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Convergence Types and Rates in Generic Karhunen-Loève Expansions with Applications to Sample Path Properties

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Abstract

We establish a Karhunen-Loève Expansion for generic centered, second order stochastic processes. We further investigate in which norms the expansion converges and derive exact rates of convergence for these norms. Moreover, we show that these results can in some situations be used to construct reproducing kernel Hilbert spaces (RKHSs) containing the paths of a version of the process. As an application, we compare the smoothness of the paths with the smoothness of the functions contained in the RKHS of the covariance function.

1 Introduction

Given a real-valued, centered stochastic process $(X_t)_{t\in T}$ with finite second moments, the covariance function $k : T \times T \to \mathbb{R}$ defined by $k(s,t) := \mathbb{E}X_sX_t$ is positive semi-definite. Consequently, there exists a reproducing kernel Hilbert space (RKHS) H on T for which k is the (reproducing) kernel. It is well-known that there are intimate relationships between H and the stochastic process.

One such relation is described by the classical Loève isometry $\Psi : L_2(X) \to H$ defined by $\Psi(X_t) = k(t,\cdot)$, where $L_2(X)$ denotes the $L_2(P)$-closure of the space spanned by $(X_t)_{t\in T}$. In particular, if $(e_i)_{i\in I}$ is an arbitrary orthonormal basis (ONB) of H, then the process enjoys the representation,

$$X_t = \sum_{i\in I} \xi_i e_i(t)\,, \tag{1}$$

where $(\xi_i)_{i\in I}$ is the family of uncorrelated random variables given by $\xi_i := \Psi^{-1}(e_i)$, and the convergence is, for each $t \in T$, unconditional in $L_2(P)$.

For Gaussian processes, the relationship between the process and its RKHS is, of course, even closer, since the finite dimensional distributions of the process are completely determined by k. Moreover, the isometry Ψ can be used to define stochastic integrals, see e.g. [16, Chapter 7]. In addition, if H is separable, the representation (1) converges also P-almost surely for each t, and $(\xi_i)_{i\in I}$ is a family of independent, standard normal random variables, see e.g. [16, Theorem 8.22]. Last but not least, in some cases we even have P-almost surely uniform convergence in t, see [2, Theorem 3.8]. Note that unlike the convergence in (1), uniform convergence in t makes it possible to represent the *paths* of the process by a series expansion.

If T is a compact metric space, ν is a strictly positive and finite Borel measure on T, and k is continuous, the famous Karhunen-Loève expansion allows to refine the expansion (1). Indeed, in this case we can find an ONB $(e_i)_{i\in I}$ of H that is also orthogonal in $L_2(\nu)$ and we additionally have

$$X(\omega) = \sum_{i\in I} \xi_i(\omega) e_i \,, \tag{2}$$

where the series converges unconditionally in $L_2(\nu)$ for P-almost all $\omega \in \Omega$. In addition, we have $X = \sum_{i\in I} \xi_i \otimes e_i$ with unconditional convergence in $L_2(P \otimes \nu)$. Like for the above mentioned example of certain Gaussian processes, the form of convergence in (2) allows for a series expansion of the paths of the process, this time, however, only with $L_2(\nu)$-convergence. However, the assumptions needed for (2) are significantly more restrictive than those for (1), and thus a natural question is to ask for weaker assumptions ensuring a path representation (2). In addition, $L_2(\nu)$-convergence is a rather weak form of convergence so that is seems to be desirable to replace it by stronger notions of convergence, e.g. by uniform convergence in t.

Another, rather different relationship between the process and its RKHS is in terms of quadratic mean smoothness. For example, if T is a metric space, then the process is continuous in quadratic mean, if and only if its kernel is continuous. Moreover, a similar statement is true for quadratic mean differentiability. We refer to [4, p. 63] and [44, p. 65ff] for details.

Of course, smoothness in quadratic mean is not related to the smoothness of the paths of the process. However, considering the path expansion (2) it seems natural to ask to which extend the paths inherit smoothness properties from H, or from the ONB $(e_i)_{i\in I}$. Probably, the first attempt in this direction is to check whether the paths are P-almost surely contained in H. Unfortunately, this is, in general not true. Indeed, for Gaussian processes with infinite dimensional RKHS the paths are P-almost surely *not* contained in H, see [22, Corollary 7.1] and also [25]. A natural next question is to look for larger RKHSs $\bar{H}$ that do contain the paths almost surely. The first result in this direction goes back to Driscoll, see [11]. Namely, he essentially showed:

Theorem 1.1. *Let (T, d) be a separable metric space and $(X_t)_{t\in T}$ be a centered and continuous Gaussian process, whose kernel k is continuous. Then for all RKHS $\bar{H}$ on T having a continuous kernel, the following statements are equivalent:*

i) Almost all paths of the process are contained in $\bar{H}$.

ii) We have $H \subset \bar{H}$ and the embedding $\mathrm{id} : H \to \bar{H}$ is Hilbert-Schmidt.

Since being Hilbert-Schmidt is already a rather strong notion of compactness, Driscoll's theorem shows that possible spaces $\bar{H}$ need to be significantly larger than H, at least for Gaussian processes satisfying the assumptions above. In particular, if we try to describe smoothness properties of the paths by a suitable RKHS $\bar{H}$, this result suggests that the paths should be rougher than the functions in H.

More recently, Lukić and Beder have shown, see [22, Theorem 5.1], that for *arbitrary* centered, second-order stochastic process $(X_t)_{t\in T}$ condition *ii)* implies the existence of a version $(Y_t)_{t\in T}$ whose paths are almost surely contained in $\bar{H}$, and for generic Gaussian processes [22, Corollary 7.1] shows *i)* $\Rightarrow$ *ii)*. Furthermore, they provide examples of non-Gaussian processes, for which the implication *i)* $\Rightarrow$ *ii)* does *not* hold, and they also present modifications *i')* and *ii')* of *i)* and *ii)*, for which we have *i')* $\Rightarrow$ *ii")* in the general case, see [22, Theorem 3.1 and Corollary 3.1] for details.

Summarizing these results, it seems fair to say that we already have reasonably good means to *test* whether a given RKHS $\bar{H}$ contains the paths of our process almost surely. Except for a couple of specific examples, however, very little is known whether such an $\bar{H}$ *exists*, or even how to *construct* such an $\bar{H}$, cf. [21, p. 255ff].

It turns out in this paper, that all these questions are related to each other by a rather general form of Mercer's theorem and its consequences, which has been recently presented in [36]. Before we go into details in the next sections let us briefly outline our main results. To this end let us assume in the following that we have a σ-finite measure ν on T and a centered, second order process $(X_t)_{t\in T}$ with $X \in \mathcal{L}_2(P \otimes \nu)$. It turns out that for such processes, H is "contained" in $L_2(\nu)$ and the "embedding" $H \to L_2(\nu)$ is Hilbert-Schmidt, which makes the results from [36] readily applicable. Here we use the quotation marks, since we actually need to consider equivalence classes to properly define the embedding. As a matter of fact, the entire theory of [36] foots on the careful differentiation between functions and their equivalence classes, and thus we need to adopt the somewhat pedantic notation of [36] later in the paper. For now, however, let us ignore these differences for the informal description of our main results:

- The Karhunen-Loève expansion (2) always holds for the process $(X_t)_{t\in T}$. In particular, no topological assumptions are needed.

- If the embedding $H \to L_2(\nu)$ is, in a certain sense, more compact than Hilbert-Schmidt, then almost all paths of the process are contained in a suitable interpolation space between $L_2(\nu)$ and H. Moreover, (2) converges in this interpolation space, too, and the average rate of this convergence can be exactly described. Finally, for Gaussian processes the results are sharp.

- Under even stronger compactness assumptions on the embedding $H \to L_2(\nu)$, the interpolation space is an RKHS and there exists a version of the process having almost all its paths in this RKHS.

- If $T \subset \mathbb{R}^d$ is a bounded and open subset with suitable boundary conditions, and H is embedded into a (fractional) Sobolev space $W^m(T)$ with $m > d/2$, then almost all paths are in the fractional Sobolev space $W^{m-d/2-\epsilon}(T)$, where $\varepsilon > 0$ is arbitrary. Moreover, for Gaussian processes this is sharp. In other words, the paths are about $d/2$-less smooth than the functions in H.

> Besides other possible applications, these results are particularly interesting for certain non-parametric statistical methods such as so-called Gaussian processes, see [28, 27, 40, 39] and the references therein, as well as spatial statistical methods, see e.g. [34, 13, 33] and the references in these articles.

The rest of this paper is organized as follows: In Section 2 some concepts from [36] are recalled and some additional results are presented. The generic Karhunen-Loève expansion is established in Section 3 and Section 4 contains the results that are related to stronger notions of convergence in the Karhunen-Loève expansion. In Section 5 we continue these investigations with the focus on instances, where the interpolation spaces are RKHSs. The Sobolev space related results are presented as applications of the general theory in Sections 4 and 5. All proofs as well as some auxiliary results can be found in Section 6.

2 Preliminaries on Kernels

Let us begin by introducing some notations. To this end, let $(T, \mathcal{B}, \nu)$ be a measure space. Recall that $\mathcal{B}$ is ν-complete, if, for every $A \subset T$ for which there exists an $N \in \mathcal{B}$ such that $A \subset N$ and $\nu(N) = 0$, we have $A \in \mathcal{B}$. In this case we say that $(T, \mathcal{B}, \nu)$ is complete.

For $S \subset T$ we denote the indicator function of S by $\mathbf{1}_S$. Moreover, for an $f : S \to \mathbb{R}$ we denote its zero-extension by $\hat{f}$, that is, $\hat{f}(t) := f(t)$ for all $t \in S$ and $\hat{f}(t) := 0$ otherwise.

As usual, $\mathcal{L}_2(\nu)$ denotes the set of all measurable functions $f : T \to \mathbb{R}$ such that $\|f\|_{\mathcal{L}_2(\nu)} := \int |f|^2 \, d\nu < \infty$. For $f \in \mathcal{L}_2(\nu)$, we further write

$$[f]_\sim := \big\{ g \in \mathcal{L}_2(\nu) : \nu(\{f \neq g\}) = 0 \big\}$$

for the ν-equivalence class of f. Let $L_2(\nu) := \mathcal{L}_2(\nu)_{/\sim}$ be the corresponding quotient space and $\|\cdot\|_{L_2(\nu)}$ be its norm. For an arbitrary, non-empty index set I and $p \in (0, \infty)$, we denote, the space of all p-summable real-valued families by $\ell_p(I)$.

In the following, we say that a Banach space F is continuously embedded into a Banach space E, if $F \subset E$ and the identity map $\mathrm{id} : F \to E$ is continuous. In this case, we sometimes write $F \hookrightarrow E$.

Let us now recall some properties of reproducing kernel Hilbert spaces (RKHSs), and their interaction with measures from [36]. To this end, let $(T, \mathcal{B}, \nu)$ be a measure space and $k : T \times T \to \mathbb{R}$ be a measurable (reproducing) kernel with RKHS H, see e.g. [4, 41, 35] for more information about these spaces. Recall that in this case the RKHS H consists of measurable functions $T \to \mathbb{R}$. In the following, we say that H is embedded into $L_2(\nu)$, if all $f \in H$ are measurable with $[f]_\sim \in L_2(\nu)$ and the linear operator

$$\begin{aligned} I_k : H &\to L_2(\nu) \\ f &\mapsto [f]_\sim \end{aligned}$$

is continuous. We write $[H]_\sim$ for its image, that is $[H]_\sim := \{[f]_\sim : f \in H\}$. Moreover, we say that H is compactly embedded into $L_2(\nu)$, if I_k is compact. For us, the most interesting class of compactly embedded RKHSs H are those whose kernel k satisfies

$$\|k\|_{\mathcal{L}_2(\nu)} := \left(\int_T k(t,t)d\nu(t)\right)^{1/2} < \infty\,. \tag{3}$$

For these kernels, the embedding $I_k :\to H$ is actually Hilbert-Schmidt, see e.g. [36, Lemma 2.3]. Finally note that $\|k\|_{\mathcal{L}_2(\nu)} < \infty$ is always satisfied for bounded kernels as long as ν is a finite measure.

Now assume that H is embedded into $L_2(\nu)$. Then one can show, see e.g. [36, Lemma 2.2], that the adjoint $S_k := I_k^* : L_2(\nu) \to H$ of the embedding I_k satisfies

$$S_k f(t) = \int_T k(t,t')f(t')d\nu(t')\,, \qquad\qquad f \in L_2(\nu), t \in T\,. \tag{4}$$

We write $T_k := I_k \circ S_k$ for the resulting integral operator $T_k : L_2(\nu) \to L_2(\nu)$. Clearly, T_k is self-adjoint and positive, and if H is compactly embedded, then T_k is also compact, so that the classical spectral theorem for compact, self-adjoint operators can be applied. In our situation, however, the spectral theorem can be refined, as we will see in Theorem 2.1 below. In order to formulate this theorem, we say that an at most countable family $(\alpha_i)_{i\in I} \subset (0,\infty)$ converges to 0 if either $I = \{1,\dots,n\}$ or $I = \mathbb{N} := \{1,2,\dots\}$ and $\lim_{i\to\infty}\alpha_i = 0$. Analogously, when we consider an at most countable family $(e_i)_{i\in I}$, we always assume without loss of generality that either $I = \{1,\dots,n\}$ or $I = \mathbb{N}$.

With these preparation we can now state the following spectral theorem for T_k, which is an abbreviated version of [36, Lemma 2.12].

Theorem 2.1. *Let $(T,\mathcal{B},\nu)$ be a measure space and k be a measurable kernel on T whose RKHS H is compactly embedded into $L_2(\nu)$. Then there exists an at most countable family $(\mu_i)_{i\in I} \subset (0,\infty)$ converging to 0 with $\mu_1 \geq \mu_2 \geq \dots > 0$ and a family $(e_i)_{i\in I} \subset H$ such that:*

i) The family $(\sqrt{\mu_i}e_i)_{i\in I}$ is an ONS in H and $([e_i]_\sim)_{i\in I}$ is an ONS in $L_2(\nu)$.

ii) The operator T_k enjoys the following spectral representation, which is convergent in $L_2(\nu)$:

$$T_k f = \sum_{i\in I}\mu_i\big\langle f,[e_i]_\sim\big\rangle_{L_2(\nu)}[e_i]_\sim\,, \qquad\qquad f \in L_2(\nu)\,. \tag{5}$$

In addition, we have

$$\mu_i e_i = S_k[e_i]_\sim\,, \qquad i \in I \tag{6}$$

$$\ker S_k = \ker T_k \tag{7}$$

$$\overline{\operatorname{ran} S_k} = \overline{\operatorname{span}\{\sqrt{\mu_i}e_i : i \in I\}} \tag{8}$$

$$\overline{\operatorname{ran} S_k^*} = \overline{\operatorname{span}\{[e_i]_\sim : i \in I\}} \tag{9}$$

$$\ker S_k^* = (\overline{\operatorname{ran} S_k})^\perp \tag{10}$$

$$\overline{\operatorname{ran} S_k^*} = (\ker S_k)^\perp\,, \tag{11}$$

where the closures and orthogonal complements are taken in the spaces the objects are naturally contained in, that is, (8) *and* (10) *are considered in* H*, while* (9) *and* (11) *are considered in* $L_2(\nu)$*.*

The following set of assumptions, which is frequently used throughout this paper, essentially summarizes some notations from Theorem 2.1.

Assumption K. *Let* $(T, \mathcal{B}, \nu)$ *be a measure space and* k *be a measurable kernel on* T *whose RKHS* H *is compactly embedded into* $L_2(\nu)$*. Furthermore, let* $(\mu_i)_{i\in I}$ *and* $(e_i)_{i\in I}$ *be as in Theorem 2.1.*

With the help of these families $(\mu_i)_{i\in I}$ and $(e_i)_{i\in I} \subset H$, some spaces and new kernels were defined in [36], which we need to recall since they are essential for this work. To begin with, [36, Equation (36)] introduced, for $\beta \in (0, 1]$, the subspace

$$[H]^\beta_\sim = \left\{ \sum_{i\in I} a_i \mu_i^{\beta/2} [e_i]_\sim : (a_i) \in \ell_2(I) \right\}$$

of $L_2(\nu)$ and equipped it with the Hilbert space norm

$$\Big\| \sum_{i\in I} a_i \mu_i^{\beta/2} [e_i]_\sim \Big\|_{[H]^\beta_\sim} := \|(a_i)\|_{\ell_2(I)} \,. \tag{12}$$

It is easy to verify that $(\mu_i^{\beta/2}[e_i]_\sim)_{i\in I}$ is an ONB of $[H]^\beta_\sim$ and that the set $[H]^\beta_\sim$ is independent of the particular choice of the family $(e_i)_{i\in I} \subset H$ in Theorem 2.1. In particular, [36, Theorem 4.6] showed that

$$[H]^\beta_\sim = \big[L_2(\nu), [H]_\sim\big]_{\beta,2} = \operatorname{ran} T_k^{\beta/2} \,, \tag{13}$$

where $T_k^{\beta/2}$ denotes the $\beta/2$-power of the operator T_k defined, as usual, by its spectral representation, and $[L_2(\nu), [H]_\sim]_{\beta,2}$ stands for the interpolation space of the standard real interpolation method, see e.g. [3, Definition 1.7 on page 299]. In addition, [36, Theorem 4.6] showed that the norms of $[H]^\beta_\sim$ and $[L_2(\nu), [H]_\sim]_{\beta,2}$ are equivalent. In other words, modulo equivalence of norms, $[H]^\beta_\sim$ *is* the interpolation space $[L_2(\nu), [H]_\sim]_{\beta,2}$.

In [36, Section 4] it was shown that under certain circumstances $[H]^\beta_\sim$ is actually the image of an RKHS under $[\,\cdot\,]_\sim$. To recall the construction of this RKHS, let us assume that we have a measurable $S \subset T$ with $\nu(T \setminus S) = 0$ and

$$\sum_{i\in I} \mu_i^\beta e_i^2(t) < \infty \,, \qquad\qquad t \in S \tag{14}$$

We write $\hat{e}_i := \mathbf{1}_S e_i$ for all $i \in I$. Clearly, this gives $\sum_{i\in I} \mu_i^\beta \hat{e}_i^2(t) < \infty$. Based on this and the fact that $([\hat{e}_i]_\sim)_{i\in I}$ is an ONS of $L_2(\nu)$, [36, Lemma 2.6] showed that

$$\hat{H}^\beta_S := \left\{ \sum_{i\in I} a_i \mu_i^{\beta/2} \, \hat{e}_i : (a_i) \in \ell_2(I) \right\} \tag{15}$$

equipped with the norm

$$\Big\| \sum_{i\in I} a_i \mu_i^{\beta/2}\, \hat{e}_i \Big\|_{\hat{H}_S^\beta} := \|(a_i)\|_{\ell_2(I)} \tag{16}$$

is a separable RKHS, which is compactly embedded into $L_2(\nu)$. Moreover, the family $(\mu_i^{\beta/2}\hat{e}_i)_{i\in I}$ is an ONB of $\hat{H}_S^\beta$ and the (measurable) kernel $\hat{k}_S^\beta$ of $\hat{H}_S^\beta$ is given by the pointwise convergent series representation

$$\hat{k}_S^\beta(t,t') = \sum_{i\in I} \mu_i^\beta \hat{e}_i(t)\hat{e}_i(t')\,, \qquad t,t' \in T\,. \tag{17}$$

Note that [36, Proposition 4.2] showed that $\hat{k}_S^\beta$ and its RKHS $\hat{H}_S^\beta$ are actually independent of the particular choice of the family $(e_i)_{i\in I} \subset H$ in Theorem 2.1, which justifies the chosen notation. Recall that in general, $\hat{k}_T^1$ does *not* equal k, and, of course, the same is true for the resulting RKHSs $\hat{H}_T^1$ and H. In fact, [36, Theorem 3.3] shows that $k = \hat{k}_T^1$ holds, if and only if $I_k : H \to L_2(\nu)$ is injective, and a sufficient condition for the latter will be presented in Lemma 2.6. Finally, for $\alpha \geq \beta$, we have $\hat{H}_S^\alpha \hookrightarrow \hat{H}_S^\beta$ by the definition of the involved norms, cf. also the proof of [36, Lemma 4.3].

In the following, we write $k_S^\beta : S \times S \to \mathbb{R}$ for the restriction of $\hat{k}_S^\beta$ onto $S \times S$ and we denote the RKHS of k_S^β by H_S^β.

Formally, the spaces $\hat{H}_S^\beta$, H_S^β and $[H]_\sim^\beta$ are different. Not surprisingly, however, they are all isometrically isomorphic to each other via natural operators. The corresponding results are collected in the following lemma.

Lemma 2.2. *Let Assumption K be satisfied, $\beta \in (0,1]$, and $R \subset S \subset T$ be measurable subsets such that R satisfies $\nu(T \setminus R) = 0$ and* (14). *Then the following operators are isometric isomorphisms:*

i) The multiplication operator $\mathbf{1}_R : \hat{H}_S^\beta \to \hat{H}_R^\beta$ defined by $f \mapsto \mathbf{1}_R f$.

ii) The zero-extension operator $\hat{\cdot} : H_S^\beta \to \hat{H}_S^\beta$.

iii) The restriction operator $\cdot_{|R} : \hat{H}_S^\beta \to H_R^\beta$.

iv) The equivalence-class operator $[\,\cdot\,]_\sim : \hat{H}_S^\beta \to [H]_\sim^\beta$.

Since in the following we need to investigate inclusions between RKHSs in more detail, let us introduce some more notations. To this end, we fix two kernels k_1, k_2 on T with corresponding RKHSs H_1 and H_2. Following [22] we say that k_2 *dominates* k_1 and write $k_1 \leq k_2$, if $H_1 \subset H_2$ and the natural inclusion operator $I_{k_1,k_2} : H_1 \to H_2$ is continuous. In this case, the adjoint operator $I_{k_1,k_2}^* : H_2 \to H_1$ exists and is continuous. In analogy to our previous notations, we write $S_{k_1,k_2} := I_{k_1,k_2}^*$. Moreover, we speak of nuclear dominance and write $k_1 \ll k_2$, if $k_1 \leq k_2$ and $I_{k_1,k_2} \circ S_{k_1,k_2}$ is nuclear.

Let us now assume that H_S^β exists for some $\beta \in (0,1)$. The preceding remarks then show that the restriction operator $\cdot_{|S} : H_T^1 \to H_S^\beta$ is well-defined and continuous. The next lemma shows that it is even compact and characterizes when it is Hilbert-Schmidt.

Lemma 2.3. *Let Assumption K be satisfied. Then, for all $\beta \in (0,1)$ and all measurable $S \subset T$ satisfying $\nu(T \setminus S) = 0$ and* (14)*, the restriction operator $\cdot_{|S} : H_T^1 \to H_S^\beta$ is compact, and the following statements are equivalent:*

i) The operator $\cdot_{|S} : H_T^1 \to H_S^\beta$ is Hilbert-Schmidt.

ii) We have $\sum_{i\in I} \mu_i^{1-\beta} < \infty$.

iii) We have $k_S^1 \ll k_S^\beta$.

Let us now recall conditions, which ensure (14) for a set S of full measure. To begin with, note that we find such an S if $\sum_{i\in I} \mu_i^\beta < \infty$, since a simple calculation based on Beppo Levi's theorem shows

$$\int_T \sum_{i\in I} \mu_i^\beta e_i^2(t)\, d\nu(t) = \sum_{i\in I} \mu_i^\beta \int_T e_i^2(t)\, d\nu(t) = \sum_{i\in I} \mu_i^\beta < \infty\,. \tag{18}$$

Moreover, in this case we obviously have $\|\hat{k}_S^\beta\|_{\mathcal{L}_2(\nu)} < \infty$. Interestingly, the converse implication is also true, namely [36, Proposition 4.4] showed that we have $\sum_{i\in I} \mu_i^\beta < \infty$, if and only if (14) holds for a set S of full measure and the resulting kernel $\hat{k}_S^\beta$ satisfies $\|\hat{k}_S^\beta\|_{\mathcal{L}_2(\nu)} < \infty$. Moreover, [36, Theorem 5.3] showed that (14) holds for a set S of full measure, if ν is a σ-finite measure for which $\mathcal{B}$ is complete and

$$[H]_\sim^\beta \hookrightarrow L_\infty(\nu)\,. \tag{19}$$

Note that this sufficient condition is particularly interesting when combined with (13), since the inclusion $[L_2(\nu), [H]_\sim]_{\beta,2} \hookrightarrow L_\infty(\nu)$ may be known in specific situations. Finally, [36, Theorem 5.3] actually showed that the inclusion (19) holds, if and only if (14) holds for a set S of full measure *and* the resulting kernel $\hat{k}_S^\beta$ is bounded.

Our next goal is to investigate under which conditions (14) even holds for $S := T$. To this end, let us now assume that we have a topology τ on T. The following definition introduces some notions of continuity.

Definition 2.4. *Let (T, τ) be a topological space and k be a kernel on T with RKHS H. Then we call k:*

i) τ-continuous, if k is continuous with respect to the product topology $\tau \otimes \tau$.

ii) separately τ-continuous, if $k(t, \cdot) : T \to \mathbb{R}$ is τ-continuous for all $t \in T$.

iii) weakly τ-continuous, if all $f \in H$ are τ-continuous.

Clearly, τ-continuous kernels are separately τ-continuous. Moreover, it is a well-known fact that given a τ-continuous kernel k its canonical feature map $\Phi : T \to H$ defined by $\Phi(t) := k(t, \cdot)$ is τ-continuous, see e.g. [35, Lemma 4.29], and hence the reproducing property $f = \langle f, \Phi(\cdot)\rangle_H$, which holds for all $f \in H$, shows that k is also weakly τ-continuous. Moreover, [35, Lemma 4.28] shows that bounded, separately τ-continuous kernels are weakly τ-continuous, too. In this regard note that even on $T = [0, 1]$ not every bounded, separately τ-continuous kernel is continuous, see [20].

Let us now introduce two topologies on T generated by k and its RKHS H. The first one is the topology τ_k generated by the well-known pseudo-metric d_k on T defined by

$$d_k(t,t') := \|\Phi(t) - \Phi(t')\|_H\,, \qquad\qquad t,t' \in T.$$

Obviously, this pseudo-metric is a metric if and only if the canonical feature map $\Phi : T \to H$ is injective, and this is also the only case in which τ_k is Hausdorff. Less known is another topology on T that is related to k, namely the initial topology $\tau(H)$ generated by the set of functions H. In other words, $\tau(H)$ is the smallest topology on T for which all $f \in H$ are continuous, that is, for which k is weakly τ-continuous. More information on these topologies can be found in Lemma 6.1.

In the following, we sometimes need measures ν that are strictly positive on all non-empty $\tau(H)$-open sets. Such measures are introduced in the following definition.

Definition 2.5. *Let $(T,\mathcal{B},\nu)$ be a measure space and k be a kernel on T with RKHS H such that $\tau(H) \subset \mathcal{B}$. Then ν is called k-positive, if, for all $O \in \tau(H)$ with $O \neq \emptyset$, we have $\nu(O) > 0$.*

The notion of k-positive measures generalizes that of strictly positive measures. Indeed, if (T,τ) is a topological space, and $\mathcal{B} := \sigma(\tau)$ is the corresponding Borel σ-algebra, then a measure ν on $\mathcal{B}$ is strictly positive, if $\nu(O) > 0$ for all non-empty $O \in \tau$. Now assume that we have a (weakly)-τ-continuous kernel k on T. Then we find $\tau(H) \subset \tau \subset \mathcal{B}$, and thus ν is also k-positive.

Note that if H is separable and k is bounded and $\mathcal{B} \otimes \mathcal{B}$-measurable, then every $f \in H$ is $\mathcal{B}$-measurable, see e.g. [35, Lemma 4.25], and hence $\sigma(H) \subset \mathcal{B}$. By part *iii)* of Lemma 6.1 we thus find $\tau(H) \subset \sigma(H) \subset \mathcal{B}$. In other words, the assumption $\tau(H) \subset \mathcal{B}$, which will occur frequently, is automatically satisfied for such H.

The following simple lemma gives a first glance at the importance of k-positive measures.

Lemma 2.6. *Let $(T,\mathcal{B},\nu)$ be a measure space and k be a kernel on T with RKHS H such that $\tau(H) \subset \mathcal{B}$. If ν is k-positive, then $I_k : H \to L_2(\nu)$ is injective and $k = k^1_T$.*

Let us now collect a set of assumptions frequently used when dealing with k-positive measures.

Assumption CK. *Let $(T,\mathcal{B},\nu)$ be a σ-finite and complete measure space and k be a kernel on T with RKHS H such that $\tau(H) \subset \mathcal{B}$ and ν is k-positive. Furthermore, Assumption K is satisfied.*

With these preparations we are now in the position to improve the result on bounded k^β_S from [36, Theorem 5.3].

Theorem 2.7. *Let Assumption CK be satisfied. Furthermore, assume that for some $0 < \beta \leq 1$, we have*

$$\big[L_2(\nu), [H]_\sim\big]_{\beta,2} \hookrightarrow L_\infty(\nu)\,. \tag{20}$$

Then, (14) *holds for $S := T$, the resulting kernel k^β_T is bounded, and $\tau(H^\beta_T) = \tau(H)$.*

Note that under the assumptions of Theorem 2.7 we also have $\sum_{i\in I}\mu_i^\beta < \infty$ provided that ν is *finite*, see [36, Theorem 5.3]. In addition, there is a partial converse, which does not need any continuity assumption. Indeed, if we have $\sup_{i\in I}\|e_i\|_\infty < \infty$, then a simple estimate shows that $\sum_{i\in I}\mu_i^\beta < \infty$ implies (14) for $S := T$, and the resulting kernel k_T^β turns out to be bounded.

To illustrate the theorem above, let us assume that (T, τ) is a topological space. In addition, let $\mathcal{B}$ be a σ-algebra on T and ν be a σ-finite and strictly positive measure on $\mathcal{B}$ such that $\mathcal{B}$ is ν-complete and $\tau \subset \mathcal{B}$. If k is a weakly τ-continuous kernel on T, we then obtain $\tau(H) \subset \tau \subset \mathcal{B}$, where H is the RKHS of k. Consequently, if H is compactly embedded into $L_2(\nu)$, and, for some $0 < \beta \leq 1$, we have (20), then the assumptions of Theorem 2.7 are satisfied, and hence k_T^β is defined and bounded. Moreover, we have $\tau(H_T^\beta) = \tau(H) \subset \tau$, that is, k_T^β is weakly τ-continuous. In other words, modulo the technical assumptions of Theorem 2.7, the embedding (20) ensures that k_T^β is defined and inherits the weak continuity from k.

Many of our results are formulated in terms of the eigenvalues $(\mu_i)_{i\in I}$, but determining these eigenvalues in a specific situation is often a very difficult task. For many results, we need, however, only the *asymptotic behavior* of the eigenvalues. It is well-known, see e.g. [7, 12], that this behavior can often be determined by entropy numbers. Our next goal is to make this statement precise. To this end, recall that the i-th (dyadic) entropy number of a compact, linear operator $T : E \to F$ between Banach spaces E and F is defined by

$$\varepsilon_i(T) := \inf\left\{\varepsilon > 0 : \exists\, y_1, \ldots, y_{2^{i-1}} \in F \text{ such that } TB_E \subset \bigcup_{j=1}^{2^{i-1}} (y_j + \varepsilon B_F)\right\}.$$

Note that in the literature these numbers are usually denoted by $e_i(T)$, instead. Since this in conflict with our notation for eigenvectors, we departed from this convention. For an introduction to these numbers we refer to the above mentioned books [7, 12].

Now the following result compares the eigenvalues $(\mu_i)_{i\in I}$ with the entropy numbers of I_k. Note that the latter are often known, see (24) below for an example.

Lemma 2.8. *Let Assumption K be satisfied. Then, for all $i \in I$, we have*

$$\mu_i \leq 4\varepsilon_i^2(I_k)\,. \tag{21}$$

Moreover, for all $\beta > 0$, there exists a constant $c_\beta > 0$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i^{2\beta}(I_k) \leq c_\beta \sum_{i\in I} \mu_i^\beta \tag{22}$$

In particular, for all $\beta > 0$ we have $\sum_{i\in I}\mu_i^\beta < \infty$ if and only if $\sum_{i=1}^\infty \varepsilon_i^{2\beta}(I_k) < \infty$.

Finally, to describe some some higher order smoothness properties of functions, we fix a non-empty open and bounded $T \subset \mathbb{R}^d$ that satisfies the strong local Lipschitz condition of [1, p. 83]. Note that the strong local Lipschitz condition is satisfied for e.g. the interior of $[0, 1]^d$ or open Euclidean balls. We write $L_2(T)$ for the L_2-space with respect to the Lebesgue measure on T. For $m \in \mathbb{N}_0$ and $p \in [1, \infty]$ we denote the Sobolev space of smoothness m by $W^{m,p}(T)$, that is

$$W^{m,p}(T) := \left\{ f \in L_p(T) : D^{(\alpha)} \text{ exists and } D^{(\alpha)} f \in L_p(T) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq m \right\},$$

where, as usual, $D^{(\alpha)} f$ denotes the weak α-partial derivative of f. For notational simplicity, we further write $W^m(T) := W^{m,2}(T)$. Recall Sobolev's embedding theorem, see e.g. [1, Theorem 4.12], which ensures $W^m(T) \hookrightarrow C(\overline{T})$ for all $m > d/2$, where $C(\overline{T})$ denotes the space of continuous functions defined on the closure $\overline{T}$ of T. For such m we can thus view $W^m(T)$ as an RKHS on T. Following tradition, we will, however, not notationally distinguish between the cases in which $W^m(T)$ is viewed as a space of equivalence classes or as a space of functions, since the meaning of the symbol $W^m(T)$ will always be clear from the context.

We further need fractional versions of Sobolev spaces and generalizations of them. To this end recall from [1, p. 230] that the Besov spaces of smoothness $s > 0$ are given by

$$B^s_{p,q}(T) := \left[L_p(T), W^{m,p}(T) \right]_{s/m,q}, \tag{23}$$

where $m > s$ is an arbitrary natural number and $p, q \in [1, \infty]$.

Recall that for $s > d/2$, we again have a continuous embedding $B^s_{2,2}(T) \hookrightarrow C(\overline{T})$, see [1, Theorem 7.37]. Moreover, we have $B^m_{2,2}(T) = W^m(T)$ for all integers $m \geq 1$, see [1, p. 230], and for this reason we often use the notation $W^s(T) := W^{s,2}(T) := B^s_{2,2}(T)$ for all $s > 0$. Note that with this notation, the equality in (23) with $p = q = 2$ actually holds for all $m > s$ by the reiteration property of the real interpolation method, see again [1, p. 230]. Finally, for $0 < s < 1$ and $p \in [1, \infty]$, we have by [37, Lemma 36.1 and p. 170]

$$B^s_{p,p}(T) = \left\{ f \in L_p(T) \, : \, \|f\|_{T,s,p} < \infty \right\},$$

where

$$\|f\|^p_{T,s,p} := \int_T \int_T \frac{|f(r) - f(t)|^p}{|r - t|^{d+sp}} \, dr \, dt$$

with the usual modification for $p = \infty$. Similarly, if $s > 1$ is not an integer, $B^s_{p,p}(T)$ equals the fractional Sobolev-Slobodeckij spaces, i.e. we have

$$B^s_{p,p}(T) = \left\{ f \in W^{\lfloor s \rfloor}_{p,p} : \|D^{(\alpha)} f\|_{T,s,p} < \infty \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| = \lfloor s \rfloor \right\}.$$

We refer to [37, p. 156] and [9] for details. In particular, $B^s_{\infty,\infty}(T)$ is the space of s-Hölder continuous functions for all $0 < s < 1$.

Let us finally recall some entropy estimates related to fractional Sobolev spaces. To this end, let $T \subset \mathbb{R}^d$ be a bounded subset that satisfies the strong local Lipschitz condition and $T = \operatorname{int} \overline{T}$, where $\operatorname{int} A$ denotes the interior of A. Then [12, p. 151] shows that, for all $s > d/2$, there exist constants c_1 and c_2 such that

$$c_1 i^{-s/d} \leq \varepsilon_i \big(\mathrm{id} : W^s(T) \to L_2(T) \big) \leq c_2 i^{-s/d} \tag{24}$$

for all $i \geq 1$, where we used the notation $W^s(T) = B^s_{2,2}(T)$ introduced above.

3 Karhunen-Loève Expansions For Generic Processes

The goal of this section is to establish a Karhunen-Loève expansion that does not require the usual assumptions such as compact index sets T or continuous kernels k. To this end, we first show that under very generic assumptions the covariance function of a centered, second-order process satisfies Assumption K, so that the theory developed in Section 2 is applicable. We then repeat the classical Karhunen-Loève approach and combine it with some aspects from Section 2.

In the following, let $(\Omega, \mathcal{A}, P)$ be a probability space and $(T, \mathcal{B}, \nu)$ be a σ-finite measure space. Given a stochastic process $(X_t)_{t\in T}$ on Ω, we denote the path $t \mapsto X_t(\omega)$ of a given $\omega \in \Omega$ by $X(\omega)$. Moreover, we call the process $(\mathcal{A}\otimes\mathcal{B})$-measurable, if the map $X : \Omega \times T \to \mathbb{R}$ defined by $(\omega, t) \mapsto X_t(\omega)$ is measurable. In this case, each path is obviously $\mathcal{B}$-measurable.

Let us assume that X is centered, second-order, that is $X_t \in \mathcal{L}_2(P)$ and $\mathbb{E}_P X_t = 0$ for all $t \in T$. Then the covariance function $k : T \times T \to \mathbb{R}$ is given by

$$k(s,t) := \mathbb{E}_P X_s X_t \,, \qquad\qquad s,t \in T\,.$$

It is well-known, see e.g. [4, p. 57], that the covariance function is symmetric and positive semi-definite, and thus a kernel by the Moore-Aronszajn theorem, see e.g. [35, Theorem 4.16].

Let us now additionally assume that ν is suitably chosen in the sense of $X \in \mathcal{L}_2(P\otimes\nu)$. For P-almost all $\omega \in \Omega$, we then have $X(\omega) \in \mathcal{L}_2(\nu)$. For such X, the following lemma collects some additional properties of the covariance function.

Lemma 3.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(T, \mathcal{B}, \nu)$ be a σ-finite measure space. In addition, let $(X_t)_{t\in T} \subset \mathcal{L}_2(P)$ be a centered and $(\mathcal{A}\otimes\mathcal{B})$-measurable stochastic process such that $X \in \mathcal{L}_2(P\otimes\nu)$. Then its covariance function $k : T \times T \to \mathbb{R}$ is measurable and we have*

$$\int_T k(t,t)\, d\nu(t) < \infty\,.$$

Consequently, the RKHS H of k is compactly embedded into $L_2(\nu)$ and the corresponding integral operator $T_k : L_2(\nu) \to L_2(\nu)$ is nuclear.

The lemma above in particular shows that for a stochastic process $X \in \mathcal{L}_2(P\otimes\nu)$ the RKHS H of its covariance function k is compactly embedded into $L_2(\nu)$. Consequently, Theorem 2.1 applies. Thus, let us assume that we have fixed families $(e_i)_{i\in I} \subset H$ and $(\mu_i)_{i\in I}$ that satisfy the assertions of Theorem 2.1. For $i \in I$ we then define $Z_i : \Omega \to \mathbb{R}$ by

$$Z_i(\omega) := \int_T X_t(\omega) e_i(t)\, d\nu(t) \tag{25}$$

for all $\omega \in \Omega\setminus N$, where $N \subset \Omega$ is a measurable subset satisfying with $P(N) = 0$ and $X(\omega) \in \mathcal{L}_2(\nu)$ for all $\omega \in \Omega\setminus N$. For $\omega \in N$ we further write $Z_i(\omega) := 0$. Clearly, each Z_i is measurable and $Z_i(\omega) = \langle [X(\omega)]_\sim, [e_i]_\sim\rangle_{L_2(\nu)}$ for P-almost all $\omega \in \Omega$.

Having finished these preparations we can now formulate our assumptions on the process X that will be used throughout the rest of this work.

Assumption X. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(T, \mathcal{B}, \nu)$ be a σ-finite measure space. In addition, let $(X_t)_{t\in T} \subset \mathcal{L}_2(P)$ be a centered and $(\mathcal{A}\otimes\mathcal{B})$-measurable stochastic process such that $X \in \mathcal{L}_2(P\otimes\nu)$. Moreover, let k be its covariance function and H be the RKHS of k. Finally, let $(e_i)_{i\in I} \subset H$ and $(\mu_i)_{i\in I}$ be as in Theorem 2.1 and $(Z_i)_{i\in I}$ be defined by* (25)*.*

The following lemma, which is somewhat folklore, shows that for processes satisfying Assumption X an expansion of the form (1) can be obtained if we replace $\xi_i := \Psi^{-1}(\sqrt{\mu_i}e_i)$ by $\mu_i^{-1/2}Z_i$. The proof of this lemma does not deviate much from the one needed for the classical Karhunen-Loève expansion, but since the traditional assumptions for this expansion are more restricted and the lemma itself is the very foundation of our following results we have included it for the sake of completeness.

Lemma 3.2. *Let Assumption X be satisfied. Then, for all $i, j \in I$ and $t \in T$, we have $Z_i \in \mathcal{L}_2(P)$ with $\mathbb{E}_P Z_i = 0$ and*

$$\mathbb{E}_P Z_i Z_j = \mu_i \delta_{i,j}\,, \tag{26}$$

$$\mathbb{E}_P Z_i X_t = \mu_i e_i(t)\,. \tag{27}$$

Moreover, for all finite $J \subset I$ and all $t \in T$ we have

$$\Big\| X_t - \sum_{j\in J} Z_j e_j(t) \Big\|^2_{\mathcal{L}_2(P)} = k(t,t) - \sum_{j\in J}\mu_j e_j^2(t)\,, \tag{28}$$

and, for a fixed $t \in T$, the following statements are equivalent:

i) With convergence in $L_2(P)$ we have

$$[X_t]_\sim = \sum_{i\in I}[Z_i]_\sim e_i(t)\,. \tag{29}$$

ii) We have

$$k(t,t) = \sum_{i\in I}\mu_i e_i^2(t)\,. \tag{30}$$

Moreover, if, for some $t \in T$, we have (29)*, then the convergence in* (29) *is necessarily unconditional in $L_2(P)$ by* (28)*. Finally, there exists a measurable $N \subset \Omega$ such that for all $\omega \in \Omega \setminus N$ we have*

$$[X(\omega)]_\sim \in (\ker T_k)^\perp = \overline{\operatorname{span}\{[e_i]_\sim : i \in I\}}^{L_2(\nu)}\,. \tag{31}$$

Recall that for continuous kernels k over compact metric spaces T and strictly positive measures ν, Equation (30) is guaranteed by the classical theorem of Mercer for *all* $t \in T$.. Moreover, since the convergence in (30) is also monotone and $t \mapsto k(t,t)$ is continuous, Dini's theorem shows in this case, that the convergence in (30), and by (28) also in (29), is *uniform* in t. In the general case, however, (30) may no longer be true. Indeed, the following proposition *characterizes* when (30) holds. In addition, it shows that for *separable* H Equation (29) holds at least ν-almost everywhere.

Proposition 3.3. *Let Assumption X be satisfied. Then the following statements are equivalent:*

i) The family $(\sqrt{\mu_i}e_i)_{i\in I}$ is an ONB of H.

ii) The operator $I_k : H \to L_2(\nu)$ is injective.

iii) For all $t \in T$ we have (29)*.*

Moreover, if H is separable, there exists a measurable $N \subset T$ with $\nu(N) = 0$ such that (29) *holds with unconditional convergence in $L_2(P)$ for all $t \in T \setminus N$.*

Note that for k-positive measures ν the injectivity of $I_k : H \to L_2(\nu)$ is automatically satisfied by Lemma 2.6, and thus we have (29) for all $t \in T$. Moreover note that the injectivity of I_k must not be confound with the injectivity of T_k. Indeed, the latter is equivalent to $I_k : H \to L_2(\nu)$ having a dense image, see (7) and (11). Moreover, the injectivity of T_k is also equivalent to $([e_i]_\sim)_{i\in I}$ being an ONB of $L_2(\nu)$, see (9).

Due to the particular version of convergence in (29), Proposition 3.3 is useful for approximating the distribution of X_t at some given time t, but useless for approximating the paths of the process X. This is addressed by the following result, which is the generic version of (2) and as such the first new result of this section.

Proposition 3.4. *Let Assumption X be satisfied. Then there exists a measurable $N \subset \Omega$ with $P(N) = 0$ such that for all $\omega \in \Omega \setminus N$ we have*

$$[X(\omega)]_\sim = \sum_{i\in I} Z_i(\omega)[e_i]_\sim\,, \tag{32}$$

where the convergence is unconditionally in $L_2(\nu)$. Moreover, for all $J \subset I$, we have

$$\int_\Omega \Big\| \, [X(\omega)]_\sim - \sum_{j\in J} Z_j(\omega)[e_j]_\sim \Big\|^2_{L_2(\nu)} dP(\omega) = \sum_{i\in I\setminus J} \mu_i\,. \tag{33}$$

In particular, with unconditional convergence in $L_2(P\otimes\nu)$, it holds

$$[X]_\sim = \sum_{i\in I} [Z_i]_\sim [e_i]_\sim\,.$$

Equation (32) shows that almost every path can be approximated using partial sums $\sum_{j\in J} Z_j[e_j]_\sim$ while (33) exactly specifies the average speed of convergence for such an approximation. In particular, (33) shows that any meaningful speed of convergence requires stronger summability assumptions on the sequence $(\mu_i)_{i\in I}$ of eigenvalues.

Corollary 3.5. *Let Assumption X be satisfied and $\Psi : L_2(X) \to H$ be the Loève isometry, where*

$$L_2(X) := \overline{\operatorname{span}\big\{[X_t]_\sim : t \in T\big\}}^{L_2(P)}$$

denotes the Cameron-Martin space. Then, for all $i \in I$, we have

$$[Z_i]_\sim = \sqrt{\mu_i}\,\Psi^{-1}(e_i)\,,$$

and the family $(\mu_i^{-1/2}[Z_i]_\sim)_{i\in I}$ is an ONS of $L_2(X)$. Moreover, it is an ONB, if and only if $(\sqrt{\mu_i}e_i)_{i\in I}$ is an ONB of H.

Let us finally consider the case of Gaussian processes. To this end, let us recall that a process $(X_t)_{t\in T}$ is called Gaussian, if, for all $n \geq 1$, $a_1, \ldots, a_n \in \mathbb{R}$, and $t_1, \ldots, t_n \in T$, the random variable $\sum_{i=1}^n a_i X_{t_i}$ has a normal distribution.

The following lemma shows that for Gaussian processes, the Z_i's are independent, normally distributed random variables.

Lemma 3.6. *Let $(X_t)_{t\in T}$ be a Gaussian process for which Assumption X is satisfied. Then the random variables $([Z_i]_\sim)_{i\in I}$ are independent and for all $i \in I$, we have $Z_i \sim \mathcal{N}(0, \mu_i)$.*

Our final result in this section in particular shows that essentially all reasonable sequences of coefficient variables $(Z_i)_{i\in I}$ can occur in the class of processes satisfying Assumption X.

Lemma 3.7. *Let Assumption K be satisfied with $\sum_{i\in I} \mu_i < \infty$, $(\Omega, \mathcal{A}, P)$ be a probability space, and $(Z_i)_{i\in I} \subset \mathcal{L}_2(P)$ be a sequence of centered random variables such that*

$$\mathbb{E}_P Z_i Z_j = \mu_i \delta_{i,j}$$

for all $i, j \in I$. For $t \in T$, we define

$$X_t := \sum_{i\in I} Z_i e_i(t)\,, \tag{34}$$

where the series convergences unconditionally in $\mathcal{L}_2(P)$. Then $(X_t)_{t\in T} \subset \mathcal{L}_2(P)$ is a centered, $(\mathcal{A} \otimes \mathcal{B})$-measurable process with $X \in \mathcal{L}_2(P \otimes \nu)$ and covariance function k_T^1. Moreover, the Z_i satisfy (25)*, i.e. we have $Z_i(\omega) = \langle [X(\omega)]_\sim, [e_i]_\sim \rangle_{L_2(\nu)}$ for P-almost all $\omega \in \Omega$ and all $i \in I$.*

4 Sample Paths Contained in Interpolation Spaces

In this section we first characterize when the paths of the process are not only contained in $L_2(\nu)$ but actually in an interpolation spaces between $L_2(\nu)$ and H. In particular, it turns out that stronger summability assumptions on the sequence $(\mu_i)_{i\in I}$ imply such path behavior, and in this case the average approximation error speed of the Karhunen-Loève expansion measured in the interpolation space can be exactly described by the behavior of $(\mu_i)_{i\in I}$. Moreover, we will see that for Gaussian processes, the summability assumption is actually *equivalent* to the path behavior. Finally, we apply the developed theory to processes whose RKHS are contained in fractional Sobolev spaces.

Let us begin with the following theorem, which characterizes when a single path is contained in a suitable interpolation space.

Theorem 4.1. *Let Assumption X be satisfied, $\beta \in (0, 1)$, and $N \subset \Omega$ be the measurable P-zero set we obtain from Proposition 3.4. Then for all $\omega \in \Omega \setminus N$ and all finite $J \subset I$ we have*

$$\Big\| \sum_{j\in J} Z_j(\omega)[e_j]_\sim \Big\|^2_{[H]_\sim^{1-\beta}} = \sum_{j\in J} \mu_j^{\beta-1} Z_j^2(\omega)\,. \tag{35}$$

Moreover, for each $\omega \in \Omega \setminus N$, the following statements are equivalent:

i) We have $\sum_{i\in I} \mu_i^{\beta-1} Z_i^2(\omega) < \infty$.

ii) We have $[X(\omega)]_\sim \in [H]_\sim^{1-\beta}$.

iii) We have $[X(\omega)]_\sim \in [L_2(\nu), [H]_\sim]_{1-\beta,2}$.

Moreover, if one and thus all statements are true for a fixed $\omega \in \Omega \setminus N$*, then* (35) *holds for all* $J \subset I$*, and the convergence in* (32) *is actually unconditional in the interpolation space* $[L_2(\nu), [H]_\sim]_{1-\beta,2}$.

By Theorem 4.1 we immediately see that almost all paths of the process X are contained in the space $[L_2(\nu), [H]_\sim]_{1-\beta,2}$, if and only if

$$\sum_{i\in I} \mu_i^{\beta-1} Z_i^2(\omega) < \infty \tag{36}$$

for P-almost all $\omega \in \Omega$. Moreover, in this case the convergence in (32) is P-almost surely unconditional in the space $[L_2(\nu), [H]_\sim]_{1-\beta,2}$. Note that in the case of $[L_2(\nu), [H]_\sim]_{1-\beta,2} \hookrightarrow L_\infty(\nu)$ the latter convergence implies $L_\infty(\nu)$-convergence of the Karhunen-Loève Expansion in (32) for P-almost all $\omega \in \Omega$. In Corollary 5.5, where we will consider this embedding situation again, we will see that significantly more can be said.

To further illustrate Theorem 4.1, let us fix an $\omega \in \Omega$ for which $[X(\omega)]_\sim \in [H]_\sim^{1-\beta}$ and (32) hold. Then, for all $\alpha \in [\beta, 1]$, we have both $[X(\omega)]_\sim \in [H]_\sim^{1-\alpha}$ and

$$[X(\omega)]_\sim = \sum_{i\in I} Z_i(\omega)[e_i]_\sim = \sum_{i\in I} \mu_i^{(\alpha-1)/2} Z_i(\omega) \left[\mu_i^{(1-\alpha)/2} e_i\right]_\sim .$$

Moreover, $([\mu_i^{(1-\alpha)/2} e_i]_\sim)_{i\in I}$ is an ONB of $[H]_\sim^{1-\alpha}$, and thus we see that, for each $m \in I$, the sum

$$\sum_{j=1}^{m} Z_j(\omega)[e_j]_\sim$$

is the best approximation of $[X(\omega)]_\sim$ in $[H]_\sim^{1-\alpha}$ for all $\alpha \in [\beta, 1]$ *simultaneously*.

Integrating (36) with respect to P and using (26) it is not hard to see that (36) is P-almost surely satisfied, if $\sum_{i\in I} \mu_i^\beta < \infty$. The following theorem characterizes this summability in terms of the path behavior of the process.

Theorem 4.2. *Let Assumption X be satisfied. Then, for* $0 < \beta < 1$*, the following statements are equivalent:*

i) We have $\sum_{i\in I} \mu_i^\beta < \infty$.

ii) There exists an $N \in \mathcal{A}$ *with* $P(N) = 0$ *such that* $[X(\omega)]_\sim \in [L_2(\nu), [H]_\sim]_{1-\beta,2}$ *holds for all* $\omega \in \Omega \setminus N$*. Furthermore,* $\Omega \setminus N \to [L_2(\nu), [H]_\sim]_{1-\beta,2}$ *defined by* $\omega \mapsto [X(\omega)]_\sim$ *is Borel measurable and we have*

$$\int_\Omega \left\| [X(\omega)]_\sim \right\|^2_{[L_2(\nu),[H]_\sim]_{1-\beta,2}} dP(\omega) < \infty .$$

Moreover, there exist constants $C_1, C_2 > 0$ such that, for all $J \subset I$, we have

$$C_1 \sum_{i\in I\setminus J} \mu_i^\beta \le \int_\Omega \Big\| [X(\omega)]_\sim - \sum_{j\in J} Z_j(\omega)[e_j]_\sim \Big\|^2_{[L_2(\nu),[H]_\sim]_{1-\beta,2}} dP(\omega) \le C_2 \sum_{i\in I\setminus J} \mu_i^\beta .$$

In general, almost sure finiteness in (36) is, of course, not equivalent to $\sum_{i\in I} \mu_i^\beta < \infty$, since by (26) this summability describes P-integrability of the random variable in (36). For Gaussian processes, however, we will see below that both conditions are in fact equivalent. The following lemma, which basically shows the equivalence of both notions under a martingale condition on $(Z_i^2)_{i\in I}$, is the key observation in this direction.

Lemma 4.3. *Let Assumption X be satisfied with $I = \mathbb{N}$. In addition, assume that, for all $i \ge 1$, we have $Z_i \in \mathcal{L}_4(P)$ and*

$$\mathbb{E}_P(Z_{i+1}^2 | \mathcal{F}_i) = \mu_{i+1} \,, \tag{37}$$

where $\mathcal{F}_i := \sigma(Z_1^2, \ldots, Z_i^2)$. Finally, assume that there exist constants $c > 0$ and $\alpha \in (0,1)$ such that

$$\mathrm{Var}\, Z_i^2 \le c\mu_i^{2-\alpha} \tag{38}$$

for all $i \ge 1$. Then, for all $\beta \in (\alpha, 1)$, the following statements are equivalent:

i) We have $\sum_{i\in I} \mu_i^\beta < \infty$.

ii) There exists an $N \in \mathcal{A}$ with $P(N) = 0$ such that (36) *holds for all $\omega \in \Omega \setminus N$.*

Combining the lemma above with Lemma 3.6 we now obtain the announced equivalence for Gaussian processes. It further shows that either almost all or almost no paths are contained in the considered interpolation space.

Corollary 4.4. *Let $(X_t)_{t\in T}$ be a Gaussian process for which Assumption X is satisfied. Then, for $0 < \beta < 1$, the following statements are equivalent:*

i) We have $\sum_{i\in I} \mu_i^\beta < \infty$.

ii) There exists an $N \in \mathcal{A}$ with $P(N) = 0$ such that (36) *holds for all $\omega \in \Omega \setminus N$.*

iii) There exists an $A \in \mathcal{A}$ with $P(A) > 0$ such that $[X(\omega)]_\sim \in [L_2(\nu), [H]_\sim]_{1-\beta,2}$ holds for all $\omega \in A$.

Moreover, all three statements are equivalent to the part ii) of Theorem 4.2.

So far, the developed theory is rather abstract. Our final goal in this section is to illustrate how our result can be used to investigate path properties of certain families of processes. These considerations will be based on the following corollary, which, roughly speaking, shows that the sample paths of a process are about $d/2$-less smooth than the functions in its RKHS.

Corollary 4.5. *Let $T \subset \mathbb{R}^d$ be a bounded subset that satisfies the strong local Lipschitz condition and $T = \operatorname{int} \overline{T}$. Moreover, let ν be the Lebesgue measure on T and $(X_t)_{t\in T}$ be a stochastic process satisfying Assumption X. Assume that $H \hookrightarrow W^m(T)$ for some $m > d/2$. Then, for all $s \in (0, m - d/2)$, we have*

$$[X(\omega)]_\sim \in B^s_{2,2}(T) \tag{39}$$

for P-almost all $\omega \in \Omega$. Moreover, there exists a constant $C > 0$ such that, for all $J \subset I$, we have

$$\int_\Omega \Big\| [X(\omega)]_\sim - \sum_{j\in J} Z_j(\omega)[e_j]_\sim \Big\|^2_{B^s_{2,2}(T)} dP(\omega) \le C \sum_{i\in I\setminus J} \mu_i^{1-s/m} ,$$

and if $H = W^m(T)$, there exist constants $C_1, C_2 > 0$ such that for all $i \in I$ we have

$$C_1 i^{-\frac{2(m-s)}{d}+1} \le \int_\Omega \Big\| [X(\omega)]_\sim - \sum_{j=1}^{i} Z_j(\omega)[e_j]_\sim \Big\|^2_{B^s_{2,2}(T)} dP(\omega) \le C_2 i^{-\frac{2(m-s)}{d}+1} .$$

Finally, if $(X_t)_{t\in T}$ is a Gaussian process with $H = W^m(T)$, then the results are sharp in the sense that (39) *does not hold with strictly positive probability for $s := m - d/2$.*

Note that for Gaussian processes with $H = W^m(T)$, Corollaries 4.5 and 4.4 show that (39) holds with some positive probability, if and only if, it holds with probability one, and the latter is also equivalent to $m > s + d/2$.

For general processes with $H = W^m(T)$ the smoothness exponent s is also sharp in the sense that (39) does not hold for $s := m - d/2$ and P-almost all $\omega \in \Omega$, provided that the process satisfies the assumptions of Lemma 4.3 for some $\alpha \in (0, \frac{d}{2m})$. The proof of this generalization is an almost literal copy of the proof of Corollary 4.5, and thus we decided to omit it.

Corollary 4.5 provides analytic properties of the sample paths in terms of $B^s_{2,2}(T)$, whenever $H \hookrightarrow W^m(T)$ is known. For *weakly stationary* processes the same result has been recently shown in [32, Theorem 3 and Remark 1] by different techniques.

Fortunately, an inclusion of the form $H \hookrightarrow W^m(T)$ is known for many classes of processes, or kernels, respectively. The following, by no means complete, list of examples illustrate this.

We begin with a class of processes which include Lévy processes.

Example 4.6. *Let $(X_t)_{t\in T}$ be a stochastic process satisfying Assumption X for $T = [0, t_0]$ and the Lebesgue measure ν on T. Furthermore, assume that the kernel is given by*

$$k(s,t) = \sigma^2 \cdot \min\{s,t\} , \qquad s,t \in [0,t_0],$$

where $\sigma > 0$ is some constant. It is well-known, see e.g. [16, Example 8.19], that the RKHS of this kernel is continuously embedded into $W^1(T)$. Consequently, for all $s \in (0, 1/2)$, we have

$$[X(\omega)]_\sim \in B^s_{2,2}(T)$$

for P-almost all $\omega \in \Omega$. Note that the considered class of processes include Lévy processes, and for these processes, it has been shown in [15] by different means that their paths are also contained in $B^s_{p,\infty}(T)$ for all $s \in (0, 1/2)$ and $p > 2$ with $sp < 1$. Interestingly, this is is equivalent to our result above.

Indeed, if we fix a pair of s and p satisfying the assumptions of [15], we have $s_0 := s-1/p+1/2 < 1/2$. For $\varepsilon > 0$ with $s_0+\varepsilon < 1/2$ we then obtain $s_0+\varepsilon-1/2 > s-1/p$ and $s_0 + \varepsilon > s$ and thus $B^{s_0+\varepsilon}_{2,2}(T) \hookrightarrow B^s_{p,\infty}(T)$ by [30, p. 82]. Consequently, our result implies that of [15]. Conversely, if we fix an $s < 1/2$, there is an $\varepsilon > 0$ with $s + 2\varepsilon < 1/2$, and for $s_0 := s + \varepsilon$ and $p_0 := (s + 2\varepsilon)^{-1}$, we have $s_0 > s$ and $s_0 - 1/p_0 > s - 1/2$, so that $B^{s_0}_{p_0,\infty}(T) \hookrightarrow B^s_{2,2}(T)$ by [30, p. 82]. Since we also have $s_0 < 1/2$, $p_0 > 2$ and $s_0 p_0 < 1$, we then see that the result of [15] implies ours.

Finally, for the Brownian motion, it is well-known that there exists a version whose sample paths are contained in $B^s_{\infty,\infty}(T)$ for all $s \in (0, 1/2)$, and finer results can be found in [29].

The following example includes the Ornstein-Uhlenbeck processes. Note that although the kernel in this example look quite different to the one of Example 4.6 the results on the smoothness properties of the paths are identical.

Example 4.7. *Let $(X_t)_{t\in T}$ be a stochastic process satisfying Assumption X for $T = [0, t_0]$ and the Lebesgue measure ν on T. Furthermore, assume that the kernel is given by*

$$k(t, t') = ae^{-\sigma|t-t'|}\,, \qquad t, t' \in [0, t_0],$$

where $a, \sigma > 0$ are some constants. It is well-known, see e.g. [4, p. 316] and [24, Example 5C], that the RKHS of this kernel equals $W^1(T)$ up to equivalent norms. Consequently, for all $s \in (0, 1/2)$, we have

$$[X(\omega)]_\sim \in B^s_{2,2}(T) \tag{40}$$

for P-almost all $\omega \in \Omega$. Note that the considered class of processes include a specific form of the Ornstein-Uhlenbeck process, see [16, Example 8.4].

By subtracting the one-dimensional C^∞-kernel $(t, t') \mapsto ae^{-\sigma(t+t')}$ from k, we see that (40) *also holds for processes having the kernel*

$$\tilde{k}(t, t') = ae^{-\sigma|t-t'|} - ae^{-\sigma(t+t')}\,, \qquad t, t' \in [0, t_0].$$

Recall that the classical Ornstein-Uhlenbeck processes belong to this class of processes.

The following example considers processes on higher dimensional domains with potentially smoother sample paths. It is in particular interesting for certain statistical methods, see [34, 28, 27, 40, 13, 39, 33], since the considered family of covariance functions allows for a high flexibility in these methods. Moreover, note that for $d = 1$ and $\alpha = 1/2$ the previous example is recovered.

Example 4.8. *Let $T \subset \mathbb{R}^d$ be an open and bounded subset satisfying the strong local Lipschitz condition and ν be the Lebesgue measure on T. Furthermore, let $(X_t)_{t\in T}$ be* *a stochastic process satisfying Assumption X. Assume that its covariance is a Matérn kernel of order $\alpha > 0$, that is*

$$k(s,t) = a\big(\sigma\|s-t\|_2\big)^{\alpha} H_\alpha\big(\sigma\|s-t\|_2\big)\,, \qquad\qquad s,t \in T,$$

where $a, \sigma > 0$ are some constants and H_α denotes the modified Bessel function of the second type of order α. Then up to equivalent norms the RKHS $H_{\alpha,\sigma}(T)$ of this kernel is $B_{2,2}^{\alpha+d/2}(T)$, see [41, Corollary 10.13] together with [31, Theorem 5.3], as well as [6] for a generalization. Consequently, for all $s \in (0,\alpha)$, we have

$$[X(\omega)]_\sim \in B_{2,2}^s(T)$$

for P-almost all $\omega \in \Omega$.

For $d = 1$ and $\alpha = k + r$ with $k \in \mathbb{N}$ and $r \in (1/2, 1]$, it was shown in [8], cf. also [14], that there exists a version of the process with k-times continuously differentiable paths. Our result improves this. Indeed, for d and α as above, we clearly find an $s \in (0,\alpha)$ with $s - k > 1/2$ and since, for this s, we have $B_{2,2}^s(T) \hookrightarrow C^k(T)$, see e.g. [38, Theorem 8.4], we see that P-almost all paths $X(\omega)$ equal ν-almost everywhere a k-times continuously differentiable function. We will show in the next section that there actually exist a version $(Y_t)_{t\in T}$ of the process with $Y(\omega) \in B_{2,2}^s(T)$ almost surely, so that our result does improve the above mentioned classical result in [8].

5 Sample Paths Contained in RKHSs

So far we have seen that, under some summability assumptions, the ν-equivalence classes of the process are contained in a suitable interpolation space. Now recall from Section 2 that these interpolation spaces can sometimes be viewed as RKHSs, too. The goal of this section is to present conditions under which a suitable version of the process has actually its paths in this RKHS. In particular, we will see that under stronger summability conditions on the eigenvalues such a path behavior occurs, in a certain sense, automatically.

Let us begin by fixing the following set of assumptions, which in particular ensure that $k_S^{1-\beta}$ can be constructed.

Assumption KS. *Let Assumption K be satisfied. Moreover, let $0 < \beta < 1$ and $S \subset T$ be a measurable set with $\nu(T \setminus S) = 0$ such that, for all $t \in S$, we have*

$$\sum_{i\in I} \mu_i e_i^2(t) = k(t,t) \tag{41}$$

$$\sum_{i\in I} \mu_i^{1-\beta} e_i^2(t) < \infty\,. \tag{42}$$

Note that if H is separable, we can always find a set S of full measure ν for which (41) holds, see [36, Corollary 3.2]. For such H, Assumption KS thus reduces to assuming that we can construct $k_S^{1-\beta}$, and the latter is possible, if, e.g. $\sum_{i\in I} \mu_i^{1-\beta} < \infty$, see (18). Moreover, recall from Lemma 2.6 that (41) holds for $S = T$ if Assumption CK is satisfied, and if, in addition, we have $[L_2(\nu), [H]_\sim]_{1-\beta,2} \hookrightarrow L_\infty(\nu)$, then Theorem 2.7 shows that (42) also holds for $S = T$.

Our first result characterizes when a suitable version of our process $(X_t)_{t\in T}$ has its paths in the corresponding RKHS $H_S^{1-\beta}$.

Theorem 5.1. *Let Assumptions X and KS be satisfied. Then the following statements are equivalent:*

i) There exists a measurable $N \subset \Omega$ with $P(N) = 0$ such that for all $\omega \in \Omega \setminus N$ we have

$$\sum_{i\in I} \mu_i^{\beta-1} Z_i^2(\omega) < \infty \,. \tag{43}$$

ii) There exists a $(\mathcal{A} \otimes \mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$ such that, for P-almost all $\omega \in \Omega$, we have

$$Y(\omega)_{|S} \in H_S^{1-\beta} \,. \tag{44}$$

Moreover, if one and thus both statements are true, we have for P-almost all $\omega \in \Omega$

$$Y(\omega)_{|S} = \sum_{i\in I} Z_i(\omega)(e_i)_{|S} \,, \tag{45}$$

where the convergence is unconditional in $H_S^{1-\beta}$.

If (43) is P-almost surely satisfied then Theorem 5.1 strengthens Theorem 4.1 in the sense that $[X(\omega)]_\sim \in [H]_\sim^{1-\beta}$ is replaced by $Y(\omega)_{|S} \in H_S^{1-\beta}$. Moreover, unlike (35), which only gives $[H]_\sim^{1-\beta}$-convergence of the Karhunen-Loève Expansion in (32), the expansion (45) converges in $H_S^{1-\beta}$, which in particular implies pointwise convergence at all $t \in S$.

We already know that the Fourier coefficient condition (43) can be ensured by a summability condition on the eigenvalues. Like in Theorem 4.2, this summability can be characterized by the path behavior of the version $(Y_t)_{t\in T}$ as the following theorem shows.

Theorem 5.2. *Let Assumptions X and KS be satisfied. Then the following statements are equivalent:*

i) We have $\sum_{i\in I} \mu_i^\beta < \infty$.

ii) We have $k_S^1 \ll k_S^{1-\beta}$.

iii) There exists a $(\mathcal{A} \otimes \mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$ such that, for P-almost all $\omega \in \Omega$, we have $Y(\omega)_{|S} \in H_S^{1-\beta}$, and

$$\int_\Omega \|Y(\omega)_{|S}\|^2_{H_S^{1-\beta}} \, dP(\omega) < \infty \,. \tag{46}$$

Let us compare the previous two theorems in the case of $S = T$ with the results of Lukić and Beder in [22]. Their Theorem 5.1 shows that $k_T^1 \ll k_T^{1-\beta}$ implies (44), and, their Corollary 3.2 conversely shows that (46) implies $k_T^1 \ll k_T^{1-\beta}$. Clearly, the difference between these two implications is exactly the difference between (46) and (44), and this difference is exactly described by Theorems 5.1 and 5.2. While in this sense, the latter two theorems completely clarify the situation for the space $H_T^{1-\beta}$, it seems fair to say that the less exact results in [22] are more general as *arbitrary* RKHS $\bar{H}$ satisfying $H \hookrightarrow \bar{H}$ are considered.

The following corollary, which considers the case of Gaussian processes, basically recovers the findings of [22, Section 7]. We mainly state it here for the sake of completeness.

Corollary 5.3. *Let $(X_t)_{t\in T}$ be a Gaussian process for which Assumptions X and KS are satisfied. Then the following statements are equivalent:*

i) We have $\sum_{i\in I} \mu_i^\beta < \infty$.

ii) We have $k_S^1 \ll k_S^{1-\beta}$.

iii) There exists a $(\mathcal{A}\otimes\mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$ such that, for P-almost all $\omega \in \Omega$, we have

$$Y(\omega)_{|S} \in H_S^{1-\beta}\,.$$

iv) There exists a $(\mathcal{A}\otimes\mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$ and an $A \in \mathcal{A}$ with $P(A) > 0$ such that, for all $\omega \in A$, we have

$$Y(\omega)_{|S} \in H_S^{1-\beta}\,.$$

For general processes satisfying the assumptions made in Lemma 4.3 for some $\alpha \in (0,1)$, the equivalences *i)* $\Leftrightarrow$ *ii)* $\Leftrightarrow$ *iii)* of Corollary 5.3 also hold for all $\beta \in (\alpha, 1)$. Indeed, the implication *iii)* $\Rightarrow$ *i)* can be shown by Lemma 4.3, and the remaining implications actually do not require the Gaussian assumption at all.

If we wish to find an RKHS $\bar{H}$ that contains the paths of a suitable version of the process by the results presented so far, we need to know the eigenvalues and eigenfunctions as well as the interpolation spaces *exactly*. However, obtaining the exact eigenvalues and -functions of T_k is often a very difficult, if not impossible, task, and the interpolation spaces may not be readily available, either. The following two corollaries address this issue by presenting a sufficient condition for the existence of such an RKHS $\bar{H}$.

Corollary 5.4. *Let Assumption X be satisfied, H be separable, and $\bar{H}$ be an RKHS on T with kernel $\bar{k}$ such that $H \hookrightarrow \bar{H}$. Let us further assume that $\bar{H}$ is compactly embedded into $L_2(\nu)$ and that*

$$\sum_{i=1}^{\infty} \varepsilon_i^\alpha (I_{\bar{k}}) < \infty \tag{47}$$

for some $\alpha \in (0,1]$. Then, for all $\beta \in [\alpha/2, 1-\alpha/2]$, there exists a measurable $S \subset T$ with $\nu(T \setminus S) = 0$ such that the following statements are true:

i) Both $H_S^{1-\beta}$ and $\bar{H}_S^{1-\beta}$ exist, and we have $H_S^{1-\beta} \hookrightarrow \bar{H}_S^{1-\beta}$.

ii) There exists a $(\mathcal{A} \otimes \mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$ such that $Y(\omega)_{|S} \in H_S^{1-\beta}$ for P-almost all $\omega \in \Omega$, and (46) *holds.*

Corollary 5.4 shows that in order to construct an RKHS containing paths on a set S of full measure ν we do not necessarily need to know the eigenvalues and -functions exactly. Instead, it suffices to have an RKHS $\bar{H}$ with $H \hookrightarrow \bar{H}$ for which we know both, entropy number estimates of the map $I_{\bar{k}}$ and the interpolation spaces of $\bar{H}$ with $L_2(\nu)$. Namely, if (47) is satisfied, then the version $(Y_t)_{t\in T}$ obtained by Corollary 5.4 satisfies $Y(\omega)_{|S} \in \bar{H}_S^{1-\beta}$ for P-almost all $\omega \in \Omega$ and combining this with $H_S^{1-\beta} \hookrightarrow \bar{H}_S^{1-\beta}$ we see that we have $\bar{H}_S^{1-\beta}$-convergence in (45). Similarly to Corollary 4.5 it is further possible to upper bound the average speed of $\bar{H}_S^{1-\beta}$-convergence in (45) with the help of the entropy numbers of $I_{\bar{k}}$. We omit the details for the sake of brevity.

Moreover, $Y(\omega)_{|S} \in H_S^{1-\beta} \subset \bar{H}_S^{1-\beta}$ for P-almost all $\omega \in \Omega$ shows that P-almost all paths also enjoy a representation of the form

$$Y(\omega)_{|S} = \sum_{j\in J} \bar{Z}_j(\omega)\bar{e}_j \,,$$

where the convergence is unconditional in $\bar{H}_S^{1-\beta}$, $(\bar{e}_j)_{j\in J}$ is the family obtained by Theorem 2.1 for the operator $T_{\bar{k}_S^1}$ and $(\bar{Z}_j)_{j\in J}$ is a suitable family of random variables such that $\sum_{j\in J} \bar{\mu}_j^{\beta-1} \bar{Z}_j^2(\omega) < \infty$ for P-almost all $\omega \in \Omega$. Following the logic above, this representation may be easier at hand than the standard Karhunen-Loève expansion, but its deeper investigation is beyond the scope of this paper.

The following corollary provides a result in the same spirit for the case $S = T$. In particular, it provides two sufficient conditions under which there exists an RKHS containing almost all paths of a suitable version. This answers a question raised in [21].

Corollary 5.5. *Let Assumption X be satisfied, H be separable, and $\bar{H}$ be an RKHS on T with kernel $\bar{k}$ such that both $H \hookrightarrow \bar{H}$ and $\bar{H}$ is compactly embedded into $L_2(\nu)$. Furthermore, assume that $(T, \mathcal{B}, \nu)$ and $\bar{k}$ satisfy Assumption CK, and that, for some $\beta \in (0, 1/2]$, one of following assumptions are satisfied:*

i) The eigenfunctions $(\bar{e}_j)_{j\in J}$ of $T_{\bar{k}}$ are uniformly bounded, i.e. $\sup_{j\in J} \|\bar{e}_j\|_\infty < \infty$, and we have

$$\sum_{i=1}^{\infty} \varepsilon_i^{2\beta}(I_{\bar{k}}) < \infty, .$$

ii) We have $[L_2(\nu), [\bar{H}]_\sim]_{1-\beta,2} \hookrightarrow L_\infty(\nu)$.

The the following statements hold:

i) The kernels $k_T^{1-\beta}$ and $\bar{k}_T^{1-\beta}$ exist, are bounded, and we have $H_T^{1-\beta} \hookrightarrow \bar{H}_T^{1-\beta}$.

ii) There exists a $(\mathcal{A}\otimes\mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$ such that $Y(\omega) \in H_T^{1-\beta}$ for all $\omega \in \Omega$, and (46) *holds.*

iii) All paths of Y are bounded and $\tau(H)$-continuous.

iv) For P-almost all $\in \Omega$, the expansion (45) *converges uniformly in t on $S = T$.*

v) If there is a separable and metrizable topology τ on T such that $\tau(H) \subset \tau$ and almost all paths of X are τ-continuous, then $X(\omega) = Y(\omega)$ for P-almost all $\omega \in \Omega$. In particular, this holds if almost all paths of X are $\tau(H)$-continuous and $\tau(H)$ is Hausdorff.

Note that in the situation of part *iv)* of Corollary 5.5 the Karhunen-Loève Expansion in (32) converges in $\ell_\infty(T)$ for P-almost all $\omega \in \Omega$. Moreover, note the $\tau(H)$-continuity of the paths obtained in *iii)* and *iv)* is potentially stronger than the τ-continuity, where τ is a "natural" topology of T.

The last result of this section improves Corollary 4.5. Note that it directly applies to the processes considered in Example 4.8.

Corollary 5.6. *Let $T \subset \mathbb{R}^d$ be a bounded subset that satisfies the strong local Lipschitz condition and $T = \operatorname{int} \overline{T}$. Moreover, let ν be the Lebesgue measure on T and $(X_t)_{t\in T}$ be a stochastic process satisfying Assumption X. Assume that $H \hookrightarrow W^m(T)$ for some $m > d$. Then the following statements hold*

i) For all $s \in (d/2, m-d/2)$, there exists a $(\mathcal{A}\otimes\mathcal{B})$-measurable version such that, for all $\omega \in \Omega$, we have

$$Y(\omega) \in B^s_{2,2}(T)\,. \tag{48}$$

Moreover, for P-almost all $\omega \in \Omega$ we have with unconditional convergence in $B^s_{2,2}(T)$:

$$Y(\omega) = \sum_{i\in I} Z_i(\omega) e_i\,. \tag{49}$$

ii) If $(X_t)_{t\in T}$ is a Gaussian process with $H = W^m(T)$, then the results are sharp in the sense that (48) *does not hold with strictly positive probability for $s := m - d/2$.*

By [1, Theorem 7.37], we immediately see that the convergence in (49) is uniform in t. Moreover, if $s > k + 1/2$ for some $k \in \mathbb{N}$, then the convergence is also in $C^k(T)$, see e.g. [38, Theorem 8.4].

Finally, like for Corollary 4.5, the sharpness result in *ii)* can be extended to a broader class of processes. We refer to our remarks following Corollary 4.5.

6 Proofs

6.1 Proofs of Preliminary Results

Proof of Lemma 2.2: *i).* Let us pick an $f \in \hat{H}^\beta_S$. Then there exists a sequence $(a_i) \in \ell_2(I)$ such that $f = \sum_{i\in I} a_i \mu_i^{\beta/2} \mathbf{1}_S e_i$, where the convergence is in $\hat{H}^\beta_S$ and thus also pointwise. Consequently, we find

$$\mathbf{1}_R f = \mathbf{1}_R \sum_{i\in I} a_i \mu_i^{\beta/2} \mathbf{1}_S e_i = \sum_{i\in I} a_i \mu_i^{\beta/2} \mathbf{1}_R \mathbf{1}_S e_i = \sum_{i\in I} a_i \mu_i^{\beta/2} \mathbf{1}_R e_i\,.$$

Now the assertion easily follows from the definitions of the spaces $\hat{H}^\beta_S$ and $\hat{H}^\beta_R$.

ii). Can be shown analogously to *i).*

ii). Again, this can be shown analogously to *i).*

iv). We obviously have $[\hat{e}_i]_\sim = [e_i]_\sim$ for all $i \in I$. Moreover, for $(a_i) \in \ell_2(I)$ we have

$$\Big[\sum_{i\in I} a_i \mu_i^{\beta/2} \hat{e}_i\Big]_\sim = \sum_{i\in I} a_i \mu_i^{\beta/2} [\hat{e}_i]_\sim$$

with convergence in $L_2(\nu)$ by the continuity of $I_{\hat{k}^\beta_S} : \hat{H}^\beta_S \to L_2(\nu)$. Combining both with the definition of the spaces $\hat{H}^\beta_S$ and $[H]^\beta_\sim$ yields the assertion. □

For the proof of Lemma 2.3, we need to recall some basics on singular numbers. To begin with, let us recall that for an arbitrary compact operator $S : H_1 \to H_2$ acting between two Hilbert spaces H_1 and H_2, the i-th singular number, see e.g. [5, p. 242] is defined by

$$s_i(S) := \sqrt{\mu_i(S^*S)}\,, \tag{50}$$

where $\mu_i(S^*S)$ denotes the i-th non-zero eigenvalue of the compact, positive and self-adjoint operator S^*S. As usual, these eigenvalues are assumed to be ordered with duplicates according to their geometric multiplicities. In addition, we extend the sequence of eigenvalues by zero, if we only have finitely many non-zero eigenvalues. Now, for a compact, self-adjoint and positive $T : H \to H$, this definition gives

$$s_i(T) = \sqrt{\mu_i(T^*T)} = \sqrt{\mu_i(T^2)} = \mu_i(T)\,, \qquad\qquad i \geq 1, \tag{51}$$

where the last equality follows from the classical spectral theorem for such T, see e.g. [17, Theorem V.2.10 on page 260] or [42, Satz VI.3.2]. For compact $S : H_1 \to H_2$ and $T := S^*S$ we thus find

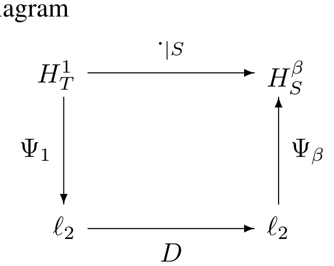
$$s_i^2(S) = \mu_i(S^*S) = \mu_i(T) = s_i(T) \tag{52}$$

for all $i \geq 1$. Consequently, we have $(s_i(S)) \in \ell_2$ if and only if $(s_i(T)) \in \ell_1$. Moreover, T is nuclear, if and only if $(s_i(T)) \in \ell_1$, see e.g. [42, Satz VI.5.5] or [5, p. 245ff], while S is Hilbert-Schmidt if and only if $(s_i(S)) \in \ell_2$, see e.g. [5, p. 250], [26, Prop. 2.11.17], or [42, p. 246].

Proof of Lemma 2.3: We first observe that, for $i \in I$, we have

$$\cdot_{|S}(\mu_i^{1/2} e_i) = \mu_i^{1/2} e_{i|S} = \mu_i^{(1-\beta)/2} \mu_i^{\beta/2} e_{i|S}\,. \tag{53}$$

Since $(\mu_i^{1/2} e_i)_{i\in I}$ and $(\mu_i^{\beta/2} e_{i|S})_{i\in I}$ are ONBs of H^1_T and H^β_S, respectively, we obtain the following commutative diagram



where Ψ_i denote the isometric isomorphisms that map each Hilbert space element to its sequence of Fourier coefficients with respect to the ONBs above, and D is the diagonal operator with respect to the sequence $(\mu_i^{(1-\beta)/2})_{i\in I}$. Since the latter sequence converges to zero, D is compact, and thus so is the restriction operator.

i) $\Leftrightarrow$ *ii).* We first observe that (53) yields

$$\big\| \cdot_{|S}(\sqrt{\mu_i}e_i)\big\|^2_{H_S^\beta} = \mu_i^{1-\beta}\,, \qquad\qquad i \in I.$$

Since $(\sqrt{\mu_i}e_i)_{i\in I}$ is an ONB of H_T^1, the equivalence *i)* $\Leftrightarrow$ *ii)* immediately follows from the fact, see e.g. [43, p. 243f], that $\cdot_{|S} : H_T^1 \to H_S^\beta$ is Hilbert-Schmidt, if and only if

$$\sum_{i\in I} \| \cdot_{|S}\, (\sqrt{\mu_i}e_i)\|^2_{\hat H_S^\beta} < \infty\,.$$

i) $\Leftrightarrow$ *iii).* The restriction operator admits the following natural factorization

$$\begin{array}{ccc} H_T^1 & \xrightarrow{\ \ \cdot_{|S}\ \ } & H_S^\beta \\ & {}_{\cdot_{|S}}\searrow \qquad \nearrow_{I_{k_S^1,k_S^\beta}} & \\ & H_S^1 & \end{array}$$

where there restriction operator $\cdot_{|S} : H_T^1 \to H_S^1$ is an isometric isomorphism. Consequently, $\cdot_{|S} : H_T^1 \to H_S^\beta$ is Hilbert-Schmidt, if and only if $I_{k_S^1,k_S^\beta}$ is Hilbert-Schmidt. In view of the desired equivalence, it suffices to show that $I_{k_S^1,k_S^\beta}$ is Hilbert-Schmidt, if and only if $I_{k_S^1,k_S^\beta} \circ S_{k_S^1,k_S^\beta}$ is nuclear. However, since $S_{k_S^1,k_S^\beta} = I^*_{k_S^1,k_S^\beta}$, this equivalence is a simple consequence of the remarks on singular numbers made in front of this proof, if we consider the compact operator $I_{k_S^1,k_S^\beta} : H_S^1 \to H_S^\beta$ for S^*. □

Lemma 6.1. *Let $(T,\mathcal{B})$ be a measure space and k be a kernel on T with RKHS H and canonical feature map $\Phi : T \to H$. Then the following statements are true:*

i) The topology τ_k is the smallest topology τ on T for which k is τ-continuous. Moreover, we have

$$\tau_k = \tau\big(\Phi : T \to (H, \|\cdot\|_H)\big)\,,$$

where $\tau(\Phi : T \to (H, \|\cdot\|_H))$ denotes the initial topology of Φ with respect to the norm-topology on H.

ii) The topology $\tau(H)$ is the smallest topology τ on T for which Φ is continuous with respect to the weak topology w on H, that is

$$\tau(H) = \tau\big(\Phi : T \to (H, w)\big)\,.$$

In particular, we have $\tau(H) \subset \tau_k$, and in general, the converse inclusion is not even true for $T = [0,1]$.

iii) If H is separable and k is bounded, then there exits a pseudo-metric on T that generates $\tau(H)$ and $\tau(H)$ is separable. Moreover, we have $\tau(H) \subset \sigma(H)$.

iv) If $\tau(H) \subset \mathcal{B}$, then all $f \in H$ are $\mathcal{B}$-measurable.

Proof of Lemma 6.1: *i).* Both assertions are shown in [35, Lemma 4.29].

ii). Let $\iota : H \to H'$ be the Fréchet-Riesz isometric isomorphism. Then we have $f = (\iota f) \circ \Phi$ for all $f \in H$ by the reproducing property. Let us first prove the inclusion "$\subset$". To this end, we fix an $f \in H$ and an open $U \subset \mathbb{R}$. We define $O := (\iota f)^{-1}(U)$. Then we have $O \in w$ and thus

$$f^{-1}(U) = \big((\iota f) \circ \Phi\big)^{-1}(U) = \Phi^{-1}\big((\iota f)^{-1}(U)\big) = \Phi^{-1}(O) \in \tau(\Phi : T \to (H, w))\,.$$

The inclusion "$\subset$" then follows from the fact that the set of considered pre-images $f^{-1}(U)$ is a sub-base of $\tau(H)$. To show the converse inclusion, we fix an $O \in w$ for which there exist an $f \in H$ and an open $U \subset \mathbb{R}$ with $O = (\iota f)^{-1}(U)$. Then we find

$$\Phi^{-1}(O) = \Phi^{-1}\big((\iota f)^{-1}(U)\big) = \big((\iota f) \circ \Phi\big)^{-1}(U) = f^{-1}(U) \in \tau(H)\,.$$

Since the set of such pre-images $\Phi^{-1}(O)$ is a sub-base of $\tau(\Phi : T \to (H, w))$ we obtained the desired inclusion.

Finally, $\tau(H) \subset \tau_k$ directly follows from combining part *i)* and *ii)* with the fact that the norm topology on H is finer than the weak topology. To show that the converse inclusion does not hold for $T = [0, 1]$, we denote the usual topology on this T by τ. Then [20] showed that there exists a bounded separately τ-continuous kernel k on T that is not τ-continuous. This gives $\tau(H) \subset \tau$ by [35, Lemma 4.28] and $\tau_k \not\subset \tau$, and thus $\tau_k \not\subset \tau(H)$.

iii). Since H' is separable, we know that for every bounded subset $A' \subset H'$ the relative topology $w^*_{|A'}$ on A', where w^* denotes the weak* topology on H', is induced be a metric, see e.g. [23, Corollary 2.6.20]. Moreover, we have $\iota^{-1}(w^*) = w$, where w is the weak topology on H. For all bounded $A \subset H$, the relative topology $w_{|A}$ on A is thus induced by a metric. Now k is bounded by assumption, and hence $A := \Phi(T)$ is bounded, see e.g. [35, p. 124]. Consequently, there exists a metric d on A that generates $w_{|A}$. Let us consider the map $\tilde{\Phi} : T \to A$, defined by $\tilde{\Phi}(t) := \Phi(t)$ for all $t \in T$. By the already proven part *ii)* and the universal property of the initial topology $\tau(\mathrm{id} : A \to (H, w)) = w_{|A}$ we then find

$$\tau(H) = \tau\big(\Phi : T \to (H, w)\big) = \tau\big(\tilde{\Phi} : T \to (A, w_{|A})\big)\,.$$

From this we easily derive that $(t, t') \mapsto d(\Phi(t), \Phi(t'))$ is the desired pseudo-metric. To see that $\tau(H)$ is separable, we recall that closed unit ball $B_{H'}$ of H' is w^*-compact by Alaoglu's theorem. Consequently, $(B_{H'}, w^*_{|B_{H'}})$ is a compact metric space, and thus separable. Arguing as above, and using that $w_{|B_H} = \iota^{-1}(w^*_{|B_{H'}})$ is metrizable, we see that $w_{|A}$ is separable for $A := \Phi(T)$, and hence so is $\tau(H)$.

Finally, since $\tau(H)$ is the initial topology of H, the collection of sets $f^{-1}(O)$, where $f \in H$ and $O \subset \mathbb{R}$ open, form a sub-base of $\tau(H)$, and since open $O \subset \mathbb{R}$ are Borel measurable, we also have $f^{-1}(O) \in \sigma(H)$ for all such f and O. Consequently, finite intersections taken from this sub-base are contained in $\sigma(H)$, too, and the collection of these intersections form a base of $\tau(H)$. Now every $\tau(H)$-open set is the union of such intersections. However, we have just seen that $\tau(H)$ is separable and generated by a pseudo-metric, which by a standard argument shows that $\tau(H)$ is second countable. Consequently, $\tau(H)$ is Lindelöf, see [18, p. 49], that is each open cover has a countable sub-cover. Consequently, each $\tau(H)$-open set is a countable union of the above intersections, and thus contained in $\sigma(H)$.

iv). From $\tau(H) \subset \mathcal{B}$ we conclude that $\sigma(H) \subset \sigma(\tau(H)) \subset \mathcal{B}$, which shows the assertion. □

Proof of Lemma 2.6: Let us pick an $f \in H$ with $f \neq 0$. Then $\{f \neq 0\}$ is $\tau(H)$-open and non-empty, and thus we have $\nu(\{f \neq 0\}) > 0$, that is $I_k f = [f]_\sim \neq 0$. Now, $k = k_T^1$ follows from [36, Theorem 3.1]. □

Lemma 6.2. *Let (T, τ) be a topological space, $I \subset \mathbb{N}$, and $(g_i)_{i \in I}$ be a family of continuous functions $g_i : T \to \mathbb{R}$. Then, all $t \in T$, the following statements hold:*

i) If $\sum_{i \in I} g_i^2(t) = \infty$, then, for all $M > 0$, there exists an open $O \subset T$ with $t \in O$ and

$$\sum_{i \in I} g_i^2(s) > M\,, \qquad\qquad s \in O\,.$$

ii) If $\sum_{i \in I} g_i^2(t) < \infty$, then, for all $\varepsilon > 0$, there exists an open $O \subset T$ with $t \in O$ and

$$\sum_{i \in I} g_i^2(s) > \sum_{i \in I} g_i^2(t) - \varepsilon\,, \qquad\qquad s \in O\,.$$

Proof of Lemma 6.2: *i).* By assumption, there exists a finite $J \subset I$ such that

$$\sum_{i \in J} g_i^2(t) > 2M\,.$$

Since the g_i^2 are continuous, there then exist, for all $i \in J$, an open $O_i \subset T$ with $t \in O_i$ and $|g_i^2(s) - g_i^2(t)| < M/|J|$ for all $s \in O_i$. For the open set $O := \bigcap_{i \in J} O_i$ and $s \in O$ we then obtain $t \in O$ and

$$\left|\sum_{i \in J} g_i^2(s) - \sum_{i \in J} g_i^2(t)\right| \leq \sum_{i \in J} |g_i^2(s) - g_i^2(t)| < M\,.$$

This yields

$$\sum_{i \in I} g_i^2(s) \geq \sum_{i \in J} g_i^2(s) > \sum_{i \in J} g_i^2(t) - M > M\,.$$

ii). Let us fix an $\varepsilon > 0$. Then there exists a finite $J \subset I$ such that

$$\sum_{i \in J} g_i^2(t) > \sum_{i \in I} g_i^2(t) - \varepsilon\,.$$

This time we pick open $O_i \subset T$ with $t \in O_i$ and $|g_i^2(s) - g_i^2(t)| < \varepsilon/|J|$ for all $s \in O_i$. Repeating the calculations above, we obtain the assertion for 2ε. □

Proof of Theorem 2.7: By our assumption and (13) we have $[H]^\beta_\sim \hookrightarrow L_\infty(\nu)$, and thus [36, Theorem 5.3] shows that there exist an $N \in \mathcal{B}$ and a constant $\kappa \in [0,\infty)$ such that $\nu(N) = 0$ and

$$\sum_{i\in I} \mu_i^\beta e_i^2(t) \leq \kappa^2 \,, \qquad\qquad t \in T \setminus N. \tag{54}$$

Moreover, by the definition of $\tau(H)$ we know that all e_i are $\tau(H)$-continuous.

Let us first show that (14) holds for $S := T$. To this end, we assume the converse, that is, there exists a $t \in T$ with

$$\sum_{i\in I} \mu_i^\beta e_i^2(t) = \infty \,.$$

By Lemma 6.2 there then exists an $O \in \tau(H)$ with $t \in O$ and

$$\sum_{i\in I} \mu_i^\beta e_i^2(s) > \kappa^2 \,, \qquad\qquad s \in O \,. \tag{55}$$

Since ν is assumed to be k-positive, we conclude that $\nu(O) > 0$, and hence there exists a $t_0 \in O \setminus N$. For this t_0 we have both (54) and (55), and thus we have found a contradiction.

To show that k_T^β is bounded, we again assume the converse. Then there exists a $t \in T$ such that

$$\sum_{i\in I} \mu_i^\beta e_i^2(t) > \kappa^2 + 1 \,,$$

so that by using $\varepsilon := 1$ in part *ii)* of Lemma 6.2 we again find an $O \in \tau(H)$ with $t \in O$ and (55). Repeating the arguments above we then obtain a contradiction.

Let us now show that $\tau(H_T^\beta) = \tau(H)$. To this end, we first fix an $f \in H_T^\beta$. Since $(\mu_i^{\beta/2} e_i)_{i\in I}$ is an ONB of H_T^β, see [36, Lemma 2.6 and Proposition 4.2], we then have

$$f = \sum_{i\in I} \langle f, \mu_i^{\beta/2} e_i \rangle_{H_T^\beta} \mu_i^{\beta/2} e_i \,,$$

where the convergence is unconditionally in H_T^β. Since k_T^β is bounded, convergence in H_T^β implies uniform convergence, see e.g. [35, Lemma 4.23], and thus the above series also converges unconditionally with respect to $\|\cdot\|_\infty$. Consequently, f is a $\|\cdot\|_\infty$-limit of a sequence of $\tau(H)$-continuous functions, and thus itself $\tau(H)$-continuous. From this we easily conclude that $\tau(H_T^\beta) \subset \tau(H)$. To show the converse inclusion $\tau(H) \subset \tau(H_T^\beta)$ let us recall that the embedding $I_k : H \to L_2(\nu)$ is injective and $H = H_T^1$ by Lemma 2.6. Now the inclusion $\tau(H) \subset \tau(H_T^\beta)$ trivially follows from the inclusion $H_T^1 \subset H_T^\beta$ established in [36, Lemma 4.3]. □

Proof of Lemma 2.8: Let us denote the i-th approximation number of a bounded linear operator $T : E \to F$ between Banach spaces E and F by $a_i(T)$, that is

$$a_i(T) := \inf\bigl\{\|T - A\| \,\big|\, A : E \to F \text{ bounded linear with } \operatorname{rank} A < i\bigr\} \,.$$

Moreover, we write $s_i(I_k)$ for the i-th singular number of I_k, see (50). Since I_k is compact, we actually have $a_i(I_k) = s_i(I_k)$ for all $i \geq 1$, see [43, Theorem 7 on p. 240], and using (51) and (52) we thus find

$$\mu_i = \mu_i(T_k) = s_i(T_k) = s_i^2(I_k) = a_i^2(I_k)$$

for all $i \in I$. Moreover, if $|I| < \infty$, then we clearly have $a_i(I_k) = 0$ for all $i > |I|$ by the spectral representation of T_k. From Carl's inequality, see [7, Theorem 3.1.2], we then obtain (22). Moreover, (21) follows from the relation

$$a_i(R : H_1 \to H_2) \leq 2\varepsilon_i(R : H_1 \to H_2)$$

that holds for all compact linear operators R between Hilbert spaces H_1 and H_2, see [7, p. 120]. □

6.2 Proofs Related to Generic KL-Expansions

Proof of Lemma 3.1: Since X is $\mathcal{A}\otimes\mathcal{B}$-measurable, the map $(\omega, s, t) \mapsto X_s(\omega)X_t(\omega)$ is $\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{B}$-measurable. From this we easily conclude that k is measurable. Moreover, a simple application of Tonelli's theorem shows

$$\int_T k(t,t)\, d\nu(t) = \int_T \mathbb{E}_P X_t^2\, d\nu(t) = \int_{\Omega\times T} X^2\, dP\otimes\nu < \infty\,.$$

The remaining assertions then follow from [36, Lemma 2.3]. □

Proof of Lemma 3.2: For $i \in I$ and $\omega \in \Omega$, we define

$$Y_i(\omega) := \int_T \bigl|X_t(\omega)e_i(t)\bigr|\, d\nu(t)\,,$$

where we note that the measurability of $(\omega, t) \mapsto X_t(\omega)e_i(t)$ together with Tonelli's theorems shows that $Y_i : \Omega \to [0, \infty]$ is measurable. Moreover, since we have $e_i \in \mathcal{L}_2(\nu)$ with $\|e_i\|_{\mathcal{L}_2(\nu)} = 1$ as well as $X(\omega) \in \mathcal{L}_2(\nu)$ for P-almost all $\omega \in \Omega$, Cauchy-Schwarz inequality implies

$$\begin{aligned}
\mathbb{E}_P Y_i^2 &= \int_\Omega \biggl(\int_T \bigl|X_t(\omega)e_i(t)\bigr|\, d\nu(t)\biggr)^2 dP(\omega) \\
&\leq \int_\Omega \biggl(\int_T X_t^2(\omega)\, d\nu(t)\biggr)\biggl(\int_T e_i^2(t)\, d\nu(t)\biggr) dP(\omega) \qquad (56) \\
&= \int_{\Omega\times T} X^2\, dP\otimes\nu \\
&< \infty\,.
\end{aligned}$$

Since $|Z_i| \leq |Y_i|$, we then obtain $Z_i \in \mathcal{L}_2(P)$. Furthermore, we have $Xe_i \in$

$\mathcal{L}_1(P \otimes \nu)$ since another application of the Cauchy-Schwarz inequality gives

$$\begin{aligned}\int_{\Omega\times T} \bigl|Xe_i\bigr|\, dP \otimes \nu &\leq \biggl(\int_{\Omega\times T} X^2\, dP\otimes\nu\biggr)^{1/2} \biggl(\int_{\Omega\times T} e_i^2 dP\otimes\nu\biggr)^{1/2} \\ &= \|X\|_{\mathcal{L}_2(P\otimes\nu)} \\ &< \infty\,. \end{aligned} \tag{57}$$

Consequently, we can apply Fubini's theorem, which yields

$$\begin{aligned}\mathbb{E}_P Z_i &= \int_\Omega \int_T X_t(\omega) e_i(t)\, d\nu(t)\, dP(\omega) \\ &= \int_T \int_\Omega X_t(\omega) e_i(t)\, dP(\omega)\, d\nu(t) \\ &= 0\,, \end{aligned}$$

where in the last step we used $\mathbb{E}_P X_t = 0$. To show (26), we first observe that

$$\begin{aligned}&\int_{\Omega\times T\times T} \bigl|X_s(\omega)e_i(s)X_t(\omega)e_j(t)\bigr|\, dP\otimes\nu\otimes\nu(w,s,t) \\ &= \int_\Omega\int_T\int_T \bigl|X_s(\omega)e_i(s)\bigr|\cdot\bigl|X_t(\omega)e_j(t)\bigr|\, d\nu(s)\, d\nu(t)\, dP(\omega) \\ &= \int_\Omega \biggl(\int_T \bigl|X_s(\omega)e_i(s)\bigr|\, d\nu(s)\biggr)\biggl(\int_T \bigl|X_t(\omega)e_i(t)\bigr|\, d\nu(t)\biggr) dP(\omega) \\ &= \mathbb{E}_P Y_i^2 < \infty\,. \end{aligned} \tag{58}$$

where in the last inequality we used the arguments from (56). Using Fubini's theorem, we then obtain

$$\begin{aligned}\mathbb{E}_P Z_i Z_j &= \int_\Omega \biggl(\int_T X_s(\omega)e_i(s)\, d\nu(s)\biggr)\biggl(\int_T X_t(\omega)e_j(t)\, d\nu(t)\biggr) dP(\omega) \\ &= \int_\Omega\int_T\int_T X_s(\omega)e_i(s)X_t(\omega)e_j(t)\, d\nu(s)\, d\nu(t)\, dP(\omega) \\ &= \int_T\int_T \mathbb{E}_P\bigl(X_s X_t\bigr) e_i(s)e_j(t)\, d\nu(s)\, d\nu(t) \\ &= \int_T\int_T k(s,t)e_i(s)e_j(t)\, d\nu(s)\, d\nu(t) \\ &= \int_T S_k([e_i]_\sim)(t)\, e_j(t)\, d\nu(t) \\ &= \int_T \mu_i e_i(t) e_j(t)\, d\nu(t) \\ &= \mu_i \delta_{i,j}\,, \end{aligned} \tag{59}$$

where in the second to last step we used (6).

Let us now show (27). To this end, note that the already established $Y_j \in \mathcal{L}_2(P)$ together with $X_t \in \mathcal{L}_2(P)$ and Tonelli's theorem implies

$$\int_{\Omega\times T} \bigl|X_t(\omega)X_s(\omega)e_j(s)\bigr|\, dP\otimes\nu(\omega,s) = \int_\Omega \bigl|X_t(\omega)Y_j(\omega)\bigr|\, dP(\omega) < \infty$$

for all $t \in T$. Consequently, the map $(\omega, s) \mapsto X_t(\omega)X_s(\omega)e_j(s)$ is $P\otimes\nu$-integrable for each $t \in T$, and by Fubini's theorem we thus obtain

$$\begin{aligned}
\mathbb{E}_P X_t Z_j &= \int_\Omega X_t(\omega) \int_T X_s(\omega)e_j(s)\, d\nu(s)\, dP(\omega)\\
&= \int_T e_j(s) \int_\Omega X_t(\omega)X_s(\omega)\, dP(\omega)\, d\nu(s)\\
&= \int_T e_j(s)k(s,t)\, d\nu(s)\\
&= S_k([e_j]_\sim)(t)\\
&= \mu_j e_j(t)\,,
\end{aligned}$$

where in the last step we used (6).

Moreover, (28) immediately follows from

$$\begin{aligned}
&\Bigl\| X_t - \sum_{j\in J} Z_j e_j(t)\Bigr\|^2_{\mathcal{L}_2(P)}\\
&= \mathbb{E}_P X_t^2 - 2\mathbb{E}_P X_t \sum_{j\in J} Z_j e_j(t) + \mathbb{E}_P \sum_{i,j\in J} Z_i e_i(t) Z_j e_j(t)\\
&= k(t,t) - 2\sum_{j\in J} \mathbb{E}_P X_t Z_j e_j(t) + \sum_{i,j\in J} e_j(t)e_i(t)\mathbb{E}_P Z_i Z_j\\
&= k(t,t) - 2\sum_{j\in J} \mu_j e_j^2(t) + \sum_{j\in J} \mu_j e_j^2(t)\,,
\end{aligned}$$

where in the last step we used the already established (26) and (27).

i) $\Leftrightarrow$ *ii)*. Follows directly from (28).

Finally, to show (31), we fix a measurable $N \subset \Omega$ with $X(\omega) \in \mathcal{L}_2(\nu)$ for all $\omega \in \Omega \setminus N$. Furthermore, we fix an $f \in \mathcal{L}_2(\nu)$ with $[f]_\sim \in \ker T_k$. Without loss of generality we may assume that $\|f\|_{\mathcal{L}_2(\nu)} = 1$. For $\omega \in N$ we now write $Z(\omega) := 0$ and

$$Z(\omega) := \int_T X_t(\omega) f(t)\, d\nu(t)$$

otherwise. Then, repeating (56) and (57) with e_i replaced by f we obtain $Z \in \mathcal{L}_2(P)$ and $Xf \in \mathcal{L}_1(P\otimes\nu)$. Moreover, repeating (58) and (59) in the same way, we obtain

$$\mathbb{E}_P Z^2 = \int_T S_k([f]_\sim)(t) f(t)\, d\nu(t) = 0$$

since $[f]_\sim \in \ker T_k = \ker S_k$ by (7). This shows that $\langle [X(\omega)]_\sim, [f]_\sim\rangle_{L_2(\nu)} = Z(\omega) = 0$ for all $\omega \in \Omega \setminus N$ and all $f \in \mathcal{L}_2(\nu)$ with $[f]_\sim \in \ker T_k$, and thus we have found the first part of (31). The second part of (31), namely,

$$(\ker T_k)^\perp = \overline{\operatorname{span}\{[e_i]_\sim : i \in I\}}^{L_2(\nu)},$$

follows from combining (7) with (11) and (9). □

Proof of Proposition 3.3: Recall that [36, Theorem 3.1] showed that both *i)* and *ii)* are equivalent to

$$k(t,t') = \sum_{i\in I} \mu_i e_i(t) e_i(t') . \tag{60}$$

for all $t, t' \in T$. In view of (28) it thus suffices to show that *iii)* $\Rightarrow$ *i)*. To show this implication we assume that (29) holds for all $t \in T$, but $(\sqrt{\mu_i} e_i)_{i\in I}$ is not an ONB of H. Let $(\tilde{e}_j)_{j\in J}$ be an ONS of H such that the union of $(\sqrt{\mu_i} e_i)_{i\in I}$ and $(\tilde{e}_j)_{j\in J}$ is an ONB of H. By assumption we know that $J \neq \emptyset$, so we can fix a $j_0 \in J$. Since $\|\tilde{e}_{j_0}\|_H = 1$, there further exists a $t \in T$ with $\tilde{e}_{j_0}(t) \neq 0$. Now, it is well-known that the kernel k can be expressed in terms of our ONB, see e.g. [35, Theorem 4.20], and hence we obtain

$$\begin{aligned} k(t,t) = \sum_{i\in I} \mu_i e_i^2(t) + \sum_{j\in J} \tilde{e}_j^2(t) &\geq \sum_{i\in I} \mu_i e_i^2(t) + \tilde{e}_{j_0}^2(t) \\ &> \sum_{i\in I} \mu_i e_i^2(t) \\ &= k(t,t) , \end{aligned}$$

where the last equality follows from the equivalence of (29) and (30). In other words, we have found a contradiction, and hence *iii)* $\Rightarrow$ *i)* is true.

Let us finally consider the case in which H is separable. By [36, Corollary 3.2 and Theorem 3.3] we then see that there exists a measurable $N \subset T$ with $\nu(N) = 0$ such that

$$k(t,t') = k_T^1(t,t') , \qquad\qquad t, t' \in T .$$

Consequently, (30) holds for all $t \in T \setminus N$, and we obtain the assertion by (28). □

Proof of Proposition 3.4: Equation (31) shows that there exists a measurable $N_1 \subset \Omega$ with $P(N_1) = 0$ such that for all $\omega \in \Omega \setminus N_1$ the path $[X(\omega)]_\sim$ is contained in the space spanned by the ONS $([e_i]_\sim)_{i\in I}$. Moreover, by the definition of Z_i there exists another measurable $N_2 \subset \Omega$ with $P(N_2) = 0$ and

$$Z_i(\omega) = \langle [X(\omega)]_\sim, [e_i]_\sim \rangle_{L_2(\nu)} \tag{61}$$

for $\omega \in \Omega \setminus N_2$. Let us define $N := N_1 \cup N_2$. For $\omega \in \Omega \setminus N$ we then obtain (32).

To show (33), we again pick an $\omega \in \Omega \setminus N$. Using Parseval's identity and (61), we obtain

$$\Big\| [X(\omega)]_\sim - \sum_{j\in J} Z_j(\omega) [e_j]_\sim \Big\|_{L_2(\nu)}^2 = \sum_{i \in I\setminus J} Z_i^2(\omega)$$

Furthermore, Lemma 3.2 implies

$$\mathbb{E}_P \sum_{i\in I\setminus J} Z_i^2 = \sum_{i\in I\setminus J} \mathbb{E}_P Z_i^2 = \sum_{i\in I\setminus J} \mu_i . \tag{62}$$

Combining both equations then yields (33) and the last assertion is a trivial consequence of (33). ☐

Proof of Corollary 3.5: Our first goal is to show that $[Z_i]_\sim \in L_2(X)$ for all $i \in I$. To this end, recall from e.g. [4, p. 65] and [16, Chapter 8.4] that the Loève isometric isomorphism $\Psi : L_2(X) \to H$ is the unique continuous extension of the well-defined linear map $\Psi_0 : \mathrm{span}\{[X_t]_\sim : t \in T\} \to \mathrm{span}\{k(t,\cdot) : t \in T\}$ described by

$$\Psi_0\Big(\sum_{i=1}^n a_i [X_{t_i}]_\sim\Big) := \sum_{i=1}^n a_i k(t_i, \cdot)\,.$$

Now let $(\tilde e_j)_{j\in J}$ be an ONS in H such that $(\sqrt{\mu_i} e_i)_{i\in I} \cup (\tilde e_j)_{j\in J}$ is an ONB of H. For an arbitrary $t \in T$ and all $i \in I$ and $j \in J$, we then find $\langle k(t,\cdot), \sqrt{\mu_i} e_i\rangle_H = \sqrt{\mu_i} e_i(t)$ and $\langle k(t,\cdot), \tilde e_j\rangle_H = \tilde e_j(t)$ and thus we obtain

$$k(t,\cdot) = \sum_{i\in I} \mu_i e_i(t) e_i + \sum_{j\in J} \tilde e_j(t)\tilde e_j\,,$$

where the series converge unconditionally in H. Applying Ψ^{-1} on both sides yields

$$[X_t]_\sim = \Psi^{-1}\big(k(t,\cdot)\big) = \sum_{i\in I} \mu_i e_i(t)\Psi^{-1}(e_i) + \sum_{j\in J} \tilde e_j(t)\Psi^{-1}(\tilde e_j)\,,$$

where the series converge unconditionally in $L_2(P)$. Let us fix $\xi_i, \tilde\xi_j \in \mathcal{L}_2(P)$ with $[\xi_i]_\sim = \mu_i \Psi^{-1}(e_i)$ and $[\tilde\xi_j]_\sim = \Psi^{-1}(\tilde e_j)$. Then our constructions ensures

$$[X_t]_\sim = \sum_{i\in I} [\xi_i]_\sim e_i(t) + \sum_{j\in J} [\tilde\xi_j]_\sim \tilde e_j(t)\,, \tag{63}$$

where, for all $t \in T$, the series converge unconditionally in $L_2(P)$. For some fixed finite sets $I_0 \subset I$ and $J_0 \subset J$, we further have

$$\begin{aligned}
&\int_\Omega \Big\| [X(\omega)]_\sim - \sum_{i\in I_0} \xi_i(\omega)[e_i]_\sim \Big\|^2_{L_2(\nu)} dP(\omega)\\
&= \int_\Omega \int_T \Big| X_t(\omega) - \sum_{i\in I_0} \xi_i(\omega) e_i(t)\Big|^2 d\nu(t)\, dP(\omega)\\
&= \int_T \Big\| [X_t]_\sim - \sum_{i\in I_0} \mu_i e_i(t)\Psi^{-1}(e_i) \Big\|^2_{L_2(P)} d\nu(t)\\
&= \int_T \Big\| k(t,\cdot) - \sum_{i\in I_0} \mu_i e_i(t) e_i \Big\|^2_H d\nu(t)\\
&= \int_T \Big(\sum_{i\in I\setminus I_0} \mu_i e_i^2(t) + \sum_{j\in J} \tilde e_j^2(t)\Big) d\nu(t)\\
&= \sum_{i\in I\setminus I_0} \mu_i \,\|\,[e_i]_\sim \|^2_{L_2(\nu)} + \sum_{j\in J} \|\,[\tilde e_j]_\sim \|^2_{L_2(\nu)}\\
&= \sum_{i\in I\setminus I_0} \mu_i\,,
\end{aligned}$$

where in the last step we used Theorem 2.1, which implies

$$\tilde{e}_j \in \overline{\operatorname{span}\{\sqrt{\mu_i}e_i : i \in I\}}^{\perp} = (\overline{\operatorname{ran} S_k})^{\perp} = \ker S_k^* = \ker I_k\,.$$

Consequently, there exists a measurable $N \subset \Omega$ with $P(N) = 0$ such that for all $\omega \in \Omega \setminus N$ we have

$$[X(\omega)]_\sim = \sum_{i\in I} \xi_i(\omega)[e_i]_\sim\,,$$

where the series converges in $L_2(\nu)$. By Proposition 3.4 we may assume without loss of generality that (32) also holds for $\omega \in \Omega \setminus N$. Since $([e_i]_\sim)_{i\in I}$ is an ONS, we then see that

$$\xi_i(\omega) = \langle [X(\omega)]_\sim, [e_i]_\sim\rangle_{L_2(P)} = Z_i(\omega)$$

for such ω, and thus we finally obtain $[Z_i]_\sim = [\xi_i]_\sim \in L_2(X)$.

Now, (26) shows that $(\mu_i^{-1/2}[Z_i]_\sim)_{i\in I}$ is an ONS of $L_2(X)$, and (28) together with Proposition 3.3 shows that it is an ONB, if and only if $(\sqrt{\mu_i}e_i)_{i\in I}$ is an ONB of H. □

Proof of Lemma 3.6: By Lemma 3.2 we know that the random variables $(Z_i)_{i\in I}$ are mutually uncorrelated and centered with $\operatorname{Var} Z_i = \mu_i$ for all $i \in I$. Moreover, by Corollary 3.5 we know $\sum_{i\in I_0}^n a_i Z_i \in L_2(X)$ for all finite $I_0 \subset I$ and $a_i \in \mathbb{R}$. Since $L_2(X)$ consists of normally distributed random variables, which can be easily checked by Lévy's continuity theorem, we conclude that $(Z_i)_{i\in I}$ are jointly normal. Consequently, they are independent, and $Z_i \sim \mathcal{N}(0, \mu_i)$ becomes obvious. □

Proof of Lemma 3.7: Let us first show that the series defining each X_t does converge. To this end, we fix a finite $J \subset I$. Then an easy calculation shows

$$\begin{aligned}\int_\Omega \biggl(\sum_{j\in J} Z_j(\omega)e_j(t)\biggr)^2 dP(\omega) &= \int_\Omega \sum_{i,j\in J} Z_i(\omega)Z_j(\omega)e_i(t)e_j(t)\,dP(\omega)\\ &= \sum_{i,j\in J} e_i(t)e_j(t)\mathbb{E}_P Z_i Z_j\\ &= \sum_{j\in J} \mu_j e_j^2(t)\,. \end{aligned} \tag{64}$$

Since the latter series converges, its sequence of partial sums is a Cauchy sequence, and hence the sequence of partial sums of the right-hand side of (34) is a Cauchy sequence in $\mathcal{L}_2(P)$. Consequently, it converges, and by repeating the argument above we see that the series also converges unconditinally. Now using, the $\mathcal{L}_2(P)$-convergence, we find

$$\mathbb{E}_P X_s X_t = \langle X_s, X_t\rangle_{\mathcal{L}_2(P)} = \sum_{i,j\in J} e_i(s)e_j(t)\mathbb{E}_P Z_i Z_j = k_T^1(s,t)$$

for all $s, t \in T$. The $(\mathcal{A}\otimes\mathcal{B})$-measurability of X is obvious, and integrating (64) with respect to ν yields

$$\int_T \int_\Omega \biggl(\sum_{j\in J} Z_j(\omega)e_j(t)\biggr)^2 dP(\omega)\,d\nu(t) = \int_T \sum_{j\in J} \mu_j e_j^2(t)\,d\nu(t) = \sum_{j\in J}\mu_j \tag{65}$$

for all finite $J \subset I$. By Beppo Levi's theorem we then conclude that (65) holds for all $J \subset I$, so that Tonelli's theorem and the assumed $\sum_{i\in I} \mu_i < \infty$ show $X \in \mathcal{L}_2(P \otimes \nu)$. Moreover, essentially the same argument gives

$$\begin{aligned}
&\int_\Omega \int_T \Bigl(X_t(\omega) - \sum_{j\in J} Z_j(\omega) e_j(t) \Bigr)^2 d\nu(t)\, dP(\omega) \\
&= \int_T \int_\Omega \Bigl(X_t(\omega) - \sum_{j\in J} Z_j(\omega) e_j(t) \Bigr)^2 dP(\omega)\, d\nu(t) \\
&= \sum_{j\in I\setminus J} \mu_j \,,
\end{aligned}$$

and hence we conclude that, for P-almost all $\omega \in \Omega$, we have

$$\bigl[X(\omega)\bigr]_\sim = \sum_{i\in I} Z_i(\omega)[e_i]_\sim$$

with convergence in $L_2(\nu)$. For these ω, we then find (25) since $([e_i]_\sim)_{i\in I}$ is an ONS in $L_2(\nu)$. □

6.3 Proofs Related to Almost Sure Paths in Interpolation Spaces

Proof of Theorem 4.1: Let us begin by some preliminary remarks. To this end, we define, for all $i \in I$, random variables $\xi_i : \Omega \to \mathbb{R}$ by

$$\xi_i(\omega) := \mu_i^{(\beta-1)/2} Z_i(\omega)\,, \qquad\qquad \omega \in \Omega. \tag{66}$$

This definition immediately yields $Z_i(\omega)[e_i]_\sim = \xi_i(\omega)\mu_i^{(1-\beta)/2}[e_i]_\sim$ for all $\omega \in \Omega$.

Let us begin by proving (35). To this end, we simply note that the definition of the norm of $[H]_\sim^{1-\beta}$ gives

$$\begin{aligned}
\Bigl\| \sum_{j\in J} Z_j(\omega)[e_j]_\sim \Bigr\|^2_{[H]_\sim^{1-\beta}} = \Bigl\| \sum_{j\in J} \xi_i(\omega)\mu_i^{(1-\beta)/2}[e_j]_\sim \Bigr\|^2_{[H]_\sim^{1-\beta}} &= \sum_{j\in J} \xi_j^2(\omega) \\
&= \sum_{j\in J} \mu_i^{\beta-1} Z_j^2(\omega)\,,
\end{aligned}$$

which shows the assertion.

i) ⇔ *ii)*. This immediately follows from (35), the definition of $[H]_\sim^{1-\beta}$, and the equality $[X(\omega)]_\sim = \sum_{i\in I} Z_i(\omega)[e_i]_\sim$.

ii) ⇔ *iii)*. This is a trivial consequence of (13).

Let us now fix an $\omega \in \Omega \setminus N$ for which we have $\sum_{i\in I} \mu_i^{\beta-1} Z_i^2(\omega) < \infty$. For an arbitrary $J \subset I$, we then have $\sum_{j\in J} \mu_j^{\beta-1} Z_j^2(\omega) < \infty$, and hence we find $\sum_{j\in J} Z_j(\omega)[e_j]_\sim \in [H]_\sim^{1-\beta}$ by using the fact that $(\mu_j^{(\beta-1)/2} Z_j(\omega))_{j\in J}$ is the sequence of Fourier coefficients of $\sum_{j\in J} Z_j(\omega)[e_j]_\sim$ in $[H]_\sim^{1-\beta}$. The definition of the norm of $[H]_\sim^{1-\beta}$ then yields (35). Finally, the unconditional convergence is a direct consequence of (35) and the fact that $[H]_\sim^{1-\beta}$ and $[L_2(\nu), [H]_\sim]_{1-\beta,2}$ have equivalent norms. □

Proof of Theorem 4.2: *i) ⇒ ii).* By our assumptions, Lemma 3.2, and Beppo Levi's theorem we obtain

$$\mathbb{E}_P \sum_{i\in I} \mu_i^{\beta-1} Z_i^2 = \sum_{i\in I} \mu_i^{\beta-1} \mathbb{E}_P Z_i^2 = \sum_{i\in I} \mu_i^{\beta} < \infty\,. \tag{67}$$

Consequently, there exists a measurable $\tilde{N} \subset \Omega$ with $P(\tilde{N}) = 0$ such that for all $\omega \in \Omega \setminus \tilde{N}$ we have $\sum_{i\in I} \mu_i^{\beta-1} Z_i^2(\omega) < \infty$. By Theorem 4.1, we then obtain

$$[X(\omega)]_\sim \in [H]_\sim^{1-\beta} = [L_2(\nu), [H]_\sim]_{1-\beta,2}$$

for all $w \in \Omega \setminus (N \cup \tilde{N})$, which shows the first assertion. Moreover, choosing $J := I$ in (35), we find

$$\int_\Omega \Big\| [X(\omega)]_\sim \Big\|^2_{[H]_\sim^{1-\beta}} dP(\omega) = \int_\Omega \sum_{i\in I} \mu_i^{\beta-1} Z_i^2(\omega)\, dP(\omega) = \sum_{i\in I} \mu_i^{\beta} < \infty\,, \tag{68}$$

where we note that measurability is not an issue as the right-hand side of (35) is measurable. Since the norms of $[L_2(\nu), [H]_\sim]_{1-\beta,2}$ and $[H]_\sim^{1-\beta}$ are equivalent as discussed around (13), it thus remains to show that the map $\Omega \setminus N \to [H]_\sim^{1-\beta}$ defined by $\omega \mapsto [X(\omega)]_\sim$ is Borel measurable. To this end, we consider the map $\xi : \Omega \setminus (N \cup \tilde{N}) \to \ell_2(I)$ defined by

$$\xi(\omega) := \big(\mu_i^{\beta-1} Z_i^2(\omega)\big)_{i\in I}$$

for all $\omega \in \Omega \setminus (N \cup \tilde{N})$. Note that our previous considerations showed that ξ indeed maps into $\ell_2(I)$. Consequently, $\langle a, \xi\rangle_{\ell_2(I)} : \Omega \setminus (N \cup \tilde{N}) \to \mathbb{R}$ is well-defined for all $a \in \ell_2(I)$. In addition, this map is clearly measurable, and since $\ell_2(I)$ is separable, the combination of Petti's measurability theorem, cf. [10, p. 9], with [10, Theorem 8 on p. 8] shows that ξ is Borel measurable. Using the isometric relation (12) we conclude that the map $\Omega \setminus (N \cup \tilde{N}) \to [H]_\sim^{1-\beta}$ defined by

$$\omega \mapsto \sum_{i\in I} \xi_i(\omega) \mu_i^{(1-\beta)/2} [e_i]_\sim = [X(\omega)]_\sim$$

is Borel measurable.

ii) ⇒ i). Let $N \subset \Omega$ be a P-zero set with $[X(\omega)]_\sim \in [L_2(\nu), [H]_\sim]_{1-\beta,2}$ for all $\omega \in \Omega \setminus N$. By Proposition 3.4 we may again assume without loss of generality that (32) is also satisfied for all $\omega \in \Omega \setminus N$. Using Beppo Levi's theorem and the discussion around (13), as well as Lemma 3.2 and (35), we then obtain

$$\sum_{i\in I} \mu_i^{\beta} = \mathbb{E}_P \sum_{i\in I} \mu_i^{\beta-1} Z_i^2 = \int_\Omega \Big\| [X(\omega)]_\sim \Big\|^2_{[H]_\sim^{1-\beta}} dP(\omega) < \infty\,.$$

Let us finally assume that *i)* and *ii)* are true. By Proposition 3.4 there then exists a measurable $N \subset \Omega$ with $P(N) = 0$ such that $\sum_{i\in I} Z_i(\omega)[e_i]_\sim = [X(\omega)]_\sim$ in $L_2(\nu)$, and $[X(\omega)]_\sim \in [L_2(\nu), [H]_\sim]_{1-\beta,2}$ for all $\omega \in \Omega \setminus N$. For these ω, Theorem 4.1 immediately yields

$$\sum_{i\in I} \mu^{\beta-1} Z_i^2(\omega) < \infty\,. \tag{69}$$

Now, to show the stronger $[L_2(\nu), [H]_\sim]_{1-\beta,2}$-convergence in (32) we observe that for all $J \subset I$ and for all $\omega \in \Omega \setminus N$ we have (35) by (69). By (69) and (35) we then conclude that the sequence of partial sums of $\sum_{i\in I} Z_i(\omega)[e_i]_\sim$ is a Cauchy sequence in $[H]_\sim^{1-\beta}$ and thus convergent in $[H]_\sim^{1-\beta}$. Moreover, since $[H]_\sim^{1-\beta} \hookrightarrow L_2(\nu)$ and $\sum_{i\in I} Z_i(\omega)[e_i]_\sim = [X(\omega)]_\sim$ in $L_2(\nu)$, its limit is $[X(\omega)]_\sim$, which shows the $[H]_\sim^{1-\beta}$-convergence in (32). Finally, because of (69), the formula (32) equals the ONB representation of $[X(\omega)]_\sim$ with respect to the ONB $(\mu_i^{(1-\beta)/2}[e_i]_\sim)_{i\in I}$ of $[H]_\sim^{1-\beta}$, and hence the convergence is also unconditionally. Now using that $[H]_\sim^{1-\beta}$ and $[L_2(\nu), [H]_\sim]_{1-\beta,2}$ have equivalent norms, we see that the convergence in (32) is indeed unconditionally in $\big[L_2(\nu), [H]_\sim\big]_{1-\beta,2}$.

To show the last assertion, we combine (35) with the just established $[H]_\sim^{1-\beta}$-convergence in (32) and a calculation that is analogous to (68) to obtain

$$\int_\Omega \Big\| \, [X(\omega)]_\sim - \sum_{j\in J} Z_j(\omega)[e_j]_\sim \Big\|^2_{[H]_\sim^{1-\beta}} dP(\omega) = \sum_{i\in I\setminus J} \mu_j^\beta\,.$$

Again, using that $[H]_\sim^{1-\beta}$ and $[L_2(\nu), [H]_\sim]_{1-\beta,2}$ have equivalent norms, we then obtain the assertion. □

Lemma 6.3. *Let $(\xi)_{i\geq 1}$ be a sequence of $\mathbb{R}$-valued random variables on some probability space $(\Omega, \mathcal{A}, P)$ and $(\mu_i)_{i\geq 1} \subset (0,\infty)$ be a monotonically decreasing sequence. We define $\mathcal{F}_i := \sigma(\xi_1^2, \dots, \xi_i^2)$ and assume that $\mathbb{E}_P \xi_1^2 = 1$ and both $\xi_i \in \mathcal{L}_4(P)$ and*

$$\mathbb{E}_P(\xi_{i+1}^2 | \mathcal{F}_i) = 1 \tag{70}$$

for all $i \geq 1$. Furthermore, assume that, for some $\beta \in (0,1)$, we have

$$\sum_{i=1}^\infty \mu_i^{2\beta} \operatorname{Var} \xi_i^2 < \infty\,. \tag{71}$$

Then, the following statements are equivalent:

- *i) We have $\sum_{i=1}^\infty \mu_i^\beta < \infty$.*
- *ii) There exists an $N \in \mathcal{A}$ with $P(N) = 0$ such that for all $\omega \in \Omega \setminus N$ we have*

$$\sum_{i=1}^\infty \mu_i^\beta \xi_i^2(\omega) < \infty\,. \tag{72}$$

Proof of Lemma 6.3: Before we begin with the actual proof we note that, for all $i \geq 1$, we have $\mathbb{E}_P \xi_{i+1}^2 = \mathbb{E}_P \mathbb{E}_P(\xi_{i+1}^2 | \mathcal{F}_i) = 1$ by (70). Moreover, for $i > j+1$ an elementary calculation shows

$$\mathbb{E}_P(\xi_i^2 | \mathcal{F}_j) = \mathbb{E}_P\big(\mathbb{E}_P(\xi_i^2 | \mathcal{F}_{i-1}) | \mathcal{F}_j\big) = 1\,, \tag{73}$$

and by (70) we thus have $\mathbb{E}_P(\xi_i^2|\mathcal{F}_j) = 1$ for all $i > j$.

i) $\Rightarrow$ *ii)*. This simply follows from

$$\mathbb{E}_P \sum_{i=1}^{\infty} \mu_i^\beta \xi_i^2 = \sum_{i=1}^{\infty} \mu_i^\beta \mathbb{E}_P \xi_i^2 = \sum_{i=1}^{\infty} \mu_i^\beta < \infty\,.$$

ii) $\Rightarrow$ *i)*. For $i, n \geq 1$, we write $X_i := \mu_i^\beta(\xi_i^2 - 1)$ and $Y_n := \sum_{i=1}^n X_i$. Then, our first simple observation is that, for $i > j$, we have

$$\mathbb{E}_P(X_i|\mathcal{F}_j) = \mu_i^\beta \mathbb{E}_P(\xi_i^2 - 1|\mathcal{F}_j) = 0 \tag{74}$$

by our preliminary considerations. Moreover, for all $n \geq 1$, the random variable Y_n is $\mathcal{F}_n$-measurable and satisfies $Y_n \in \mathcal{L}_2(P)$. In addition, we have

$$\mathbb{E}_P(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}_P(X_{n+1}|\mathcal{F}_n) + Y_n = Y_n$$

by (74), and thus $(Y_n)_{n\geq 1}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$. Our next goal is to show that it is uniformly bounded in $\mathcal{L}_2(P)$. To this end, we first observe that for $i > j$ we have

$$\mathbb{E}_P(X_i X_j) = \mathbb{E}_P \mathbb{E}_P(X_i X_j|\mathcal{F}_j) = \mathbb{E}_P\big(X_j \mathbb{E}_P(X_i|\mathcal{F}_j)\big) = 0$$

since X_j is $\mathcal{F}_j$-measurable and (74). Consequently, we obtain

$$\begin{aligned}
\mathbb{E}_P Y_n^2 = \sum_{i=1}^n \mathbb{E}_P X_i^2 + 2\sum_{i=1}^n \sum_{j=1}^{i-1} \mathbb{E}_P(X_i X_j) &= \sum_{i=1}^n \mu_i^{2\beta} \mathbb{E}_P(\xi_i^2 - 1)^2 \\
&\leq \sum_{i=1}^\infty \mu_i^{2\beta} \operatorname{Var} \xi_i^2\,,
\end{aligned}$$

which by (71) shows that $(Y_n)_{n\geq 1}$ is indeed uniformly bounded in $\mathcal{L}_2(P)$. By martingale convergence, see e.g. [19, Theorem 11.10], there thus exists a random variable $Y_\infty \in \mathcal{L}_2(P)$ such that $Y_n \to Y_\infty$ in $\mathcal{L}_2(P)$ and P-almost surely. In particular, there exists an $\omega \in \Omega$ with $Y_\infty(\omega) \in \mathbb{R}$ such that we have both (72) and $Y_n(\omega) \to Y_\infty(\omega)$, where the latter simply means that $\sum_{i=1}^\infty X_i(\omega)$ converges. For this ω, we thus obtain

$$\begin{aligned}
\sum_{i=1}^\infty \mu_i^\beta = \sum_{i=1}^\infty \mu_i^\beta \big(\xi_i^2(\omega) - \xi_i^2(\omega) + 1\big) &= \sum_{i=1}^\infty \mu_i^\beta \xi_i^2(\omega) - \sum_{i=1}^\infty \mu_i^\beta \big(\xi_i^2(\omega) - 1\big) \\
&= \sum_{i=1}^\infty \mu_i^\beta \xi_i^2(\omega) - Y_\infty(\omega)\,,
\end{aligned}$$

and since the last difference is a real number we have proven the assertion. □

Proof of Lemma 4.3: *i)* $\Rightarrow$ *ii)*. Follows from a literal repetition of (67).

ii) $\Rightarrow$ *i)*. Our first goal is to show that the random variables $\xi_i := \mu_i^{-1/2} Z_i$ satisfy the assumptions of Lemma 6.3. Indeed, we clearly, have $\xi_i \in \mathcal{L}_4(P)$ and the definition of the σ-algebras $\mathcal{F}_i$ is consistent with Lemma 6.3. Moreover, (37) implies (70), and, for all $\beta \in (0, 1)$, condition (36) implies (72). Furthermore, our definitions yields

$$\operatorname{Var} \xi_i^2 = \mu_i^{-2} \operatorname{Var} Z_i^2 \leq c\mu_i^{-\alpha} \tag{75}$$

for all $i \geq 1$, and consequently, we find

$$\sum_{i=1}^{\infty} \mu_i^{2\beta} \operatorname{Var} \xi_i^2 \leq c \sum_{i=1}^{\infty} \mu_i^{2\beta-\alpha} < \infty$$

whenever $2\beta \geq \alpha + 1$, i.e. (71) is satisfied for such β. Using Lemma 6.3, we then see that the implication *ii)* $\Rightarrow$ *i)* is true for all $\beta \in [\beta_1, 1)$, where $\beta_1 := (\alpha + 1)/2$. To treat the case $\beta \in (\alpha, \beta_1)$, we define a sequence $(\beta_n)_{n\geq 1}$ by $\beta_{n+1} := (\alpha + \beta_n)/2$ for all $n \geq 1$. By induction and the definition of β_1, we then see that

$$\beta_n = 2^{-n} + \alpha \sum_{i=1}^{n} 2^{-i}$$

for all $n \geq 1$. Consequently, we have both $\beta_n \in (\alpha, 1)$ for all $n \geq 1$ and $\beta_n \searrow \alpha$.

Our next goal is to show that the implication *ii)* $\Rightarrow$ *i)* is true for all β_n. To this end, we first observe that we have already seen that the implication is true for β_1. To proceed by induction, we now assume that the implication is true for β_n, so that our goal is to show that it is also true for β_{n+1}. To this end, let us assume that there exists a measurable $N \subset \Omega$ with $P(N) = 0$ such that (36), and thus (72), holds for β_{n+1} and all $\omega \in \Omega \setminus N$. Here we note that in the absence of such an N there is nothing to prove. Now, since $\mu_i \to 0$ and $\beta_n > \beta_{n+1}$, it is easy to see that (36) also holds for β_n and all $\omega \in \Omega \setminus N$, and hence our induction hypothesis yields $\sum_{i=1}^{\infty} \mu_i^{\beta_n} < \infty$. This in turn shows

$$\sum_{i=1}^{\infty} \mu_i^{2\beta_{n+1}} \operatorname{Var} \xi_i^2 = \sum_{i=1}^{\infty} \mu_i^{\alpha+\beta_n} \operatorname{Var} \xi_i^2 \leq c \sum_{i=1}^{\infty} \mu_i^{\alpha+\beta_n} \mu_i^{-\alpha} < \infty \tag{76}$$

by (75). Consequently, applying Lemma 6.3 gives $\sum_{i=1}^{\infty} \mu_i^{\beta_{n+1}} < \infty$, which finishes the induction.

Finally, let us fix a $\beta \in (\alpha, \beta_1)$ for which there exists a measurable $N \subset \Omega$ with $P(N) = 0$ such that (36) holds for β and all $\omega \in \Omega \setminus N$. By the construction of (β_n), there then exists an $n \geq 1$ such that $\beta \in [\beta_{n+1}, \beta_n)$. Using the same arguments as above, we then see that (36) also holds for β_n and all $\omega \in \Omega \setminus N$, and hence we find $\sum_{i=1}^{\infty} \mu_i^{\beta_n} < \infty$ by our preliminary result. Repeating (76), we find

$$\sum_{i=1}^{\infty} \mu_i^{2\beta} \operatorname{Var} \xi_i^2 \leq \sum_{i=1}^{\infty} \mu_i^{2\beta_{n+1}} \operatorname{Var} \xi_i^2 \leq \sum_{i=1}^{\infty} \mu_i^{\alpha+\beta_n} \mu_i^{-\alpha} < \infty\,,$$

and consequently Lemma 6.3 gives $\sum_{i=1}^{\infty} \mu_i^{\beta} < \infty$. □

Proof of Corollary 4.4: Clearly, if I is finite, there is nothing to prove, and hence we solely focus on the case $I = \mathbb{N}$.

i) ⇔ *ii)*. By Lemma 3.6 we know that the $(Z_i)_{i\in I}$ are independent, and thus we find $\mathbb{E}_P(Z_{i+1}^2|\mathcal{F}_i) = \mathbb{E}_P Z_{i+1}^2 = \mu_{i+1}$ by Lemma 3.2. Consequently, (37) is satisfied. Moreover, since we have $Z_i \sim \mathcal{N}(0,\mu_i)$ for all $i \in I$ by Lemma 3.6 there exists a constant $c > 0$ such that

$$\mu^{-2}\,\mathrm{Var}\, Z_i^2 = \mathrm{Var}(\mu_i^{-1/2} Z_i)^2 \leq c$$

for all $i \in I$. This shows that (38) holds for all $\alpha \in (0,1)$. Applying Lemma 4.3 then yields the assertion.

ii) ⇒ *iii)*. trivial.

iii) ⇒ *ii)*. Assume that there exists an $A \in \mathcal{A}$ with $P(A) > 0$ such that $[X(\omega)]_\sim \in [L_2(\nu),[H]_\sim]_{1-\beta,2}$ holds for all $\omega \in A$. Without loss of generality we may additionally assume that $A \subset \Omega \setminus N$, where $N \subset \Omega$ is the measurable P-zero set obtained from Proposition 3.4. By Theorem 4.1 we then know that $\sum_{i\in I}\mu^{\beta-1}Z_i^2(\omega) < \infty$ for all $\omega \in A$, and hence

$$P\Big(\Big\{\sum_{i\in I}\mu^{\beta-1}Z_i^2 < \infty\Big\}\Big) > 0\,.$$

However, the $(Z_i)_{i\in I}$ are independent by Lemma 3.6 and hence we conclude by Kolmogorov's zero-one law that $\sum_{i\in I}\mu^{\beta-1}Z_i^2(\omega) < \infty$ actually holds for P-almost all $\omega \in \Omega$. □

Proof of Corollary 4.5: Let us write I for the embedding $H \hookrightarrow W^m(T)$. Using (24) and the multiplicativity of the dyadic entropy numbers, see [7, p. 21], we then find

$$\varepsilon_i\big(I_k : H \to L_2(\nu)\big) \leq \|I\| \cdot \varepsilon_i\big(\mathrm{id} : W^m(T) \to L_2(\nu)\big) \leq c\, i^{-m/d}\,,$$

where $c > 0$ is a suitable constant. Lemma 2.8 then gives $\mu_i \leq 4c\, i^{-2m/d}$ for all $i \geq 1$, and hence we have $\sum_{i\in I}\mu_i^\beta < \infty$ for all $\beta > \frac{d}{2m}$. Let us fix an $0 < s < m - d/2$. For $\beta := 1 - s/m$, we then have $\beta \in (\frac{d}{2m}, 1)$, and by Theorem 4.2 we conclude that

$$[X(\omega)]_\sim \in [L_2(T),[H]_\sim]_{1-\beta,2} \subset \big[L_2(T), W^m(T)\big]_{1-\beta,2} = B_{2,2}^{(1-\beta)m}(T) = B_{2,2}^s(T)$$

for P-almost all $\omega \in \Omega$. Moreover, the first norm estimate, including the implicitly assumed measurability of the integrand, also follows from Theorem 4.2. The second norm estimate follows by combining Theorem 4.2 with (24) and Lemma 2.8, which is possible by the assumed $H = W^m(T)$.

Finally, let us assume that $(X_t)_{t\in T}$ is a Gaussian process with $H = W^m(T)$ but (39) *does* hold for $s := m - d/2$ with strictly positive probability P. Then we have

$$[X(\omega)]_\sim \in B_{2,2}^s(T) = \big[L_2(T), W^m(T)\big]_{s/m,2} = \big[L_2(T), W^m(T)\big]_{1-\beta,2}\,,$$

where $\beta := \frac{d}{2m}$. By Corollary 4.4 we then see that $\sum_{i\in I}\mu_i^\beta < \infty$, and thus

$$\sum_{i\in I}\varepsilon_i^{d/m}\big(\mathrm{id} : W^m(T) \to L_2(T)\big) = \sum_{i\in I}\varepsilon_i^{2\beta}\big(I_k : H \to L_2(T)\big) < \infty$$

by Lemma 2.8. However, this contradicts (24). □

6.4 Proofs Related to Almost Sure Paths in RKHSs

Lemma 6.4. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(T, \mathcal{B}, \nu)$ be a measure space, and $(X_t)_{t\in T} \subset \mathcal{L}_2(P)$ be a $(\mathcal{A}\otimes\mathcal{B})$-measurable stochastic process with $X \in \mathcal{L}_2(P\otimes\nu)$. Then, for every $(\mathcal{A}\otimes\mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$, we have both $(Y_t)_{t\in T} \subset \mathcal{L}_2(P)$ and $Y \in \mathcal{L}_2(P\otimes\nu)$, and, for P-almost all $w \in \Omega$, we further have*

$$[Y(\omega)]_\sim = [X(\omega)]_\sim \,.$$

Proof of Lemma 6.4: Since $(Y_t)_{t\in T} \subset \mathcal{L}_2(P)$ is a version of $(X_t)_{t\in T} \subset \mathcal{L}_2(P)$, we have

$$P(Y_t = X_t) = 1\,, \qquad\qquad t\in T,$$

and thus we find both $(Y_t)_{t\in T} \subset \mathcal{L}_2(P)$ and $\|Y_t - X_t\|_{\mathcal{L}_2(P)} = 0$ for all $t \in T$. Using the measurability of $Y : \Omega\times T \to \mathbb{R}$ and Tonelli's theorem, we thus find

$$\begin{aligned}\int_P \big\| \,[Y(\omega)]_\sim - [X(\omega)]_\sim \,\big\|^2_{L_2(\nu)} dP(\omega) &= \int_P \int_T \big|Y_t(\omega) - X_t(\omega)\big|^2 d\nu(t)\, dP(\omega)\\ &= \int_T \int_P \big|Y_t(\omega) - X_t(\omega)\big|^2 dP(\omega)\, d\nu(t)\\ &= 0\,.\end{aligned}$$

This shows $[Y(\omega)]_\sim = [X(\omega)]_\sim$ for P-almost all $w \in \Omega$, and since another application of Tonelli's theorem yields

$$\int_{\Omega\times T} \big|Y_t(\omega) - X_t(\omega)\big|^2 dP\otimes\nu(\omega,t) = \int_T \int_P \big|Y_t(\omega) - X_t(\omega)\big|^2 dP(\omega)\, d\nu(t) = 0\,,$$

we also obtain $Y \in \mathcal{L}_2(P\otimes\nu)$. □

Proof of Theorem 5.1: *i) ⇒ ii).* As in the proof of Theorem 4.1, we define, for all $i \in I$, random variables $\xi_i : \Omega \to \mathbb{R}$ by

$$\xi_i(\omega) := \mu_i^{(\beta-1)/2} Z_i(\omega)\,, \qquad\qquad \omega\in\Omega.$$

For $t \in S$, we further define Y_t by

$$Y_t(\omega) := \sum_{i\in I} \xi_i(\omega) \mu_i^{(1-\beta)/2} e_i(t)\,, \qquad\qquad \omega \in \Omega\setminus N \tag{77}$$

and $Y_t(\omega) := 0$ otherwise. Moreover, for $t \in T \setminus S$ we simply write $Y_t := X_t$. Obviously, this construction guarantees the $(\mathcal{A}\otimes\mathcal{B})$-measurability of $Y : \Omega\times T \to \mathbb{R}$.

Let us first show that $(Y_t)_{t\in T}$ is a version of $(X_t)_{t\in T}$. Clearly, it suffices to show that

$$P\big(X_t = Y_t\big) = 1$$

for all $t \in S$. However, this immediately follows from

$$\|\, X_t - Y_t \,\|^2_{\mathcal{L}_2(P)} = \Big\|\, X_t - \sum_{i\in I} Z_i e_i(t)\,\Big\|^2_{\mathcal{L}_2(P)} = k(t,t) - \sum_{i\in I} \mu_i e_i^2(t) = 0\,,$$

where we used both (28) and (41).

Let us now show that all paths of Y restricted to S are contained in $H_S^{1-\beta}$. Clearly, for $\omega \in N$ our definition yields $Y(\omega)_{|S} = 0$, and hence there is nothing to prove for such ω. Moreover, in the case $\omega \in \Omega \setminus N$, we first observe that the family of functions $((\mu_i^{(1-\beta)/2}\hat{e}_i)_{|S})_{i\in I}$ forms an ONB of $H_S^{1-\beta}$ since the restriction operator

$$\cdot_{|S} : \hat{H}_S^{1-\beta} \to H_S^{1-\beta}$$

is a isometric isomorphism by Lemma 2.2. Using $(\mu_i^{(1-\beta)/2}\hat{e}_i)_{|S} = (\mu_i^{(1-\beta)/2}e_i)_{|S}$ and $(\xi_i(\omega))_{i\in I} \in \ell_2(I)$, where the latter follows from (43), we then find $Y(\omega)_{|S} \in H_S^{1-\beta}$ by the definition (77) of the random variables Y_t for $t \in S$.

ii) ⇒ i). By Lemma 6.4 we find a measurable $N_1 \subset \Omega$ with $P(N_1) = 0$ such that $Y(\omega)_{|S} \in H_S^{1-\beta}$ and

$$[Y(\omega)]_\sim = [X(\omega)]_\sim$$

for all $\omega \in \Omega \setminus N_1$. Let us fix an $\omega \in \Omega \setminus N_1$. Since $Y(\omega)_{|S} \in H_S^{1-\beta}$ there then exists a sequence $(a_i)_{i\in I} \subset \ell_2(I)$ such that

$$Y(\omega)_{|S} = \sum_{i\in I} a_i \mu_i^{(1-\beta)/2}(e_i)_{|S}\,, \tag{78}$$

where the convergence is in $H_S^{1-\beta}$. Let us write $\hat{Y}(\omega) := \mathbf{1}_S Y(\omega)$. Then we find $\hat{Y}(\omega) \in \hat{H}_S^{1-\beta}$ and

$$\hat{Y}(\omega) = \sum_{i\in I} a_i \mu_i^{(1-\beta)/2}\hat{e}_i\,,$$

where the convergence is in $\hat{H}_S^{1-\beta}$. Since $\hat{H}_S^{1-\beta}$ is compactly embedded into $L_2(\nu)$, the operator $[\,\cdot\,]_\sim : \hat{H}_S^{1-\beta} \to L_2(\nu)$ is continuous, which in turn yields

$$[X(\omega)]_\sim = [Y(\omega)]_\sim = [\hat{Y}(\omega)]_\sim = \sum_{i\in I} a_i \mu_i^{(1-\beta)/2}[\hat{e}_i]_\sim = \sum_{i\in I} a_i \mu_i^{(1-\beta)/2}[e_i]_\sim\,,$$

where the convergence is in $L_2(\nu)$. On the other hand, Proposition 3.4 showed that there exists a measurable $N_2 \subset \Omega$ with $P(N_2) = 0$ such that for all $\omega \in \Omega \setminus N_2$ we have

$$[X(\omega)]_\sim = \sum_{i\in I} Z_i(\omega)[e_i]_\sim\,,$$

where again the convergence is in $L_2(\nu)$. Using that $([e_i]_\sim)$ is an ONS in $L_2(\nu)$, we thus find $Z_i(\omega) = a_i\mu_i^{(1-\beta)/2}$ for all $\omega \notin N_1 \cup N_2$. Now (45) follows from (78), and since $(a_i)_{i\in I} \in \ell_2(I)$ we also obtain *i)* for $N := N_1 \cup N_2$. □

Proof of Theorem 5.2: *i) ⇔ ii)*. This has already been shown in Lemma 2.3.

Before we prove the remaining implications, let us assume that we have an $(\mathcal{A}\otimes\mathcal{B})$-measurable version $(Y_t)_{t\in T}$ of $(X_t)_{t\in T}$ such that $Y(\omega)_{|S} \in H_S^{1-\beta}$ for P-almost all $\omega \in \Omega$. By Lemma 6.4 we then conclude that

$$[\hat{Y}(\omega)_{|S}]_\sim = [Y(\omega)]_\sim = [X(\omega)]_\sim$$

for P-almost all $\omega \in \Omega$, where $\hat{Y}(\omega)$ denotes the zero-extension of $Y(\omega)_{|S}$ to T. In addition, we have $\|Y(\omega)_{|S}\|_{H_S^{1-\beta}} = \|[\hat{Y}(\omega)_{|S}]_\sim\|_{[H_S^{1-\beta}]_\sim}$ by Lemma 2.2. Together, this yields

$$\int_\Omega \big\| Y(\omega)_{|S} \big\|^2_{H_S^{1-\beta}} \, dP(\omega) = \int_\Omega \big\| [X(\omega)]_\sim \big\|^2_{[H]_\sim^{1-\beta}} \, dP(\omega) = \sum_{i\in I} \mu_i^\beta \tag{79}$$

where the last identity follows by a repetition of (68). Moreover, note that all three quantities may simultaneously be infinite.

i) ⇒ iii). We have

$$\int_\Omega \sum_{i\in I} \mu_i^{\beta-1} Z_i^2(\omega)\, dP(\omega) = \sum_{i\in I} \mu_i^{\beta-1} \int_\Omega Z_i^2(\omega)\, dP(\omega) = \sum_{i\in I} \mu_i^\beta < \infty\,,$$

and hence we find a measurable $N \subset \Omega$ with $P(N) = 0$ such that for all $\omega \in \Omega \setminus N$ we have (43). Now the assertion follows from Theorem 5.1 and (79).

iii) ⇒ i). Follows directly from (79). □

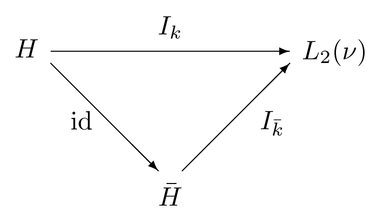
Proof of Corollary 5.3: *i) ⇔ ii).* This has already been shown in Lemma 2.3, see also Theorem 5.2.

i) ⇒ iii). Repeating (67), we see yet another time that (43) holds for P-almost all $\omega \in \Omega$. Applying Theorem 5.1 then yields the assertion.

iii) ⇒ iv). trivial

iv) ⇒ i). For $\omega \in A$ we have $[X(\omega)]_\sim = [\hat{Y}(\omega)_{|S}]_\sim \in [H_S^{1-\beta}]_\sim = [H]_\sim^{1-\beta}$ and hence *i)* follows by Corollary 4.4. □

Proof of Corollary 5.4: Before we begin with the actual proof, let us first note that the factorization



together with the multiplicativity of the dyadic entropy numbers, see [7, p. 21], yields

$$\varepsilon_i(I_k) \le \|\,\mathrm{id} : H \to \bar{H}\|\, \varepsilon_i(I_{\bar{k}})$$

for all $i \ge 1$, and therefore we find $\sum_{i=1}^\infty \varepsilon_i^\alpha(I_k) < \infty$. Applying Lemma 2.8 then shows both $\sum_{j\in J} \bar{\mu}_j^{\alpha/2} < \infty$ and $\sum_{i\in I} \mu_i^{\alpha/2} < \infty$, where $(\bar{\mu}_j)_{j\in J}$ is the sequence of non-zero eigenvalues of $T_{\bar{k}}$ obtained by Theorem 2.1.

Moreover, for $\beta \in [\alpha/2, 1-\alpha/2]$, we have $\alpha/2 \le 1-\beta$, and thus we find both $\sum_{j\in J} \bar{\mu}_j^{1-\beta} < \infty$ and $\sum_{i\in I} \mu_i^{1-\beta} < \infty$. Analogously, $\beta \ge \alpha/2$ implies $\sum_{j\in J} \bar{\mu}_j^{\beta} < \infty$ and $\sum_{i\in I} \mu_i^{\beta} < \infty$.

i). Let us pick a $\beta \in [\alpha/2, 1-\alpha/2]$. Then, our preliminary considerations showed both $\sum_{j\in J} \bar{\mu}_j^{1-\beta} < \infty$ and $\sum_{i\in I} \mu_i^{1-\beta} < \infty$. By (18) we then see that we find a measurable $S_0 \subset T$ with $\nu(T \setminus S_0) = 0$ such that both $H_{S_0}^{1-\beta}$ and $\bar{H}_{S_0}^{1-\beta}$ exist.

Our next goal is to find a subset S of S_0 with $\nu(T \setminus S) = 0$ and $H_S^{1-\beta} \subset \bar{H}_S^{1-\beta}$. To this end, note that (13) together with $[H]_\sim \subset [\bar{H}]_\sim \subset L_2(\nu)$ and the definition of interpolation norms shows

$$[H]_\sim^{1-\beta} = \bigl[L_2(\nu), [H]_\sim\bigr]_{1-\beta,2} \hookrightarrow \bigl[L_2(\nu), [\bar{H}]_\sim\bigr]_{1-\beta,2} = [\bar{H}]_\sim^{1-\beta}\,,$$

and hence the inclusion operator $I : [H]_\sim^{1-\beta} \to [\bar{H}]_\sim^{1-\beta}$ is continuous. Now consider the situation

$$H_{S_0}^{1-\beta} \xrightarrow{\;[\,\hat{\cdot}\,]_\sim\;} [H]_\sim^{1-\beta} \xrightarrow{\;I\;} [\bar{H}]_\sim^{1-\beta} \xleftarrow{\;[\,\hat{\cdot}\,]_\sim\;} \bar{H}_{S_0}^{1-\beta}$$

where the operators $[\,\hat{\cdot}\,]_\sim$ are isometric isomorphisms by Lemma 2.2. Consequently, for all $f \in H_{S_0}^{1-\beta}$ there exists a unique $g_f \in \bar{H}_{S_0}^{1-\beta}$ such that $[\hat{f}]_\sim = [\hat{g}_f]_\sim$, and the map $f \mapsto g_f$ is linear and continuous. In other words, for all $f \in H_{S_0}^{1-\beta}$, there exists a a measurable $N_f \subset S_0$ with $\nu(N_f) = 0$ and $f(t) = g_f(t)$ for all $t \in S_0 \setminus N_f$.

Let us find such a ν-zero set N that is an independent of f. To this end, we fix a countable dense $D \subset H_{S_0}^{1-\beta}$ and define $N := \bigcup_{f\in D} N_f$, where we note that such a D exists since $H_{S_0}^{1-\beta}$ is separable by construction. Now the definition of N immediately yields $N \subset S_0$ and $\nu(N) = 0$, as well as

$$f(t) = g_f(t)\,, \qquad t \in S_0 \setminus N \tag{80}$$

for all $f \in D$. To show the latter for all $f \in H_{S_0}^{1-\beta}$, we fix such an f and a sequence $(f_n) \subset D$ with $f_n \to f$ in $H_{S_0}^{1-\beta}$. Then we have $g_{f_n} \to g_f$ in $\bar{H}_{S_0}^{1-\beta}$ by the above mentioned continuity of $f \mapsto g_f$, and since both spaces are reproducing kernel Hilbert spaces, we obtain $f_n(t) \to f(t)$ and $g_{f_n}(t) \to g_f(t)$ for all $t \in S_0$. Using $f_n(t) = g_{f_n}(t)$ for all $t \in S_0 \setminus N$ and $n \geq 1$, we thus find (80). Defining $S := S_0 \setminus N$ then gives $H_S^{1-\beta} \subset \bar{H}_S^{1-\beta}$ and the continuity of this embedding follows from the continuity of I and Lemma 2.2.

ii). Our goal is to apply Theorem 5.2. To this end, we first observe that (41) holds for a set $\tilde{S} \subset T$ with $\nu(T \setminus \tilde{S}) = 0$ by the assumed separability of H and [36, Corollary 3.2]. Consequently, we may assume without loss of generality that (41) holds for the set S found in part *i)*. Moreover, we have already seen in part *i)* that we have $\sum_{i\in I} \mu_i^{1-\beta} < \infty$, which in turn implies (42) by (18). Finally, our preliminary considerations showed that $\beta \geq \alpha/2$ implies $\sum_{i\in I} \mu_i^{\beta} < \infty$, and thus Theorem 5.2 is applicable. □

Proof of Corollary 5.5: We first show that assumption *i)* implies assumption *ii)*, so that in the remainder of this proof is suffices to work with the latter. To this end, note that

$$\sum_{j\in J}\bar{\mu}_j^{1-\beta}\bar{e}_j^2(t)\leq\sup_{j\in J}\|\bar{e}_j\|_\infty\sum_{j\in J}\bar{\mu}_j^{1-\beta}\leq\sup_{j\in J}\|\bar{e}_j\|_\infty\sum_{j\in J}\bar{\mu}_j^{\beta}$$
$$\leq 4\sup_{j\in J}\|\bar{e}_j\|_\infty\sum_{i=1}^{\infty}\varepsilon_i^{2\beta}(I_{\bar{k}})<\infty\,,$$

where we used $1-\beta\leq\beta$ and Lemma 2.8. Consequently, $\bar{k}_T^{1-\beta}$ exists and is bounded, and from the latter we immediately obtain $[L_2(\nu),[\bar{H}]_\sim]_{1-\beta,2}=[\bar{H}_T^{1-\beta}]\hookrightarrow L_\infty(\nu)$.

i). We first note that $H\subset\bar{H}$ implies $\tau(H)\subset\tau(\bar{H})$, and hence Assumption CK is satisfied for k, too. Moreover, the continuity of the inclusion operator $I:[H]_\sim^{1-\beta}\to[\bar{H}]_\sim^{1-\beta}$ considered in the proof of part *i)* of Corollary 5.4 implies $[L_2(\nu),[H]_\sim]_{1-\beta,2}\hookrightarrow L_\infty(\nu)$. By Theorem 2.7, we then see that both $H_T^{1-\beta}$ and $\bar{H}_T^{1-\beta}$ do exist. Moreover, the kernels $k_T^{1-\beta}$ and $\bar{k}_T^{1-\beta}$ are bounded by Theorem 2.7.

To show that $H_T^{1-\beta}\subset\bar{H}_T^{1-\beta}$, we consider the map $f\mapsto g_f$ from the proof of part *i)* of Corollary 5.4. Then we have seen above that (80) holds for $S_0=T$ and all $f\in H_T^{1-\beta}$. Let us assume that there exists an $f\in H_T^{1-\beta}$ and a $t\in T$ such that $f(t)\neq g_f(t)$. Then we have $\{|f-g_f|>0\}\neq\emptyset$ and $\{|f-g_f|>0\}\in\tau(\bar{H})$, which together imply $\nu(\{|f-g_f|>0\})>0$, since ν is assumed to be $\bar{k}$-positive. In other words, (80) does not hold for f, which contradicts our earlier findings. This shows $f=g_f$ for all $f\in H_T^{1-\beta}$ and thus $H_T^{1-\beta}\subset\bar{H}_T^{1-\beta}$. The continuity of the corresponding embedding again follows from the continuity of I.

ii). Considering the proof of part *ii)* of Corollary 5.4, we easily see that it suffices to check that (41) holds for $S:=T$. The latter, however, follows from Lemma 2.6.

iii). All $f\in H_T^{1-\beta}$ are bounded since the kernel $k_T^{1-\beta}$ is bounded. Moreover, all $f\in H_T^{1-\beta}$ are $\tau(H_T^{1-\beta})$-continuous by the very definition of this topology, and since Theorem 2.7 showed $\tau(H_T^{1-\beta})=\tau(H)$, they are also $\tau(H)$-continuous. Now the additional assertions on the paths of Y follow from $Y(\omega)\in H_T^{1-\beta}$ for all $\omega\in\Omega$.

iv). Since $\bar{k}_T^{1-\beta}$ is bounded, we have $\bar{H}_T^{1-\beta}\hookrightarrow\ell_\infty(T)$, see e.g. [35, Lemma 4.23]. Now the $\ell_\infty(T)$-convergence of (45) follows from the $\bar{H}_T^{1-\beta}$-convergence established in Theorem 5.1.

v). Let us fix a countable, τ-dense subset $D\subset T$. Since Y is a version of X, we then have $P(\{Y_t\neq X_t\})=0$ for all $t\in D$, and hence there exists a P-zero set $N\in\mathcal{A}$ such that $X_t(\omega)=Y_t(\omega)$ for all $t\in D$ and $\omega\in\Omega\setminus N$. Without loss of generality we may also assume that $X(\omega)$ is τ-continuous for all $\omega\in\Omega\setminus N$. and since $\tau(H)\subset\tau$, we further see by part *iii)* that all paths of Y are τ-continuous, too. Now the assertion follows by a simple limit argument.

By Lemma 2.6 the operator $I_{\bar{k}}$ is injective, and thus [36, Theorem 3.1] shows that $(\bar{e}_j)_{j\in J}$ is an ONB of $\bar{H}$. Consequently, $\bar{H}$ is separable and Lemma 6.1 shows that $\tau(\bar{H})$ is separable and generated by a pseudo-metric. If $\tau(H)$ is Hausdorff, this pseudo-metric becomes a metric and the assertion follows from the first part. □

Proof of Corollary 5.6: *i).* Let us consider Corollary 5.4 for $\bar{H}=W^m(T)$. Then (24) shows that

$$\sum_{i=1}^{\infty} \varepsilon_i^{\alpha}(I_{\bar{k}}) < \infty$$

holds for all $\alpha > d/m$. Let us pick an $s \in (d/2, m - d/2)$ and define $\beta := 1 - s/m$. This gives $\frac{d}{2m} < \beta < 1 - \frac{d}{2m}$, and hence β satisfies the assumptions of Corollary 5.4 for a suitable $\alpha \in (d/m, 1]$ with $\beta \in [\alpha/2, 1 - \alpha/2]$. Moreover, we have

$$[L_2(T), [H]_\sim]_{1-\beta,2} \hookrightarrow \big[L_2(T), W^m(T)\big]_{1-\beta,2} = B_{2,2}^{(1-\beta)m}(T) = B_{2,2}^{s}(T) \hookrightarrow L_\infty(\nu)$$

by Sobolev's embedding theorem for Besov spaces, see e.g. [1, Theorem 7.34], and hence we can apply part *iii)* of Corollary 5.4 and Theorem 5.1.

ii). This follows from Corollary 4.5 since (48) implies (39). □

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