frac{\partial W}{\partial F_{ij}}(\mathbf{F}) - \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(\mathbf{F})(\tilde{F}_{kl} - F_{kl})\Big] \\
&\qquad + \gamma\Delta F_{ij}\partial_t(\tilde{C}^\alpha_{ij} - \tilde{F}^\alpha_{ij})\,\mathrm{d}\,\mathbf{x}. \quad (4.11)
\end{aligned}
$$

Proof :
The proof is analogous to the proof of Lemma 3.1. For completeness it is given in the Appendix. ■

Corollary 4.4 (Reformulated relative entropy) *Under the assumptions of Proposition 4.3 the following equation is fulfilled*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d} W(\tilde{\mathbf{F}}^\alpha) - W(\mathbf{F}) - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})(\tilde{F}^\alpha_{ij} - F_{ij}) + \frac{\alpha\gamma}{2}|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha|^2 + \frac{\gamma}{2}|\nabla\tilde{\mathbf{C}}^\alpha - \nabla\mathbf{F}|^2 + \frac{1}{2}|\tilde{\mathbf{v}}^\alpha - \mathbf{v}|^2\,\mathrm{d}\,\mathbf{x} \\
&= \int_{\mathbb{T}^d} \gamma\Delta F_{ij}\partial_t(\tilde{C}^\alpha_{ij} - \tilde{F}^\alpha_{ij}) - \mu|D\tilde{\mathbf{v}}^\alpha - D\mathbf{v}|^2 + \partial_j v_i\Big[\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) - \frac{\partial W}{\partial F_{ij}}(\mathbf{F}) - \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(\mathbf{F})(\tilde{F}_{kl} - F_{kl})\Big]\,\mathrm{d}\,\mathbf{x}. \quad (4.12)
\end{aligned}
$$

Proof :
The proof is immediate upon rearranging the velocity and gradient terms in (4.11). ■

As in the 1-dimensional case we need to remove the W-terms from the left hand side of (4.12).

Corollary 4.5 (Reduced relative entropy) *Under the assumptions of Proposition 4.3 the following estimate is satisfied*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d} \frac{\alpha\gamma}{2}|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha|^2 + \frac{\gamma}{2}|\nabla\tilde{\mathbf{C}}^\alpha - \nabla\mathbf{F}|^2 + \frac{1}{2}|\tilde{\mathbf{v}}^\alpha - \mathbf{v}|^2\,\mathrm{d}\,\mathbf{x} \\
&\qquad \le \int_{\mathbb{T}^d} \gamma\Delta F_{ij}\partial_t(\tilde{C}^\alpha_{ij} - \tilde{F}^\alpha_{ij}) + \frac{1}{4\mu}\Big(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\Big)^2\,\mathrm{d}\,\mathbf{x}. \quad (4.13)
\end{aligned}
$$

Proof :
A straightforward calculation shows

$$
\begin{aligned}
&\partial_t\big(W(\tilde{\mathbf{F}}^\alpha)-W(\mathbf{F})-\frac{\partial W}{\partial F_{ij}}(\mathbf{F})(\tilde{F}^\alpha_{ij}-F_{ij})\big)-\partial_j v_i\big(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)-\frac{\partial W}{\partial F_{ij}}(\mathbf{F})-\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(\mathbf{F})(\tilde{F}_{kl}-F_{kl})\big)\\
&=\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_j\tilde{v}^\alpha_i-\frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j v_i-\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(\mathbf{F})\partial_l v_k(\tilde{F}^\alpha_{ij}-F_{ij})-\frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j\tilde{v}^\alpha_i+\frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j v_i\\
&\quad-\partial_j v_i\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)+\partial_j v_i\frac{\partial W}{\partial F_{ij}}(\mathbf{F})+\partial_j v_i\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(\mathbf{F})(\tilde{F}_{kl}-F_{kl})\big)\\
&=\big(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha))-\frac{\partial W}{\partial F_{ij}}(\mathbf{F})\big)\big(\partial_j\tilde{v}^\alpha_i-\partial_j v_i\big).
\end{aligned}
\tag{4.14}
$$

Inserting (4.14) into (4.12) implies

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d}\frac{\alpha\gamma}{2}|\tilde{\mathbf{F}}^\alpha-\tilde{\mathbf{C}}^\alpha|^2+\frac{\gamma}{2}|\nabla\tilde{\mathbf{C}}^\alpha-\nabla\mathbf{F}|^2+\frac{1}{2}|\tilde{\mathbf{v}}^\alpha-\mathbf{v}|^2\,\mathrm{d}\,x\\
&=\int_{\mathbb{T}^d}-\mu|D\tilde{\mathbf{v}}^\alpha-D\mathbf{v}|^2-\big(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)-\frac{\partial W}{\partial F_{ij}}(\mathbf{F})\big)\big(\partial_j\tilde{v}^\alpha_i-\partial_j v_i\big)+\gamma\Delta F_{ij}\partial_t(\tilde{C}^\alpha_{ij}-\tilde{F}^\alpha_{ij})\,\mathrm{d}\,\mathbf{x}.
\end{aligned}
\tag{4.15}
$$

The assertion of the corollary follows from (4.15) upon using Young's inequality. ∎

Lemma 4.6 (Estimate on time derivatives) *Under the assumptions of Proposition 4.3 it holds*

$$
\big|\int_{\mathbb{T}^d}\Delta F_{ij}\partial_t(\tilde{C}^\alpha_{ij}-\tilde{F}^\alpha_{ij})\,\mathrm{d}\,\mathbf{x}\big|\le\sqrt{\frac{2}{\alpha}}|\mathbf{F}|_{H^3(\mathbb{T}^d)}|\tilde{\mathbf{v}}^\alpha|_{H^1(\mathbb{T}^d)}.
$$

Proof :
Because of Lemma 3.11 we have

$$
\begin{aligned}
\big|\int_{\mathbb{T}^d}\Delta F_{ij}\partial_t(\tilde{C}^\alpha_{ij}-\tilde{F}^\alpha_{ij})\,\mathrm{d}\,\mathbf{x}\big|&=\big|\int_{\mathbb{T}^d}\Delta F_{ij}(G_\alpha[\partial_i\tilde{v}^\alpha_j]-\partial_i v_j)\,\mathrm{d}\,\mathbf{x}\big|\\
&=\big|\int_{\mathbb{T}^d}\partial_{ikk}F_{ij}(G_\alpha[\tilde{v}^\alpha_j]-\tilde{v}^\alpha_j)\,\mathrm{d}\,\mathbf{x}\big|\le\sqrt{\frac{2}{\alpha}}|\mathbf{F}|_{H^3(\mathbb{T}^d)}|\tilde{\mathbf{v}}^\alpha|_{H^1(\mathbb{T}^d)}.
\end{aligned}
\tag{4.16}
$$

∎

Let us denote the Poincaré constant on $\mathbb{T}^d$ by K_p such that

$$
\|\mathbf{F}(t,\cdot)-\tilde{\mathbf{C}}^\alpha(t,\cdot)\|^2_{L^2(\mathbb{T}^d)}\le K_p|\mathbf{F}(t,\cdot)-\tilde{\mathbf{C}}^\alpha(t,\cdot)|^2_{H^1(\mathbb{T}^d)}\quad\forall t\in[0,T],
\tag{4.17}
$$

because of

$$
\int_{\mathbb{T}^d}\mathbf{F}(t,\mathbf{x})-\tilde{\mathbf{C}}^\alpha(t,\mathbf{x})\,\mathrm{d}\,\mathbf{x}=\int_{\mathbb{T}^d}\mathbf{F}(t,\mathbf{x})-\tilde{\mathbf{F}}^\alpha(t,\mathbf{x})\,\mathrm{d}\,\mathbf{x}=\int_{\mathbb{T}^d}\mathbf{F}(0,\mathbf{x})-\tilde{\mathbf{F}}^\alpha(0,\mathbf{x})\,\mathrm{d}\,\mathbf{x}=0.
$$

Proposition 4.7 (Growth of reduced relative entropy) *Provided the assumptions of Proposition 4.3 are satisfied, then for α large enough*

$$\frac{d}{dt}\Big(\frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha\|^2_{L^2(\mathbb{T}^d)} + \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha - \mathbf{F}|^2_{H^1(\mathbb{T}^d)} + \frac{1}{2}\|\tilde{\mathbf{v}}^\alpha - \mathbf{v}\|^2_{L^2(\mathbb{T}^d)}\Big)$$
$$\leq \frac{\bar{W}^2 K_p}{\mu\gamma}\Big(\frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha\|^2_{L^2(\mathbb{T}^d)} + \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha - \mathbf{F}|^2_{H^1(\mathbb{T}^d)}\Big) + \gamma\sqrt{\frac{2}{\alpha}}|\mathbf{F}|_{H^3(\mathbb{T}^d)}|\tilde{\mathbf{v}}^\alpha|_{H^1(\mathbb{T}^d)}, \quad (4.18)$$

with $\bar{W}$ defined in (4.2) *and K_p defined in* (4.17)

Proof :
Combining (4.13) and Lemma 4.6 we find

$$\frac{d}{dt}\Big(\frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha\|^2_{L^2(\mathbb{T}^d)} + \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha - \mathbf{F}|^2_{H^1(\mathbb{T}^d)} + \frac{1}{2}\|\tilde{\mathbf{v}}^\alpha - \mathbf{v}\|^2_{L^2(\mathbb{T}^d)}\Big)$$
$$\leq \frac{\bar{W}^2}{2\mu}\Big(\|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha\|^2_{L^2(\mathbb{T}^d)} + \|\tilde{\mathbf{C}}^\alpha - \mathbf{F}\|^2_{L^2(\mathbb{T}^d)}\Big) + \gamma\sqrt{\frac{2}{\alpha}}|\mathbf{F}|_{H^3(\mathbb{T}^d)}\|\tilde{\mathbf{v}}^\alpha\|_{H^1(\mathbb{T}^d)}$$
$$\leq \max\Big\{\frac{1}{\alpha}, K_p\Big\}\frac{\bar{W}^2}{\gamma\mu}\Big(\frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha - \tilde{\mathbf{C}}^\alpha\|^2_{L^2(\mathbb{T}^d)} + \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha - \mathbf{F}|^2_{H^1(\mathbb{T}^d)}\Big)$$
$$+ \gamma\sqrt{\frac{2}{\alpha}}|\mathbf{F}|_{H^3(\mathbb{T}^d)}|\tilde{\mathbf{v}}^\alpha|_{H^1(\mathbb{T}^d)}. \quad (4.19)$$

Equation (4.19) proves (4.18) for α sufficiently large.

■

Let us define the multi-dimensional reduced relative energy

$$\eta^\alpha(t) := \frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha(t,\cdot) - \tilde{\mathbf{C}}^\alpha(t,\cdot)\|^2_{L^2(\mathbb{T}^d)} + \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha(t,\cdot) - \mathbf{F}(t,\cdot)|^2_{H^1(\mathbb{T}^d)} + \frac{1}{2}\|\tilde{\mathbf{v}}^\alpha(t,\cdot) - \mathbf{v}(t,\cdot)\|^2_{L^2(\mathbb{T}^d)}. \quad (4.20)$$

Then, we can write the assertion of Proposition 4.7 as

$$(\eta^\alpha)'(t) \leq \frac{\bar{W}^2 K_p}{\mu\gamma}\eta^\alpha(t) + \gamma\sqrt{\frac{2}{\alpha}}|\mathbf{F}(t,\cdot)|_{H^3(\mathbb{T}^d)}|\tilde{\mathbf{v}}^\alpha(t,\cdot)|_{H^1(\mathbb{T}^d)}. \quad (4.21)$$

We would like to use Gronwall's inequality to derive a bound for η^α. In order to do this we need to estimate $\eta^\alpha(0)$.

Proposition 4.8 (Estimate on initial relative entropy) *Provided the assumptions of Proposition 4.3 are fulfilled, then η^α defined in* (4.20) *satisfies*

$$\eta^\alpha(0) \leq \frac{\gamma}{\alpha}\|\bar{\mathbf{F}}\|^2_{H^3(\mathbb{T}^d)}.$$

Proof :
We have

$$\eta^\alpha(0) = \frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha(0,\cdot) - \tilde{\mathbf{C}}^\alpha(0,\cdot)\|^2_{L^2(\mathbb{T}^d)} + \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha(0,\cdot) - \mathbf{F}(0,\cdot)|^2_{H^1(\mathbb{T}^d)}. \tag{4.22}$$

Using Lemma 3.11 we find

$$\begin{aligned}\frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha(0,\cdot) - \tilde{\mathbf{C}}^\alpha(0,\cdot)\|^2_{L^2(\mathbb{T}^d)} &\le \frac{\gamma}{2\alpha}|\bar{\mathbf{F}}|^2_{H^2(\mathbb{T}^d)}\\ \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha(0,\cdot) - \mathbf{F}(0,\cdot)|^2_{H^1(\mathbb{T}^d)} &\le \frac{\gamma}{2\alpha^2}|\bar{\mathbf{F}}|^2_{H^3(\mathbb{T}^d)}.\end{aligned} \tag{4.23}$$

■

Theorem 4.9 (Multi-dimensional model convergence) *Let the assumptions of Proposition 4.3 be fulfilled. Then, for α large enough the strong solution $(\mathbf{F},\mathbf{v})$ of (4.1),(4.8) and the strong solution $(\tilde{\mathbf{F}}^\alpha, \tilde{\mathbf{v}}^\alpha, \tilde{\mathbf{C}}^\alpha)$ of (4.3),(4.8) fulfill the following estimate for all $t \in (0,T)$:*

$$\begin{aligned}&\frac{\alpha\gamma}{2}\|\tilde{\mathbf{F}}^\alpha(t,\cdot) - \tilde{\mathbf{C}}^\alpha(t,\cdot)\|^2_{L^2(\mathbb{T}^d)} + \frac{\gamma}{2}|\tilde{\mathbf{C}}^\alpha(t,\cdot) - \mathbf{F}(t,\cdot)|^2_{H^1(\mathbb{T}^d)} + \frac{1}{2}\|\tilde{\mathbf{v}}^\alpha(t,\cdot) - \mathbf{v}(t,\cdot)\|^2_{L^2(S^1)}\\ &\qquad \le \frac{\gamma}{\alpha}\|\bar{\mathbf{F}}\|^2_{H^3(S^1)}e^{Kt} + \gamma\sqrt{\frac{2}{\alpha}}e^{Kt}\Big(\|\mathbf{F}\|^2_{L^2(0,T;H^3(\mathbb{T}^d))} + \frac{E_0}{\mu} + \frac{\gamma}{\mu\alpha}\|\bar{\mathbf{F}}\|^2_{H^3(\mathbb{T}^d)}\Big)\end{aligned} \tag{4.24}$$

with

$$E_0 := \int_{\mathbb{T}^d} W(\bar{\mathbf{F}}) + \frac{\gamma}{2}|\nabla\bar{\mathbf{F}}|^2 + \frac{1}{2}|\bar{\mathbf{v}}|^2 \,\mathrm{d}\,\mathbf{x} \quad \textit{and} \quad K := \frac{\bar{W}^2 K_p}{\gamma\mu}.$$

Proof :
Applying Gronwall's inequality to (4.21) we find

$$\eta^\alpha(t) \le \eta^\alpha(0)e^{Kt} + \gamma\sqrt{\frac{2}{\alpha}}\int_0^t e^{K(s-t)}|\mathbf{F}(s,\cdot)|_{H^3(\mathbb{T}^d)}|\tilde{\mathbf{v}}^\alpha(s,\cdot)|_{H^1(\mathbb{T}^d)}\,\mathrm{d}\,s.$$

Using Proposition 4.8 and Young's inequality we find

$$\eta^\alpha(t) \le \frac{\gamma}{\alpha}\|\bar{\mathbf{F}}\|^2_{H^3(\mathbb{T}^d)}e^{Kt} + \gamma\sqrt{\frac{2}{\alpha}}e^{Kt}\Big(\|\mathbf{F}\|^2_{L^2(0,T;H^3(\mathbb{T}^d))} + |\tilde{\mathbf{v}}^\alpha|^2_{L^2(0,T;H^1_s(\mathbb{T}^d))}\Big). \tag{4.25}$$

This completes the proof as

$$\mu|\tilde{\mathbf{v}}^\alpha|^2_{L^2(0,T;H^1_s(\mathbb{T}^d))} \le E_0 + \eta^\alpha(0) \le E_0 + \frac{\gamma}{\alpha}\|\bar{\mathbf{F}}\|^2_{H^3(\mathbb{T}^d)}$$

by equation (4.6) analogous to Proposition 3.11.

■

Remark 4.10 (Higher order convergence) *If we assumed $\|\tilde{\mathbf{v}}^\alpha\|_{L^2(0,T;H^2(\mathbb{T}^d))}$ was bounded uniformly in α we would obtain a higher order convergence result analogous to Theorem 3.16.*

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Appendix

Proof of Lemma 4.3:
We begin with a direct computation

$$
\begin{aligned}
A := \int_{\mathbb{T}^d} & \partial_t \Big(W(\tilde{\mathbf{F}}^\alpha) + \frac{\alpha\gamma}{2}(\tilde{F}^\alpha_{ij} - \tilde{C}^\alpha_{ij})^2 + \frac{\gamma}{2}(\partial_k \tilde{C}^\alpha_{ij})^2 + \frac{1}{2}|\tilde{\mathbf{v}}^\alpha|^2 - W(\mathbf{F}) - \frac{\gamma}{2}(\partial_k F_{ij})^2 - \frac{1}{2}|\mathbf{v}|^2 \\
& \quad - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})(\tilde{F}^\alpha_{ij} - F_{ij}) - \gamma \partial_k F_{ij} \partial_k (\tilde{C}^\alpha_{ij} - F_{ij}) - v_i(\tilde{v}^\alpha_i - v_i) \Big) \\
& + \partial_j \Big(- \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\tilde{v}^\alpha_i + \alpha\gamma(\tilde{C}^\alpha_{ij} - \tilde{F}^\alpha_{ij})\tilde{v}^\alpha_i + \frac{\partial W}{\partial F_{ij}}(\mathbf{F})(\tilde{v}^\alpha_i - v_i) + \gamma \partial_k F_{ij} \partial_k (\tilde{v}^\alpha_i - v_i) \\
& \quad + \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha) v_i - \gamma \partial_{kk} \tilde{C}^\alpha_{ij} v_i \Big) \,\mathrm{d}\,\mathbf{x} \\
= \int_{\mathbb{T}^d} & \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_t \tilde{F}^\alpha_{ij} - \gamma \partial_{kk}\tilde{C}^\alpha_{ij}(\partial_t \tilde{F}^\alpha_{ij} - \partial_t \tilde{C}^\alpha_{ij}) + \gamma \partial_k \tilde{C}^\alpha_{ij} \partial_k \partial_t \tilde{C}^\alpha_{ij} + \tilde{v}^\alpha_i \partial_t \tilde{v}^\alpha_i \\
& - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_t F_{ij} - \gamma \partial_k F_{ij} \partial_k \partial_t F_{ij} - v_i \partial_t v_i - \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(\mathbf{F}) \partial_t F_{kl} (\tilde{F}^\alpha_{ij} - F_{ij}) \\
& - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_t \tilde{F}^\alpha_{ij} + \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_t F_{ij} - \gamma \partial_t \partial_k F_{ij} \partial_k \tilde{C}^\alpha_{ij} + \gamma \partial_t \partial_k F_{ij} \partial_k F_{ij} \\
& - \gamma \partial_k F_{ij} \partial_t \partial_k \tilde{C}^\alpha_{ij} + \gamma \partial_k F_{ij} \partial_t \partial_k \tilde{F}^\alpha_{ij} - \gamma \partial_k F_{ij} \partial_t \partial_k \tilde{F}^\alpha_{ij} + \gamma \partial_k F_{ij} \partial_k \partial_t F_{ij} \\
& - \partial_t v_i \tilde{v}^\alpha_i + \partial_t v_i v_i - v_i \partial_t \tilde{v}^\alpha_i + v_i \partial_t v_i \\
& - \partial_j (\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha))\tilde{v}^\alpha_i - \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_j \tilde{v}^\alpha_i + \alpha\gamma \partial_j \tilde{C}^\alpha_{ij} \tilde{v}^\alpha_i - \alpha\gamma \partial_j \tilde{F}^\alpha_{ij} \tilde{v}^\alpha_i + \gamma \partial_{kk} \tilde{C}^\alpha_{ij} \partial_j \tilde{v}^\alpha_i \\
& + \partial_j (\frac{\partial W}{\partial F_{ij}}(\mathbf{F}))\tilde{v}^\alpha_i - \partial_j (\frac{\partial W}{\partial F_{ij}}(\mathbf{F})) v_i + \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j \tilde{v}^\alpha_i - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j v_i \\
& + \gamma \partial_{kj} F_{ij} \partial_k \tilde{v}^\alpha_i - \gamma \partial_{kj} F_{ij} \partial_k v_i + \gamma \partial_k F_{ij} \partial_{kj} \tilde{v}^\alpha_i - \gamma \partial_k F_{ij} \partial_{kj} v_i \\
& + \partial_j (\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)) v_i + \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_j v_i - \gamma \partial_{kkj} \tilde{C}^\alpha_{ij} v_i - \gamma \partial_{kk} \tilde{C}^\alpha_{ij} \partial_j v_i \,\mathrm{d}\,\mathbf{x}.
\end{aligned}
\tag{4.26}
$$

In (4.26) several terms cancel and we get

$$
\begin{aligned}
A = \int_{\mathbb{T}^d} & \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_j \tilde{v}_i^\alpha - \gamma \partial_{kk}\tilde{C}_{ij}^\alpha \partial_j \tilde{v}_i^\alpha + \gamma \partial_k(\partial_k \tilde{C}_{ij}^\alpha \partial_t \tilde{C}_{ij}^\alpha) + \tilde{v}_i^\alpha \partial_j(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)) \\
& - \gamma \tilde{v}_i^\alpha \partial_{jkk}\tilde{C}_{ij}^\alpha + \mu \tilde{v}_i^\alpha \Delta \tilde{v}_i^\alpha - \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(\mathbf{F})\partial_l v_k(\tilde{F}_{ij}^\alpha - F_{ij}) - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j \tilde{v}_i^\alpha \\
& - \gamma \partial_{kj} v_i \partial_k \tilde{C}_{ij}^\alpha + \gamma \partial_{kj} v_i \partial_k F_{ij} - \gamma \partial_k\big(\partial_k F_{ij}(\partial_t \tilde{C}_{ij}^\alpha - \partial_j \tilde{v}_i^\alpha)\big) + \gamma \partial_{kk} F_{ij} \partial_t(\tilde{C}_{ij}^\alpha - \tilde{F}_{ij}^\alpha) \\
& - \gamma \partial_k F_{ij} \partial_{kj} \tilde{v}_i^\alpha - \tilde{v}_i^\alpha \partial_j(\frac{\partial W}{\partial F_{ij}}(\mathbf{F})) + \gamma \tilde{v}_i^\alpha \partial_{jkk} F_{ij} - \mu \tilde{v}_i^\alpha \Delta v_i + v_i \partial_j(\frac{\partial W}{\partial F_{ij}}(\mathbf{F})) \\
& - \gamma v_i \partial_{jkk} F_{ij} + \mu v_i \Delta v_i - v_i \partial_j(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)) + \gamma v_i \partial_{jkk}\tilde{C}_{ij}^\alpha - \mu v_i \Delta \tilde{v}_i^\alpha \\
& - \partial_j(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha))\tilde{v}_i^\alpha - \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_j \tilde{v}_i^\alpha + \gamma \partial_{jkk}\tilde{C}_{ij}^\alpha \tilde{v}_i^\alpha + \gamma \partial_{kk}\tilde{C}_{ij}^\alpha \partial_j \tilde{v}_i^\alpha + \partial_j(\frac{\partial W}{\partial F_{ij}}(\mathbf{F}))\tilde{v}_i^\alpha \\
& - \partial_j(\frac{\partial W}{\partial F_{ij}}(\mathbf{F}))v_i + \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j \tilde{v}_i^\alpha - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j v_i + \gamma \partial_{kj} F_{ij} \partial_k \tilde{v}_i^\alpha - \gamma \partial_{kj} F_{ij} \partial_k v_i + \gamma \partial_k F_{ij} \partial_{kj} \tilde{v}_i^\alpha \\
& - \gamma \partial_k F_{ij} \partial_{kj} v_i + \partial_j(\frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha))v_i + \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_j v_i - \gamma \partial_{kkj}\tilde{C}_{ij}^\alpha v_i - \gamma \partial_{kk}\tilde{C}_{ij}^\alpha \partial_j v_i \,\mathrm{d}\mathbf{x}.
\end{aligned}
\tag{4.27}
$$

Once more several terms cancel and others may be combined to obtain conservative terms such that

$$
\begin{aligned}
A = \int_{\mathbb{T}^d} & \gamma \partial_k(\partial_k \tilde{C}_{ij}^\alpha \partial_t \tilde{C}_{ij}^\alpha) + \mu(\tilde{v}_i^\alpha - v_i)\Delta(\tilde{v}_i^\alpha - v_i) - \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(\mathbf{F})\partial_j v_i(\tilde{F}_{kl} - F_{kl}) - \gamma \partial_k(\partial_j v_i \partial_k \tilde{C}_{ij}^\alpha) \\
& - \gamma \partial_k\big(\partial_k F_{ij}(\partial_t \tilde{C}_{ij}^\alpha - \partial_j \tilde{v}_i^\alpha)\big) + \gamma \partial_{kk} F_{ij} \partial_t(\tilde{C}_{ij}^\alpha - \tilde{F}_{ij}^\alpha) + \gamma \partial_k(\tilde{v}_i^\alpha \partial_{jk} F_{ij}) - \partial_j(\partial_k \tilde{v}_i^\alpha \partial_k F_{ij}) \\
& - \gamma \partial_k(v_i \partial_{kj} F_{ij}) - \frac{\partial W}{\partial F_{ij}}(\mathbf{F})\partial_j v_i + \gamma \partial_j(\partial_k F_{ij} \partial_k \tilde{v}_i^\alpha) + \frac{\partial W}{\partial F_{ij}}(\tilde{\mathbf{F}}^\alpha)\partial_j v_i \,\mathrm{d}\mathbf{x}.
\end{aligned}
\tag{4.28}
$$

The assertion of the Proposition follows from (4.26) and (4.28), upon using Gauss' Theorem and the periodicity.

■

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