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L_2- and $S^r_{p,q}B$-discrepancy of (order 2) digital nets

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L_2- and $S^r_{p,q}B$-discrepancy of (order 2) digital nets

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Dick proved that all order 2 digital nets satisfy optimal upper bounds of the L_2-discrepancy. We give an alternative proof for this fact using Haar bases. Furthermore, we prove that all digital nets satisfy optimal upper bounds of the $S^r_{p,q}B$-discrepancy for a certain parameter range and enlarge that range for order 2 digitals nets. L_p-, $S^r_{p,q}F$- and S^r_pH-discrepancy is considered as well.

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1 Introduction and results

Let N be some positive integer and let $\mathcal{P}$ be a point set in the unit cube $[0,1)^d$ with N points. Then the discrepancy function $D_{\mathcal{P}}$ is defined as

$$D_{\mathcal{P}}(x) = \frac{1}{N} \sum_{z \in \mathcal{P}} \chi_{[0,x)}(z) - x_1 \cdot \ldots \cdot x_d. \tag{1}$$

for any $x = (x_1, \ldots, x_d) \in [0,1)^d$. By $\chi_{[0,x)}$ we mean the characteristic function of the interval $[0,x) = [0,x_1) \times \ldots \times [0,x_d)$, so the term $\sum_z \chi_{[0,x)}(z)$ is equal to $\#(\mathcal{P} \cap [0,x))$.

This means that $D_{\mathcal{P}}$ measures the deviation of the number of points of $\mathcal{P}$ in $[0,x)$ from the fair number of points $N|[0,x)| = N\,x_1 \cdot \ldots \cdot x_d$ which would be achieved by a (practically impossible) perfectly uniform distribution of the points of $\mathcal{P}$, normalized by the total number of points.

Usually one is interested in calculating the norm of the discrepancy function in some normed space of functions on $[0,1)^d$ to which the discrepancy function belongs. A well known result concerns $L_p([0,1)^d)$-spaces for $1 < p < \infty$. There exists a constant $c_{p,d} > 0$ such that for every positive integer N and all point sets $\mathcal{P}$ in $[0,1)^d$ with N points, we have

$$\left\| D_{\mathcal{P}} | L_p([0,1)^d) \right\| \geq c_{p,d} \, \frac{(\log N)^{(d-1)/2}}{N}. \tag{2}$$

It was proved by Roth [R54] for $p = 2$ and by Schmidt [S77] for arbitrary $1 < p < \infty$. The best value for $c_{2,d}$ can be found in [HM11]. Furthermore, there exists a constant $C_{p,d} > 0$ such that for every positive integer N, there exists a point set $\mathcal{P}$ in $[0,1)^d$ with N points such that

$$\left\| D_{\mathcal{P}} | L_p(\mathbb{Q}^d) \right\| \leq C_{p,d} \, \frac{(\log N)^{(d-1)/2}}{N}. \tag{3}$$

It was proved by Davenport [D56] for $p = 2, d = 2$, by Roth [R80] for $p = 2$ and arbitrary d and finally by Chen [C80] in the general case. The best value for $C_{2,d}$ can be found in [DP10] and [FPPS10].

There are results for $L_1([0,1)^d)$- and star ($L_\infty([0,1)^d)$-) discrepancy though there are still gaps between lower and upper bounds, see [H81], [S72], [BLV08]. As general references for studies of the discrepancy function we refer to the monographs [DP10], [NW10], [M99], [KN74] and surveys [B11], [Hi14], [M13c].

Roth's and Chen's original proofs of (3) were probabilistic. Explicit constructions of point sets with good L_p-discrepancy in arbitrary dimension have not been known for a long time. Chen and Skriganov [CS02] (see also [CS08] and [DP10]) gave constructions with optimal bound of the L_2-discrepancy and Skriganov [S06] later proved the L_p bound. The constructions of Chen and Skriganov were order 1 digital nets with large Hamming weight. Dick and Pillichshammer [DP14a] (see also [DP14b]) gave alternative constructions. Their constructions are order 3 digital nets. Dick [D14] proved then the following result.

Theorem 1.1. *There exists a constant $C_{d,b,v} > 0$ such that for every positive integer n*

and every order 2 digital (v,n,d)-net $\mathcal{P}_n^b$ in base b we have

$$\left\| D_{\mathcal{P}_n^b} | L_2([0,1)^d) \right\| \leq C_{d,b,v} \, \frac{n^{(d-1)/2}}{b^n}.$$

In this work we give an alternative proof for this fact.

Furthermore, there are results for the discrepancy in other function spaces, like Hardy spaces, logarithmic and exponential Orlicz spaces, weighted L_p-spaces, BMO (see [B11] for results and further literature).

Here, we are interested in Besov ($S^r_{p,q}B([0,1)^d)$), Triebel-Lizorkin ($S^r_{p,q}F([0,1)^d)$) and Sobolev ($S^r_pH([0,1)^d)$) spaces with dominating mixed smoothness. Triebel [T10] proved that for all $1 \leq p,q \leq \infty$ with $q < \infty$ if $p = \infty$ and all $r \in \mathbb{R}$ satisfying $1/p - 1 < r < 1/p$, there exists a constant $c_{p,q,r,d} > 0$ such that for every integer $N \geq 2$ and all point sets $\mathcal{P}$ in $[0,1)^d$ with N points, we have

$$\left\| D_{\mathcal{P}} | S^r_{p,q}B([0,1)^d) \right\| \geq c_{p,q,r,d} \, N^{r-1} \, (\log N)^{(d-1)/q} \tag{4}$$

and with the additional condition that $q > 1$ if $p = \infty$ there exists a constant $C_{p,q,r,d} > 0$ such that for every positive integer N, there exists a point set $\mathcal{P}$ in $[0,1)^d$ with N points and we have

$$\left\| D_{\mathcal{P}} | S^r_{p,q}B([0,1)^d) \right\| \leq C_{p,q,r,d} \, N^{r-1} \, (\log N)^{(d-1)(1/q+1-r)} .$$

Hinrichs [Hi10] proved for $d = 2$ that for all $1 \leq p,q \leq \infty$ and all $0 \leq r < 1/p$ there exists a constant $C_{p,q,r} > 0$ such that for every integer $N \geq 2$ there exists a point set $\mathcal{P}$ in $[0,1)^2$ with N points such that

$$\left\| D_{\mathcal{P}} | S^r_{p,q}B([0,1)^2) \right\| \leq C_{p,q,r} \, N^{r-1} \, (\log N)^{1/q} .$$

Markhasin [M13b] proved that for all $1 \leq p,q \leq \infty$ and all $0 < r < 1/p$ there exists a constant $C_{p,q,r,d} > 0$ such that for every integer $N \geq 2$ there exists a point set $\mathcal{P}$ in $[0,1)^d$ with N points such that

$$\left\| D_{\mathcal{P}} | S^r_{p,q}B([0,1)^d) \right\| \leq C_{p,q,r,d} \, N^{r-1} \, (\log N)^{(d-1)/q} . \tag{5}$$

Explicit point sets with optimal bounds of $S^r_{p,q}B$-discrepancy used in [M13b] are the already mentioned point sets by Chen and Skriganov. In $d = 2$ also (generalized) Hammersley point sets can be used (see [Hi10], [M13a]). Our goal is to prove that there are way more point sets with optimal bounds of the $S^r_{p,q}B$-discrepancy. Furthermore there are results for spaces $S^r_{p,q}F([0,1)^d)$ and $S^r_pH([0,1)^d)$ in [M13c].

Theorem 1.2. *Let $1 \le p < \infty$, $1 \le q \le \infty$ and $0 < r < 1/p$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every integer n and every order 1 digital (v,n,d)-net $\mathcal{P}^b_n$ in base b we have*

$$\left\| D_{\mathcal{P}^b_n} | S^r_{p,q}B([0,1)^d) \right\| \le C_{p,q,r,d,b,v}\, b^{n(r-1)}\, n^{(d-1)/q}.$$

Theorem 1.3. *Let $1 \le p, q \le \infty$, ($q > 1$ if $p = \infty$) and $0 \le r < 1/p$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer n and every order 2 digital (v,n,d)-net $\mathcal{P}^b_n$ in base b we have*

$$\left\| D_{\mathcal{P}^b_n} | S^r_{p,q}B([0,1)^d) \right\| \le C_{p,q,r,d,b,v}\, b^{n(r-1)}\, n^{(d-1)/q}.$$

Corollary 1.4. *Let $1 \le p, q < \infty$ and $0 < r < 1/\max(p,q)$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer n and every order 1 digital (v,n,d)-net $\mathcal{P}^b_n$ in base b we have*

$$\left\| D_{\mathcal{P}^b_n} | S^r_{p,q}F([0,1)^d) \right\| \le C_{p,q,r,d,b,v}\, b^{n(r-1)}\, n^{(d-1)/q}.$$

Corollary 1.5. *Let $1 \le p, q < \infty$ and $0 \le r < 1/\max(p,q)$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer n and every order 2 digital (v,n,d)-net $\mathcal{P}^b_n$ in base b we have*

$$\left\| D_{\mathcal{P}^b_n} | S^r_{p,q}F([0,1)^d) \right\| \le C_{p,q,r,d,b,v}\, b^{n(r-1)}\, n^{(d-1)/q}.$$

Corollary 1.6. *Let $1 \le p < \infty$ and $0 < r < 1/\max(p,2)$. There exists a constant $C_{p,r,d,b,v} > 0$ such that for every positive integer n and every order 1 digital (v,n,d)-net $\mathcal{P}^b_n$ in base b we have*

$$\left\| D_{\mathcal{P}^b_n} | S^r_pH([0,1)^d) \right\| \le C_{p,r,d,b,v}\, b^{n(r-1)}\, n^{(d-1)/2}.$$

Corollary 1.7. *Let $1 \le p < \infty$ and $0 \le r < 1/\max(p,2)$. There exists a constant $C_{p,r,d,b,v} > 0$ such that for every positive integer n and every order 2 digital (v,n,d)-net $\mathcal{P}^b_n$ in base b we have*

$$\left\| D_{\mathcal{P}^b_n} | S^r_pH([0,1)^d) \right\| \le C_{p,r,d,b,v}\, b^{n(r-1)}\, n^{(d-1)/2}.$$

Theorem 1.8. *Let $1 \le p < \infty$. There exists a constant $C_{p,d,b,v} > 0$ such that for every* *positive integer n and every order 2 digital (v,n,d)-net $\mathcal{P}_n^b$ in base b we have*

$$\left\| D_{\mathcal{P}_n^b} | L_p([0,1)^d) \right\| \leq C_{p,d,b,v} \, \frac{n^{(d-1)/2}}{b^n}.$$

We point out that obviously Theorem 1.1 is a consequence of Theorem 1.8. Nevertheless, we will prove them independently, so that readers without a background in function spaces with dominating mixed smoothness (which is required for the proof of Theorem 1.8) will be able to understand the proof of the L_2 bound.

Theorems 1.2 and1.3 are consistent with older results. Chen-Skriganov point sets are order 1 digital (v,n,d)-nets while (generalized) Hammersley point sets are order 2 digital $(0,n,2)$-nets

2 Function spaces with dominating mixed smoothness

We define the spaces $S^r_{p,q}B([0,1)^d)$, $S^r_{p,q}F([0,1)^d)$ and $S^r_pH([0,1)^d)$ according to [T10]. Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions on $\mathbb{R}^d$. Let $\varphi_0 \in \mathcal{S}(\mathbb{R})$ satisfy $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\varphi_0(x) = 0$ for $|x| > \frac{3}{2}$. Let $\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x)$ where $x \in \mathbb{R}$, $k \in \mathbb{N}$ and $\varphi_k(x) = \varphi_{k_1}(x_1)\dots\varphi_{k_d}(x_d)$ where $k = (k_1,\dots,k_d) \in \mathbb{N}_0^d$, $x = (x_1,\dots,x_d) \in \mathbb{R}^d$. The functions φ_k are a dyadic resolution of unity since

$$\sum_{k\in\mathbb{N}_0^d} \varphi_k(x) = 1$$

for all $x \in \mathbb{R}^d$. The functions $\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)$ are entire analytic functions for every $f \in \mathcal{S}'(\mathbb{R}^d)$.

Let $0 < p,q \leq \infty$ and $r \in \mathbb{R}$. The Besov space with dominating mixed smoothness $S^r_{pq}B(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with finite quasi-norm

$$\left\| f | S^r_{pq}B(\mathbb{R}^d) \right\| = \left(\sum_{k\in\mathbb{N}_0^d} 2^{r(k_1+\ldots+k_d)q} \left\| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f) | L_p(\mathbb{R}^d) \right\|^q \right)^{\frac{1}{q}} \tag{6}$$

with the usual modification if $q = \infty$.

Let $0 < p < \infty$, $0 < q \leq \infty$ and $r \in \mathbb{R}$. The Triebel-Lizorkin space with dominating mixed smoothness $S^r_{pq}F(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with finite quasi-norm

$$\left\| f|S^r_{pq}F(\mathbb{R}^d) \right\| = \left\| \left(\sum_{k\in\mathbb{N}_0^d} 2^{r(k_1+\ldots+k_d)q} \left| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f)(\cdot) \right|^q \right)^{\frac{1}{q}} |L_p(\mathbb{R}^d) \right\| \tag{7}$$

with the usual modification if $q = \infty$.

Let $\mathcal{D}([0,1)^d)$ consist of all complex-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support in the interior of $[0,1)^d$ and let $\mathcal{D}'([0,1)^d)$ be its dual space of all distributions in $[0,1)^d$. The Besov space with dominating mixed smoothness $S^r_{pq}B([0,1)^d)$ consists of all $f \in \mathcal{D}'([0,1)^d)$ with finite quasi-norm

$$\left\| f|S^r_{pq}B([0,1)^d) \right\| = \inf \left\{ \left\| g|S^r_{pq}B(\mathbb{R}^d) \right\| : \ g \in S^r_{pq}B(\mathbb{R}^d),\ g|_{[0,1)^d} = f \right\}. \tag{8}$$

The Triebel-Lizorkin space with dominating mixed smoothness $S^r_{pq}F([0,1)^d)$ consists of all $f \in \mathcal{D}'([0,1)^d)$ with finite quasi-norm

$$\left\| f|S^r_{pq}F([0,1)^d) \right\| = \inf \left\{ \left\| g|S^r_{pq}F(\mathbb{R}^d) \right\| : \ g \in S^r_{pq}F(\mathbb{R}^d),\ g|_{[0,1)^d} = f \right\}. \tag{9}$$

The spaces $S^r_{pq}B(\mathbb{R}^d)$, $S^r_{pq}F(\mathbb{R}^d)$, $S^r_{pq}B([0,1)^d)$ and $S^r_{pq}F([0,1)^d)$ are quasi-Banach spaces. We define the Sobolev space with dominating mixed smoothness as

$$S^r_p H([0,1)^d) = S^r_{p2}F([0,1)^d). \tag{10}$$

If $r \in \mathbb{N}_0$ then it is denoted by $S^r_p W([0,1)^d)$ and is called classical Sobolev space with dominating mixed smoothness. An equivalent norm for $S^r_p W([0,1)^d)$ is

$$\sum_{\alpha\in\mathbb{N}_0^d;\, 0\le\alpha_i\le r} \left\| D^\alpha f | L_p([0,1)^d) \right\|.$$

Of special interest is the case $r = 0$ since

$$S^0_p H([0,1)^d) = L_p([0,1)^d).$$

The Besov and Triebel-Lizorkin spaces can be embedded in each other (see [T10] or [M13c, Corollary 1.13]). We point out that the following embedding is a combination of well known results and might look odd at the first glance.

Lemma 2.1. *Let* $0 < p, q < \infty$ *and* $r \in \mathbb{R}$. *Then we have*

$$S^r_{\max(p,q),q}B([0,1)^d) \hookrightarrow S^r_{pq}F([0,1)^d) \hookrightarrow S^r_{\min(p,q),q}B([0,1)^d).$$

3 Haar and Walsh bases

We denote $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. Let $b \geq 2$ be an integer. We denote $\mathbb{D}_j = \{0, 1, \ldots, b^j - 1\}$ and $\mathbb{B}_j = \{1, \ldots, b-1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$ and $\mathbb{B}_{-1} = \{1\}$. For $j = (j_1, \ldots, j_d) \in \mathbb{N}^d_{-1}$ let $\mathbb{D}_j = \mathbb{D}_{j_1} \times \ldots \times \mathbb{D}_{j_d}$ and $\mathbb{B}_j = \mathbb{B}_{j_1} \times \ldots \times \mathbb{B}_{j_d}$. For a real a we write $a_+ = \max(a, 0)$ and for $j \in \mathbb{N}^d_{-1}$ we write $|j|_+ = j_{1+} + \ldots + j_{d+}$.

For $j \in \mathbb{N}_0$ and $m \in \mathbb{D}_j$ we call the interval

$$I_{j,m} = \big[b^{-j}m, b^{-j}(m+1)\big)$$

the m-th b-adic interval in $[0,1)$ on level j. We put $I_{-1,0} = [0,1)$ and call it the 0-th b-adic interval in $[0,1)$ on level -1. For any $k = 0, \ldots, b-1$ let $I^k_{j,m} = I_{j+1,bm+k}$. We put $I^{-1}_{-1,0} = I_{-1,0} = [0,1)$. For $j \in \mathbb{N}^d_{-1}$ and $m = (m_1, \ldots, m_d) \in \mathbb{D}_j$ we call

$$I_{j,m} = I_{j_1,m_1} \times \ldots \times I_{j_d,m_d}$$

the m-th b-adic interval in $[0,1)^d$ on level j. We call the number $|j|_+$ the order of the b-adic interval $I_{j,m}$. Its volume is $b^{-|j|_+}$.

Let $j \in \mathbb{N}_0$, $m \in \mathbb{D}_j$ and $l \in \mathbb{B}_j$. Let $h_{j,m,l}$ be the function on $[0,1)$ with support in $I_{j,m}$ and the constant value $\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}lk}$ on $I^k_{j,m}$ for any $k = 0, \ldots, b-1$. We put $h_{-1,0,1} = \chi_{I_{-1,0}}$ on $[0,1)$.

Let $j \in \mathbb{N}^d_{-1}$, $m \in \mathbb{D}_j$ and $l = (l_1, \ldots, l_d) \in \mathbb{B}_j$. The function $h_{j,m,l}$ given as the tensor product

$$h_{j,m,l}(x) = h_{j_1,m_1,l_1}(x_1) \ldots h_{j_d,m_d,l_d}(x_d)$$

for $x = (x_1, \ldots, x_d) \in [0,1)^d$ is called a b-adic Haar function on $[0,1)^d$. The functions $h_{j,m,l}$, $j \in \mathbb{N}^d_{-1}$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$ are called b-adic Haar basis on $[0,1)^d$.

The following result is [M13c, Theorem 2.1].

Theorem 3.1. *The system*

$$\left\{ b^{\frac{|j|_+}{2}} h_{j,m,l} \, : \, j \in \mathbb{N}^d_{-1}, \, m \in \mathbb{D}_j, \, l \in \mathbb{B}_j \right\}$$

is an orthonormal basis of $L_2([0,1)^d)$, *an unconditional basis of* $L_p([0,1)^d)$ *for* $1 < p < \infty$

and a conditional basis of $L_1([0,1)^d)$. *For any function* $f \in L_2([0,1)^d)$ *we have*

$$\left\| f|L_2([0,1)^d)\right\|^2 = \sum_{j\in\mathbb{N}_{-1}^d} b^{|j|} \sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j} |\langle f, h_{j,m,l}\rangle|^2.$$

The following result is [M13c, Theorem 2.11].

Theorem 3.2. *Let* $0 < p, q \le \infty$, *(*$q > 1$ *if* $p = \infty$*) and* $1/p - 1 < r < \min(1/p, 1)$. *Let* $f \in \mathcal{D}'([0,1)^d)$. *Then* $f \in S^r_{pq}B([0,1)^d)$ *if and only if it can be represented as*

$$f = \sum_{j\in\mathbb{N}_{-1}^d} b^{|j|_+} \sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j} \mu_{j,m,l}\, h_{j,m,l} \tag{11}$$

for some sequence $(\mu_{j,m,l})$ *satisfying*

$$\left(\sum_{j\in\mathbb{N}_{-1}^d} b^{|j|_+(r-1/p+1)q} \left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j} |\mu_{j,m,l}|^p \right)^{q/p} \right)^{1/q} < \infty. \tag{12}$$

The convergence of (11) *is unconditional in* $\mathcal{D}'([0,1)^d)$ *and in any* $S^\rho_{pq}B([0,1)^d)$ *with* $\rho < r$. *The representation* (11) *of* f *is unique with the b-adic Haar coefficients* $\mu_{j,m,l} = \langle f, h_{j,m,l}\rangle$. *The expression* (12) *is an equivalent quasi-norm on* $S^r_{pq}B([0,1)^d)$.

For $\alpha \in \mathbb{N}$ with the b-adic expansion $\alpha = \beta_{a_1-1}b^{a_1-1} + \ldots + \beta_{a_\nu-1}b^{a_\nu-1}$ with $0 < a_1 < a_2 < \ldots < a_\nu$ and digits $\beta_{a_1-1}, \ldots, \beta_{a_\nu-1} \in \{1, \ldots, b-1\}$, the NRT weight of order $\sigma \in \mathbb{N}$ is given by

$$\varrho_\sigma(\alpha) = a_\nu + a_{\nu-1} + \ldots + a_{\max(\nu-\sigma+1,1)}.$$

Furthermore, $\varrho_\sigma(0) = 0$.

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, the NRT weight of order σ is given by

$$\varrho_\sigma(\alpha) = \varrho_\sigma(\alpha_1) + \ldots + \varrho_\sigma(\alpha_d).$$

Let $\alpha \in \mathbb{N}$. The α-th b-adic Walsh function $\mathrm{wal}_\alpha : [0,1) \to \mathbb{C}$ is given by

$$\mathrm{wal}_\alpha(x) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\left(\beta_{a_1-1}x_{a_1} + \ldots + \beta_{a_\nu-1}x_\nu\right)}$$

for $x \in [0,1)$ with b-adic expansion $x = x_1b^{-1} + x_2b^{-2} + \ldots$. Furthermore, $\mathrm{wal}_0 = \chi_{[0,1)}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Then the α-th b-adic Walsh function wal_α on $[0,1)^d$ is given as the tensor product

$$\mathrm{wal}_\alpha(x) = \mathrm{wal}_{\alpha_1}(x^1) \ldots \mathrm{wal}_{\alpha_d}(x^d)$$

for $x = (x^1, \ldots, x^d) \in [0,1)^d$. The functions wal_α, $\alpha \in \mathbb{N}_0^d$ are called b-adic Walsh basis on $[0,1)^d$.

The b-adic Walsh function wal_α is constant on b-adic intervals $I_{(\varrho_1(\alpha_1),\ldots,(\varrho_1(\alpha_d)),m}$ for every $m \in \mathbb{D}_{(\varrho_1(\alpha_1),\ldots,(\varrho_1(\alpha_d))}$. The following result is [DP10, Theorem A.11].

Lemma 3.3. *The system*

$$\left\{\mathrm{wal}_\alpha \,:\, \alpha \in \mathbb{N}_0^d\right\}$$

is an orthonormal basis of $L_2([0,1)^d)$.

4 Digital (v,n,d)-nets

We quote from [DP14a] and [D07] to describe the digital construction method and properties of resulting digital nets. We also refer to [N87] and [NP01].

For an integer $b \geq 2$ let $\mathbb{Z}_b$ denote the commutative ring of integers modulo b. For $s, n \in \mathbb{N}$ with $s \geq n$ let $C_1, \ldots, C_d$ be $s \times n$ matrices with entries from $\mathbb{Z}_b$. For $\nu \in \{0, 1, \ldots, b^n - 1\}$ with the b-adic expansion $\nu = \nu_0 + \nu_1 b + \ldots + \nu_{n-1} b^{n-1}$ with digits $\nu_0, \nu_1, \ldots, \nu_{n-1} \in \{0, 1, \ldots, b-1\}$ the b-adic digit vector $\bar{\nu}$ is given as $\bar{\nu} = (\nu_0, \nu_1, \ldots, \nu_{n-1})^\top \in \mathbb{Z}_b^n$. Then we compute $C_i \bar{\nu} = (x_{i,\nu,1}, x_{i,\nu,2}, \ldots, x_{i,\nu,s})^\top \in \mathbb{Z}_b^s$ for $1 \leq i \leq d$. Finally we define

$$x_{i,\nu} = x_{i,\nu,1} b^{-1} + x_{i,\nu,2} b^{-2} + \ldots + x_{i,\nu,s} b^{-s} \in [0,1)$$

and $x_\nu = (x_{1,\nu}, \ldots, x_{d,\nu})$. We call the point set $\mathcal{P}_n^b = \{x_0, x_1, \ldots, x_{b^n-1}\}$ a digital net in base b.

Now let $\sigma \in \mathbb{N}$ and suppose $s \geq \sigma n$. Let $0 \leq v \leq \sigma n$ be an integer. For every $1 \leq i \leq d$ we write $C_i = (c_{i,1}, \ldots, c_{i,s})^\top$ where $c_{i,1}, \ldots, c_{i,s} \in \mathcal{P}_b^n$ are the row vectors of C_i. If for all $1 \leq \lambda_{i,1} < \ldots < \lambda_{i,\eta_i} \leq s$, $1 \leq i \leq d$ with

$$\lambda_{1,1} + \ldots + \lambda_{1,\min(\eta_1,\sigma)} + \ldots + \lambda_{d,1} + \ldots + \lambda_{d,\min(\eta_d,\sigma)} \leq \sigma n - v$$

the vectors $c_{1,\lambda_{1,1}}, \ldots, c_{1,\lambda_{1,\eta_1}}, \ldots, c_{d,\lambda_{d,1}}, \ldots, c_{d,\lambda_{d,\eta_d}}$ are linearly independet over $\mathbb{Z}_b$, then $\mathcal{P}_n^b$ is called an order σ digital (v,n,d)-net in base b.

Lemma 4.1.

(i) *Let $v < \sigma n$. Then every order σ digital (v, n, d)-net in base b is an order σ digital $(v + 1, n, d)$-net in base b. In particular every point set $\mathcal{P}_n^b$ constructed with the digital method is at least an order σ digital $(\sigma n, n, d)$-net in base b.*

(ii) *Let $1 \leq \sigma_1 \leq \sigma_2$. Then every order σ_2 digital (v, n, d)-net in base b is an oder σ_1 digital $(\lceil v\sigma_1/\sigma_2 \rceil, n, d)$-net in base b.*

Lemma 4.2. *Let $\mathcal{P}_n^b$ be an order σ digital (v, n, d)-net in base b then every b-adic interval of order $n - v$ contains exactly b^v points of $\mathcal{P}_n^b$.*

Let $t \in \mathbb{N}_0$ with b-adic expansion $t = \tau_0 + \tau_1 b + \tau_2 b^2 + \ldots$. We put $\bar{t} = (\tau_0, \tau_1, \ldots, \tau_{s-1})^\top \in \mathbb{Z}_b^s$ and define

$$\mathfrak{D}(\mathfrak{C}) = \left\{ t = (t_1, \ldots, t_d) \in \mathbb{N}_0^d \setminus \{(0, \ldots, 0)\} : C_1^\top \bar{t}_1 + \ldots + C_d^\top \bar{t}_d = (0, \ldots, 0) \in \mathbb{Z}_b^n \right\}.$$

Lemma 4.3. *$\mathcal{P}_n^b$ is an order σ digital (v, n, d)-net in base b if and only if $\varrho_\sigma(t) > \sigma n - v$ for all $t \in \mathfrak{D}(\mathfrak{C})$.*

Lemma 4.4. *Let $\mathcal{P}_n^b$ be an order σ digital (v, n, d)-net in base b with generating matrices $C_1, \ldots, C_d$. Then*

$$\sum_{z \in \mathcal{P}_n^b} \mathrm{wal}_t(z) = \begin{cases} b^n & \text{if } t \in \mathfrak{D}(\mathfrak{C}), \\ 0 & \text{otherwise.} \end{cases}$$

We consider the Walsh series expansion of the function $\chi_{[0,x)}$,

$$\chi_{[0,x)}(y) = \sum_{t=0}^{\infty} \hat{\chi}_{[0,x)}(t) \, \mathrm{wal}_t(y), \tag{13}$$

where for $t \in \mathbb{N}_0$ the t-th Walsh coefficient is given by

$$\hat{\chi}_{[0,x)}(t) = \int_0^1 \chi_{[0,x)}(y) \overline{\mathrm{wal}_t(y)} \mathrm{d}y = \int_0^x \overline{\mathrm{wal}_t(y)} \mathrm{d}y.$$

Lemma 4.5. *Let $\mathcal{P}_n^b$ be an order σ digital (v, n, d)-net in base b with generating matrices $C_1, \ldots, C_d$. Then*

$$D_{\mathcal{P}_n^b}(x) = \sum_{t \in \mathfrak{D}(\mathfrak{C})} \hat{\chi}_{[0,x)}(t).$$

Proof. For $t = (t_1, \ldots, t_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in [0, 1)^d$, we have

$$\hat{\chi}_{[0,x)}(t) = \hat{\chi}_{[0,x_1)}(t_1) \cdot \ldots \cdot \hat{\chi}_{[0,x_d)}(t_d).$$

Applying Lemma 4.4 we get

$$\begin{aligned} D_{\mathcal{P}}(x) &= \frac{1}{b^n} \sum_{z\in\mathcal{P}_n^b} \sum_{t_1,\ldots,t_d=0}^{\infty} \hat{\chi}_{[0,x)}(t)\,\mathrm{wal}_t(z) - \hat{\chi}_{[0,x)}((0,\ldots,0)) \\ &= \sum_{\substack{t_1,\ldots,t_d=0 \\ (t_1,\ldots t_d)\neq(0,\ldots,0)}}^{\infty} \hat{\chi}_{[0,x)}(t) \frac{1}{b^n} \sum_{z\in\mathcal{P}} \mathrm{wal}_t(z) \\ &= \sum_{t\in\mathfrak{D}(\mathfrak{C})} \hat{\chi}_{[0,x)}(t). \end{aligned}$$

□

Several constructions of order σ digital (v,n,d)-nets are known. For details, examples and further literature we refer to [DP14b]. There are especially constructions with a good quality parameter v, e. g. we can construct order 2 digital (d,n,d)-nets in base b as well as order 1 digital $(0,n,d)$-nets.

5 Proofs of the results

For two sequences a_n and b_n we will write $a_n \preceq b_n$ if there exists a constant $c>0$ such that $a_n \leq c\,b_n$ for all n. For $t>0$ with b-adic expansion $t=\tau_0+\tau_1 b+\ldots+\tau_{\varrho_1(t)-1}b^{\varrho_1(t)-1}$, we put $t=t'+\tau_{\varrho_1(t)-1}b^{\varrho_1(t)-1}$.

The following result is [M13b, Lemma 5.1].

Lemma 5.1. *Let $f(x)=x_1\cdot\ldots\cdot x_d$ for $x=(x_1,\ldots,x_d)\in[0,1)^d$. Let $j\in\mathbb{N}_{-1}^d$, $m\in\mathbb{D}_j, l\in\mathbb{B}_j$. Then $|\langle f,h_{j,m,l}\rangle|\preceq b^{-2|j|_+}$.*

The following result is [M13b, Lemma 5.2].

Lemma 5.2. *Let $z=(z_1,\ldots,z_d)\in[0,1)^d$ and $g(x)=\chi_{[0,x)}(z)$ for $x=(x_1,\ldots,x_d)\in[0,1)^d$. Let $j\in\mathbb{N}_{-1}^d$, $m\in\mathbb{D}_j, l\in\mathbb{B}_j$. Then $\langle g,h_{j,m,l}\rangle=0$ if z is not contained in the interior of the b-adic interval $I_{j,m}$. If z is contained in the interior of $I_{j,m}$ then $|\langle g,h_{j,m,l}\rangle|\preceq b^{-|j|_+}$.*

The following result is [M13b, Lemma 5.9].

Lemma 5.3. *Let $j\in\mathbb{N}_{-1}^d$, $m\in\mathbb{D}_j$, $l\in\mathbb{B}_j$ and $\alpha\in\mathbb{N}_0^d$. Then*

$$|\langle h_{j,m,l},\mathrm{wal}_\alpha\rangle|\preceq b^{-|j|_+}.$$

If $\varrho_1(\alpha_i) \neq j_i + 1$ *for some* $1 \leq i \leq d$ *then*

$$\langle h_{j,m,l}, \mathrm{wal}_\alpha \rangle = 0.$$

The following result is [M13b, Lemma 5.10].

Lemma 5.4. *Let* $t, \alpha \in \mathbb{N}_0$*. Then*

$$|\langle \hat{\chi}_{[0,\cdot)}(t), \mathrm{wal}_\alpha \rangle| \preceq b^{-\max(\varrho_1(t), \varrho_1(\alpha))}.$$

If $\alpha \neq t'$ *and* $\alpha \neq t$ *and* $\alpha' \neq t$ *then*

$$\langle \hat{\chi}_{[0,\cdot)}(t), \mathrm{wal}_\alpha \rangle = 0.$$

Lemma 5.5. *Let* $C_1, \ldots, C_d \in \mathbb{Z}_b^{s\times n}$ *generate an order* 1 *digital* (v, n, d)*-net in base* b*. Let* $\lambda_1, \ldots, \lambda_d, \gamma_1, \ldots, \gamma_d \in \mathbb{N}_0$*. Let* $\omega_{\gamma_1,\ldots,\gamma_d}^{\lambda_1,\ldots,\lambda_d}(\mathfrak{C})$ *denote the cardinality of such* $t \in \mathfrak{D}(\mathfrak{C})$ *with* $\varrho_1(t_i) = \gamma_i$ *for all* $1 \leq i \leq d$ *that either* $\gamma_i \leq \lambda_i$ *or* $\varrho_1(t_i') = \lambda_i$*. If* $\lambda_1, \ldots, \lambda_d \leq s$ *then*

$$\omega_{\gamma_1,\ldots,\gamma_d}^{\lambda_1,\ldots,\lambda_d}(\mathfrak{C}) \leq (b-1)^d \, b^{(\min(\lambda_1,\gamma_1-1)+\ldots+\min(\lambda_d,\gamma_d-1)-n+v)_+}.$$

Proof. Let $t = (t_1, \ldots, t_d) \in \mathfrak{D}(\mathfrak{C})$ with $\varrho_1(t_i) = \gamma_i$ for all $1 \leq i \leq d$ and either $\gamma_i \leq \lambda_i$ or $\varrho_1(t_i') = \lambda_i$. Let t_i have b-adic expansion $t_i = \tau_{i,0} + \tau_{i,1} b + \tau_{i,2} b^2 + \ldots$. Let $C_i = (c_{i,1}, \ldots, c_{i,s})^\top$, put $\lambda_i^* = \min(\lambda_i, \gamma_i - 1)$ and $c_{i,\gamma_i} = (0, \ldots, 0)$ if $\gamma_i > s$, $1 \leq i \leq d$. Then we have

$$\begin{aligned}
&c_{1,1}^\top \tau_{1,0} + \ldots + c_{1,\lambda_1^*}^\top \tau_{1,\lambda_1^*-1} + c_{1,\gamma_1}^\top \tau_{1,\gamma_1-1} + \\
&\vdots \\
+&c_{d,1}^\top \tau_{d,0} + \ldots + c_{d,\lambda_d^*}^\top \tau_{d,\lambda_d^*-1} + c_{d,\gamma_d}^\top \tau_{d,\gamma_d-1} = (0\ldots,0)^\top \in \mathbb{Z}_b^n.
\end{aligned} \tag{14}$$

We put

$$A = (c_{1,1}^\top, \ldots, c_{1,\lambda_1^*}^\top, \ldots, c_{d,1}^\top, \ldots, c_{d,\lambda_d^*}^\top) \in \mathbb{Z}_b^{n\times(\lambda_1^*+\ldots+\lambda_d^*)},$$

$$y = (\tau_{1,0}, \ldots, \tau_{1,\lambda_1^*-1}, \ldots, \tau_{d,0}, \ldots, \tau_{d,\lambda_d^*-1})^\top \in \mathbb{Z}_b^{(\lambda_1^*+\ldots+\lambda_d^*)\times 1}$$

and

$$w = -c_{1,\gamma_1}^\top \tau_{1,\gamma_1-1} - \ldots - c_{d,\gamma_d}^\top \tau_{d,\gamma_d-1} \in \mathbb{Z}_b^{n\times 1}.$$

Then (14) corresponds to $Ay = w$ and we have

$$\omega_{\gamma_1,\ldots,\gamma_d}^{\lambda_1,\ldots,\lambda_d}(\mathfrak{C}) = \#\{y \in \mathbb{Z}_b^{\lambda_1^*+\ldots+\lambda_d^*} : \, Ay = w\}.$$

Since $C_1, \dots, C_d$ generate an order 1 digital (v,n,d)-net, the rank of A is $\lambda_1^* + \dots + \lambda_d^*$ if $\lambda_1^* + \dots + \lambda_d^* \leq n - v$. In this case the solution space of the homogeneous system $Ay = (0,\dots,0)$ has dimension 0. If $\lambda_1^* + \dots + \lambda_d^* > n - v$ then $\operatorname{rank}(A) \geq n - v$ and the dimension of the solution space of the homogeneous system is $\lambda_1^* + \dots + \lambda_d^* - \operatorname{rank}(A) \leq \lambda_1 + \dots + \lambda_d - n + v$. This means that for a given w the system $Ay = w$ has at most 1 solution if $\lambda_1^* + \dots + \lambda_d^* \leq n - v$ and at most $b^{\lambda_1^* + \dots + \lambda_d^* - n + v}$ otherwise. Finally, there are $(b-1)^d$ possible choices for w since none of the numbers $\tau_{1,\gamma_1 - 1}, \dots, \tau_{d,\gamma_d - 1}$ can be 0. □

We point out that the condition $\lambda_1, \dots, \lambda_d \leq s$ is not necessary. It just reduces the technicalities but the results would be the same without it. One would have to define $\lambda_i^{**} = \min(\lambda_i *, s)$ and in the case where $\lambda_i * > s$ we would get an additional factor $b^{\lambda_i * - s}$ compensating the restriction.

Proposition 5.6. *Let $\mathcal{P}_n^b$ be an order 1 digital (v,n,d)-net in base b. Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$.*

(i) If $|j|_+ \geq n - v$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \preceq b^{-|j|_+ - n + v}$ and $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \preceq b^{-2|j|_+}$ for all but at most b^n values of m.

(ii) If $|j|_+ < n - v$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \preceq b^{-|j|_+ - n + v} \left(n - v - |j|_+\right)^{d-1}$.

Proof. For (i), let $|j|_+ \geq n - v$. Since $\mathcal{P}_n^b$ contains exactly b^n points, there are no more than b^n such m for which $I_{j,m}$ contains a point of $\mathcal{P}_n^b$ meaning that at least all but b^n intervals contain no points at all. Thus the second statement follows from Lemma 5.1. The remaining intervals contain at most b^v points of $\mathcal{P}_n^b$ (Lemma 4.2) so the first statement follows from Lemmas 5.1 and 5.2.

We now prove (ii) so let $|j|_+ < n - v$ and $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$. The function $h_{j,m,l}$ can be given (Lemma 3.3) as

$$h_{j,m,l} = \sum_{\alpha \in \mathbb{N}_0^d} \langle h_{j,m,l}, \mathrm{wal}_\alpha \rangle \, \mathrm{wal}_\alpha \, .$$

We apply Lemmas 4.5, 5.3 and 5.4 and get

$$\begin{aligned} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| &= \left| \left\langle \sum_{t \in \mathfrak{D}(\mathfrak{C})} \hat{\chi}_{[0,\cdot)}(t), \sum_{\alpha \in \mathbb{N}_0^d} \langle h_{jml}, \mathrm{wal}_\alpha \rangle \, \mathrm{wal}_\alpha \right\rangle \right| \\ &\leq \sum_{t \in \mathfrak{D}(\mathfrak{C})} \sum_{\alpha \in \mathbb{N}_0^d} \left| \left\langle \hat{\chi}_{[0,\cdot)}(t), \mathrm{wal}_\alpha \right\rangle \right| \left| \langle h_{jml}, \mathrm{wal}_\alpha \rangle \right| \end{aligned}$$

$$\leq b^{-|j|_+} \sum_{t\in\mathfrak{D}(\mathfrak{C})} \sum_{\substack{\alpha\in\mathbb{N}_0^d\\ \varrho_1(\alpha_i)=j_i+1\\ 1\leq i\leq d}} \left|\left\langle \hat{\chi}_{[0,\cdot)}(t), \mathrm{wal}_\alpha \right\rangle\right|$$

$$\leq b^{-|j|_+} \sum_{t\in\mathfrak{D}(\mathfrak{C})} \sum_{\substack{\alpha\in\mathbb{N}_0^d\\ \alpha_i=t_i' \vee \alpha=t_i \vee \alpha_i'=t_i\\ \varrho_1(\alpha_i)=j_i+1,\, 1\leq i\leq d}} b^{-\max(\varrho_1(\alpha_1),\varrho_1(t_1))-\ldots-\max(\varrho_1(\alpha_1),\varrho_1(t_d))}$$

$$= b^{-|j|_+} \sum_{\substack{t\in\mathfrak{D}(\mathfrak{C})\\ \varrho_1(t_i)\leq j_i+1 \vee \varrho_1(t_i')=j_i+1\\ 1\leq i\leq d}} b^{-\max(j_1+1,\varrho_1(t_1))-\ldots-\max(j_d+1,\varrho_1(t_d))}$$

$$= b^{-|j|_+} \sum_{\gamma_1,\ldots,\gamma_d=0}^{\infty} b^{-\max(j_1+1,\gamma_1)-\ldots-\max(j_d+1,\gamma_d)}\, \omega_{\gamma_1,\ldots,\gamma_d}^{j_1+1,\ldots,j_d+1}(\mathfrak{C}). \tag{15}$$

By Lemma 5.5 we get

$$\omega_{\gamma_1,\ldots,\gamma_d}^{j_1+1,\ldots,j_d+1}(\mathfrak{C}) \leq (b-1)^d\, b^d$$

since $j_1+1,\ldots,j_d+1 \leq n-v \leq s$ and $j_1+1+\ldots+j_d+1 \leq |j|_+ + d < n-v+d$.

We recall that we have $\varrho_1(t) > n-v$ for all $t \in \mathfrak{D}(\mathfrak{C})$. This means that $\omega_{\gamma_1,\ldots,\gamma_d}^{j_1+1,\ldots,j_d+1}(\mathfrak{C}) = 0$ whenever $\gamma_1+\ldots+\gamma_d \leq n-v$. Therefore $\omega_{\gamma_1,\ldots,\gamma_d}^{j_1+1,\ldots,j_d+1}(\mathfrak{C}) = 0$ if $\gamma_i \leq j_i$ for all $1 \leq i \leq d$. For any $I \subset \{1,\ldots,d\}$ let $I^c = \{1,\ldots,d\} \setminus I$. We perform an index shift to get

$$|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle| \preceq b^{-|j|_+} \sum_{\substack{\gamma_1,\ldots,\gamma_d=0\\ \gamma_1+\ldots+\gamma_d>n-v}}^{\infty} b^{-\max(j_1+1,\gamma_1)-\ldots-\max(j_d+1,\gamma_d)}$$

$$= b^{-|j|_+} \sum_{I\subsetneq\{1,\ldots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)} \sum_{\substack{0\leq\gamma_{i_1}\leq j_{i_1}\\ i_1\in I}} \sum_{\substack{\gamma_{i_2}\geq j_{i_2}+1\\ i_2\in I^c\\ \gamma_1+\ldots+\gamma_d\geq\max\left(n-v+1,\, \sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)\right)}} b^{-\sum_{\kappa_2\in I^c}\gamma_{\kappa_2}}$$

$$= b^{-|j|_+} \sum_{I\subsetneq\{1,\ldots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \cdots$$

$$\cdots \sum_{\substack{0\leq\gamma_{i_1}\leq j_{i_1}\\ i_1\in I}} \sum_{\substack{\gamma_{i_2}\geq 0,\, i_2\in I^c\\ \sum_{\kappa_2\in I^c}\gamma_{\kappa_2}\geq\left(n-v-\sum_{\kappa_1\in I}\gamma_{\kappa_1}-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)_+}} b^{-\sum_{\kappa_2\in I^c}\gamma_{\kappa_2}}$$

$$\leq b^{-|j|_+} \sum_{I\subsetneq\{1,\ldots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \cdots$$

$$
\begin{aligned}
&\qquad \dots \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1}\\ i_1\in I}} \sum_{r=\left(n-v-\sum_{\kappa_1\in I}\gamma_{\kappa_1}-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)_+}^{\infty} b^{-r}(r+1)^{d-1-\#I} \\
&\le b^{-|j|_+} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1}\\ i_1\in I}} b^{-n+v+\sum_{\kappa_1\in I}\gamma_{\kappa_1}+\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \times \\
&\qquad \times \left(n-v-\sum_{\kappa_1\in I}\gamma_{\kappa_1}-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)_+^{d-1-\#I} \\
&\le b^{-|j|_+-n+v} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)} \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1}\\ i_1\in I}} b^{\sum_{\kappa_1\in I}\gamma_{\kappa_1}} \times \\
&\qquad \times \left(n-v-\sum_{\kappa_1\in I}\gamma_{\kappa_1}-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)_+^{d-1} \\
&\le b^{-|j|_+-n+v} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)} b^{\sum_{\kappa_1\in I}(j_{\kappa_1}+1)} \\
&\qquad \times \left(n-v-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)_+^{d-1} \\
&\preceq b^{-|j|_+-n+v}\,(n-v-|j|_+)^{d-1}.
\end{aligned}
$$

□

Proposition 5.7. *Let $\mathcal{P}_n^b$ be an order 2 digital (v,n,d)-net in base b. Let $j\in\mathbb{N}_{-1}^d$, $m\in\mathbb{D}_j$, $l\in\mathbb{B}_j$.*

(i) If $|j|_+\ge n-\lceil v/2\rceil$ then $|\langle D_{\mathcal{P}_n^b},h_{j,m,l}\rangle|\preceq b^{-|j|_+-n+v/2}$ and $|\mu_{j,m,l}(D_{\mathcal{P}_n^b})|\preceq b^{-2|j|_+}$ for all but b^n values of m.

(ii) If $|j|_+< n-\lceil v/2\rceil$ then $|\langle D_{\mathcal{P}_n^b},h_{j,m,l}\rangle|\preceq b^{-2n+v}\,(2n-v-2|j|_+)^{d-1}$.

Proof. According to Lemma 4.1, $\mathcal{P}_n^b$ is an order 1 digital $(\lceil v/2\rceil,n,d)$-net. Hence (i) follows from Proposition 5.6.

We now prove (ii) so let $|j|_+<n-\lceil v/2\rceil$ and $m\in\mathbb{D}_j$, $l\in\mathbb{B}_j$. We start at (15) and recall that we have $\varrho_2(t)>2n-v$ for all $t\in\mathfrak{D}(\mathfrak{C})$. This means that $\omega_{\gamma_1,\dots,\gamma_d}^{j_1+1,\dots,j_d+1}(\mathfrak{C})=0$ whenever $\gamma_1+\min(\gamma_1,j_1+1)+\ldots+\gamma_d+\min(\gamma_d,j_d+1)\le 2n-v$. We argue similarly to the proof of Proposition 5.6 to get

$$
\begin{aligned}
|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle| &\preceq b^{-|j|_+} \sum_{\gamma_1,\dots,\gamma_d=0}^{\infty} b^{-\max(j_1+1,\gamma_1)-\dots-\max(j_d+1,\gamma_d)}\, \omega^{j_1+1,\dots,j_d+1}_{\gamma_1,\dots,\gamma_d} \\
&\preceq b^{-|j|_+} \sum_{\substack{\gamma_1,\dots,\gamma_d=0 \\ \gamma_1+\min(\gamma_1,j_1+1)+\dots+\gamma_d+\min(\gamma_d,j_d+1)>2n-v}}^{\infty} b^{-\max(j_1+1,\gamma_1)-\dots-\max(j_d+1,\gamma_d)} \\
&= b^{-|j|_+} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)} \dots \\
&\qquad \dots \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1} \\ i_1\in I}} \sum_{\substack{\gamma_{i_2}\ge j_{i_2}+1 \\ i_2\in I^c \\ 2\sum_{\kappa_1\in I}\gamma_{\kappa_1}+\sum_{\kappa_2\in I^c}(\gamma_{\kappa_2}+j_{\kappa_2}+1)\ge\max\left(2n-v+1,2\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)\right)}} b^{-\sum_{\kappa_2\in I^c}\gamma_{\kappa_2}} \\
&= b^{-|j|_+} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \dots \\
&\qquad \dots \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1} \\ i_1\in I}} \sum_{\substack{\gamma_{i_2}\ge 0,\, i_2\in I^c \\ \sum_{\kappa_2\in I^c}\gamma_{\kappa_2}\ge\left(2n-v-2\sum_{\kappa_1\in I}\gamma_{\kappa_1}-2\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)_+}} b^{-\sum_{\kappa_2\in I^c}\gamma_{\kappa_2}} \\
&\le b^{-|j|_+} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \dots \\
&\qquad \dots \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1} \\ i_1\in I}} \sum_{r=\left(2n-v-2\sum_{\kappa_1\in I}\gamma_{\kappa_1}-2\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)_+}^{\infty} b^{-r}(r+1)^{d-1-\#I} \\
&\le b^{-|j|_+} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)-\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1} \\ i_1\in I}} b^{-2n+v+2\sum_{\kappa_1\in I}\gamma_{\kappa_1}+2\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \times \\
&\qquad \times \left(2n-v-2\sum_{\kappa_1\in I}\gamma_{\kappa_1}-2\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)^{d-1-\#I} \\
&\le b^{-|j|_+-2n+v} \sum_{I\subsetneq\{1,\dots,d\}} b^{-\sum_{\kappa_1\in I}(j_{\kappa_1}+1)+\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)} \sum_{\substack{0\le\gamma_{i_1}\le j_{i_1} \\ i_1\in I}} b^{2\sum_{\kappa_1\in I}\gamma_{\kappa_1}} \times
\end{aligned}
$$

$$\times \left(2n - v - 2\sum_{\kappa_1\in I}\gamma_{\kappa_1} - 2\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)^{d-1}$$

$$\leq b^{-|j|_+-2n+v}\sum_{I\subsetneq\{1,\ldots,d\}} b^{\sum\limits_{\kappa_1\in I}(j_{\kappa_1}+1)+\sum\limits_{\kappa_2\in I^c}(j_{\kappa_2}+1)}\times$$

$$\times \left(2n - v - 2\sum_{\kappa_1\in I}(j_{\kappa_1}+1) - 2\sum_{\kappa_2\in I^c}(j_{\kappa_2}+1)+1\right)^{d-1}$$

$$\preceq b^{-2n+v}\,(2n-v-2|j|_+)^{d-1}.$$

□

We are now ready to prove the theorems.

Proof of Theorem 1.1. Let $D_{\mathcal{P}_n^b}$ be an order 2 digital (v,n,d)-net in base b. We apply Theorem 3.1, hence we need to prove

$$\sum_{j\in\mathbb{N}_{-1}^d} b^{|j|_+}\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^2 \preceq b^{-2n+v}\, n^{d-1}\, v.$$

We recall that $\#\mathbb{D}_j = b^{|j|_+}$, $\#\mathbb{B}_j = b-1$. We split the sum and apply Proposition 5.7 to get

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}_{-1}^d\\ |j|_+<n-\lceil v/2\rceil}} b^{|j|_+}\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^2\\
&\preceq \sum_{\substack{j\in\mathbb{N}_{-1}^d\\ |j|_+<n-\lceil v/2\rceil}} b^{|j|_+}\, b^{|j|_+}\, b^{-4n+2v}\,(2n-v-2|j|_+)^{2(d-1)}\\
&\leq b^{-4n+2v}\sum_{\kappa=0}^{n-v/2-1} b^{2\kappa}\,(2n-v-2\kappa)^{2(d-1)}\,(\kappa+1)^{d-1}\\
&\leq b^{-4n+2v}\, b^{2n-v}\,(2n-v-2n+v)^{2(d-1)}\,(n-v/2)^{d-1}\\
&\preceq b^{-2n+v}\, n^{d-1}
\end{aligned}$$

for big intervals,

$$\sum_{\substack{j\in\mathbb{N}_{-1}^d\\ n>|j|_+\geq n-\lceil v/2\rceil}} b^{|j|_+}\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^2$$

$$\begin{aligned}
&\preceq \sum_{\substack{j\in\mathbb{N}_{-1}^d \\ n>|j|_+\geq n-\lceil v/2\rceil}} b^{|j|_+}\, b^{|j|_+}\, b^{-2|j|_+-2n+v} \\
&\leq b^{-2n+v} \sum_{\kappa=n-\lceil v/2\rceil}^{n-1} (\kappa+1)^{d-1} \\
&\preceq b^{-2n+v}\, n^{d-1}\, v
\end{aligned}$$

for middle intervals and

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}_{-1}^d \\ |j|_+\geq n}} b^{|j|_+} \sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^2 \\
&\preceq \sum_{\substack{j\in\mathbb{N}_{-1}^d \\ |j|_+\geq n}} b^{|j|_+}\, b^n\, b^{-2|j|_+-2n+v} + \sum_{\substack{j\in\mathbb{N}_{-1}^d \\ |j|_+\geq n}} b^{|j|_+}\, (b^{|j|_+} - b^n)\, b^{-4|j|_+} \\
&\leq b^{-n+v} \sum_{\kappa=n}^{\infty} b^{-\kappa}\, (\kappa+1)^{d-1} + \sum_{\kappa=n}^{\infty} b^{-2\kappa}\, (\kappa+1)^{d-1} \\
&\preceq b^{-2n+v}\, n^{d-1}
\end{aligned}$$

for small intervals. □

Proof of Theorem 1.2. Let $D_{\mathcal{P}_n^b}$ be an order 1 digital (v,n,d)-net in base b. We apply Theorem 3.2, hence we need to prove

$$\sum_{j\in\mathbb{N}_{-1}^d} b^{|j|_+(r-1/p+1)q} \left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^p \right)^{q/p} \preceq b^{n(r-1)q}\, n^{(d-1)}\, b^{vq}.$$

We recall that $\#\mathbb{D}_j = b^{|j|_+}$, $\#\mathbb{B}_j = b-1$. We split the sum and apply Minkowski's inequality and Proposition 5.6 to get

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}_{-1}^d \\ |j|_+<n-v}} b^{|j|_+(r-1/p+1)q} \left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^p \right)^{q/p} \\
&\preceq \sum_{\substack{j\in\mathbb{N}_{-1}^d \\ |j|_+<n-v}} b^{|j|_+(r-1/p+1)q}\, b^{|j|_+q/p}\, b^{(-|j|_+-n+v)q}\, (n-v-|j|_+)^{(d-1)q} \\
&\leq b^{(-n+v)q} \sum_{\kappa=0}^{n-v-1} b^{\kappa rq}\, (n-v-\kappa)^{(d-1)q}\, (\kappa+1)^{d-1}
\end{aligned}$$

$$\le b^{(-n+v)q}\, b^{(n-v)rq}\,(n-v+1)^{d-1}$$
$$\preceq b^{n(r-1)q}\, n^{d-1}\, b^{v(1-r)q}$$

for big intervals,

$$\sum_{\substack{j\in\mathbb{N}_{-1}^d\\ n>|j|_+\ge n-v}} b^{|j|_+(r-1/p+1)q}\left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^p\right)^{q/p}$$
$$\preceq \sum_{\substack{j\in\mathbb{N}_{-1}^d\\ n>|j|_+\ge n-v}} b^{|j|_+(r-1/p+1)q}\, b^{|j|_+q/p}\, b^{(-|j|_+-n+v)q}$$
$$\le b^{(-n+v)q}\sum_{\kappa=n-v}^{n-1} b^{\kappa rq}(\kappa+1)^{d-1}$$
$$\preceq b^{(-n+v)q}\, b^{nrq}\, n^{d-1}$$
$$\le b^{n(r-1)q}\, n^{(d-1)}\, b^{vq}$$

for middle intervals and considering the range of r

$$\sum_{\substack{j\in\mathbb{N}_{-1}^d\\ |j|_+\ge n}} b^{|j|_+(r-1/p+1)q}\left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^p\right)^{q/p}$$
$$\preceq \sum_{\substack{j\in\mathbb{N}_{-1}^d\\ |j|_+\ge n}} b^{|j|_+(r-1/p+1)q}\, b^{nq/p}\, b^{(-|j|_+-n+v)q} + \sum_{\substack{j\in\mathbb{N}_{-1}^d\\ |j|_+\ge n}} b^{|j|_+(r-1/p+1)q}\,(b^{|j|_+}-b^n)^{q/p}\, b^{-2|j|_+q}$$
$$\le b^{nq/p}\, b^{(-n+v)q}\sum_{\kappa=n}^{\infty} b^{\kappa(r-1/p)q}\,(\kappa+1)^{d-1} + \sum_{\kappa=n}^{\infty} b^{\kappa(r-1)q}\,(\kappa+1)^{d-1}$$
$$\preceq b^{nq/p}\, b^{(-n+v)q}\, b^{n(r-1/p)q} n^{d-1} + b^{n(r-1)q}\, n^{d-1}$$
$$\preceq b^{n(r-1)q}\, n^{(d-1)}\, b^{vq}$$

for small intervals. □

Proof of Theorem 1.3. Let $D_{\mathcal{P}_n^b}$ be an order 2 digital (v,n,d)-net in base b. The proof is similar to the proof of Theorem 1.2. We apply Proposition 5.7 instead of 5.6 to get

$$\sum_{\substack{j\in\mathbb{N}_{-1}^d\\ |j|_+<n-\lceil v/2\rceil}} b^{|j|_+(r-1/p+1)q}\left(\sum_{m\in\mathbb{D}_j,\, l\in\mathbb{B}_j}|\langle D_{\mathcal{P}_n^b}, h_{j,m,l}\rangle|^p\right)^{q/p}$$

$$
\begin{aligned}
&\preceq \sum_{\substack{j\in\mathbb{N}_{-1}^d \\ |j|_+<n-\lceil v/2\rceil}} b^{|j|_+(r-1/p+1)q}\, b^{|j|_+q/p}\, b^{(-2n+v)q}\,(2n-v-2|j|_+)^{(d-1)q} \\
&\leq b^{(-2n+v)q} \sum_{\kappa=0}^{n-v/2-1} b^{\kappa(r+1)q}\,(2n-v-2\kappa)^{(d-1)q}\,(\kappa+1)^{d-1} \\
&\leq b^{(-2n+v)q}\, b^{(n-v/2)(r+1)q}\,(n-v/2+1)^{d-1} \\
&\preceq b^{n(r-1)q}\, n^{d-1}\, b^{v/2(1-r)q}
\end{aligned}
$$

and analogous results for the other subsums. □

Proof of Corollaries 1.4, 1.5, 1.6 and 1.7. The results for the Triebel-Lizorkin spaces follow from Theorem 2.1 and Theorems 1.2 and 1.3, respectively. The results for the Sobolev spaces then follow in the case $q=2$. □

Proof of Theorem 1.8. The result follows from Corollary 1.7 in the case $r=0$. □

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