Finite Groups with Sylow numbers $\{q^x, a, b\}$

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by Iris Köster

Let G be a finite group and let $\pi(G)$ be the set of prime divisors of the order of G. The number of Sylow p-subgroups in G is denoted by $n_p(G)$ and $sn(G) = \{n_p(G) \mid p \in \pi(G)\}$ is called the set of Sylow numbers of G.

A series of articles studies finite groups with given Sylow numbers in order to get criterions for solvability. Groups with odd or prime power Sylow numbers [Guo96] are solvable as well as groups with less then 3 Sylow numbers [Luc98].

Florian Luca stated in [Luc98] the following result.

Theorem. A finite group G is solvable provided one of the following conditions holds.

a) $sn(G) = \{a, b\}$,

b) $sn(G) = \{1, a, b\}$,

c) $sn(G) = \{q^x, a, b\}$, where q is a prime number and either $\gcd(a, b) = 1$ or $q \nmid ab$.

Unfortunately the proof is based on [Zha95, Theorem 1] which has been disproved by a counterexample in [Chi98].

In [Mor12] Alexander Moretó verified part a) of the theorem. The purpose of this note is to establish parts b) and c). The second assertion has been proved in my Master thesis ([Koe13, p. 30]) whileas the proof of the third part is new.

Note that [Zha95, Theorem 1] and [Zha95, Theorem 3] are valid for groups which do not have a composition factor isomorphic to the projective special unitary group $U_3(2^f)$, where f is even and not divisible by 3. For such groups Luca's proof given in [Luc98] remains vaild. Thus it suffices to establish the following fact.

Proposition 1. Let G be a group with Sylow numbers $sn(G) = \{q^x, a, b\}$. Then G does not have a composition factor isomorphic with $U_3(r)$ where $r = 2^f$, f is even and not divisible by 3.

Proof of part Proposition 1. In [Chi98, Proof of Theorem 1] the Sylow numbers of $U_3(r)$ are given as

$$sn(U_3(r)) = \left\{(r+1)(r^2-r+1), \frac{r^3(r^3+1)}{2}, \frac{r^3(r-1)(r^2-r+1)}{6}, \frac{r^3(r+1)^2(r-1)}{3}\right\}.$$

We have to prove the following claim: every Sylow number $n_p(U_3(r))$ of the projective special unitary group is divided by at least two different primes. It is obvious, that $r^3(r^3+1)/2 = 2^{3f-1}(2^{3f}+1)$, $r^3(r-1)(r^2-r+1)/6$ and $r^3(r+1)^2(r-1)/3$ are divided by 2 and at least one odd prime.

We analyse the greatest common divisor of $(r+1)$ and (r^2-r+1): $\gcd(r+1, r^2-r+1) = \gcd(r+1, r^2-r+1 \mod r+1) = \gcd(r+1, 3 \mod r+1)$ either is 1 or 3. If $\gcd(r+1, r^2-r+1) = 1$ the claim follows immediately. If $\gcd(r+1, r^2-r+1) = 3$, then either $r+1$ or r^2-r+1 is divided by 3 but not by 9. $r+1$ as well as r^2-r+1 are greater than 5, so one of the factors owns a prime divisor $s \neq 3$.

Suppose $H \cong U_3(r)$, $r = 2^f$, f even and $3 \nmid f$, is a composition factor of G. By [Zha95, Lemma 1] $n_p(H)$ divides $n_p(G)$. None of the Sylow numbers of H are of prime power order. Therefore $n_p(H)|ab$.

Note, there exist primes $p_1, p_2 \in \pi(H)$ with $n_{p_1}(H)|a$ and $n_{p_2}(H)|b$: Assume $p_2 \in \pi(n_{p_1}(H))$ for a $p_1 \in \pi(H)$ and $n_{p_1}(H)|a$. Then $n_{p_1}(G) = a$. Therefore $n_{p_2}(G) \equiv 1 \mod p_2$ cannot divide a. Due to the fact that $n_{p_2}(H)|n_{p_2}(G)$ is no prime power, $n_{p_2}(G)$ cannot be a prime power and $n_{p_2}(G) = b$. Because of $n_{p_2}(H)|n_{p_2}(G) = b$ the sylow number $n_{p_1}(H)$ has to divide b.

Assume $\gcd(a, b) = 1$. Without loss of generality we suppose $n_p(H) = (r^3(r^3+1)/2)|a$ for one $p \in \pi(H)$. There exist a sylow number $n_s(H)$ in H with $n_s(H)|b$ for one $s \in \pi(H)$.

We consider $n_s(H) = (r+1)(r^2-r+1)$. Then $\gcd((r+1)(r^2-r+1), r^3(r^3+1)/2) = r^3+1 \neq 1$. Choose $k \in \pi(r^3+1)$. k divides $n_p(H)$ and $n_s(H)$ and it follows $k| \gcd(n_p(H), n_s(H))| \gcd(a, b) = 1$. So r^3-1 can't divide b.

Suppose $n_s(H) = r^3(r-1)(r^2-r+1)/6$. Then $r^3(r^2-r+1)/6 \neq 1$ divides $\gcd(r^3(r-1)(r^2-r+1)/6, r^3(r^3+1)/2)$. Choose $k \in \pi(r^3(r^2-r+1)/6)$. Then k divides $\gcd(n_p(H)|n_s(H))| \gcd(a, b) = 1$ and $k = 1$. It follows $n_s(H) \neq r^3(r-1)(r^2-r+1)/6$.

Assume $n_s(H) = r^3(r+1)^2(r-1)/3$. Then $r^3(r+1)/3$ is a divisor of $\gcd(r^3(r+1)^2(r-1)/3, r^3(r^3+1)/2)$. Choose $k \in \pi(r^3(r+1)/3)$. As before it is $k = 1$. We conclude, all sylow numbers of H divide a and do not divide b, a contradiction to the fact, there has to be one sylow number which divides b.

Suppose $q \nmid ab$. There exist sylow numbers $n_{p_1}(H), n_{p_2}(H)$ with $n_{p_1}(H)|a$ and $n_{p_2}(H)|b$. Note, that $\gcd(n_{p_1}(H), n_{p_2}(H)) \neq 1$ for any sylow numbers of H. Choose $k \in \pi(\gcd(n_{p_1}(H), n_{p_2}(H)))$. It is $n_k(H) \equiv 1 \mod k$. Because of $k| \gcd(n_{p_1}(H), n_{p_2}(H))| \gcd(a, b)$ and $n_k(H)|n_k(G) \in \{q^x, a, b\}$ we conclude that $n_k(H)|q^x$, a contradiction to the fact, that $n_k(H) \in sn(H)$ can't be of prime power order.

The conclusion of the proposition holds as well provided $sn(G) = \{1, a, b\}$. So this gives in the same way a proof of part b) of the theorem. However in order to prove b) we give a proof based on M.Hall's formula for the behaviour of Sylow numbers under group extensions as it is given in [Koe13].

Theorem. ([Hal67, Theorem 2.1] Let G be a finite group and $M \trianglelefteq G$. Then

$$n_p(G) = n_p(G/M)n_p(M)n_p(\mathrm{N}_{PM}(P \cap M)/P \cap M).$$

The proof of part b) is based on the following

Proposition 2. Let G be a group with $sn(G) = \{1, a, b\}$. Let Q be a normal Sylow q-subgroup of G. Then the Sylow numbers of G/Q either are $sn(G/Q) = \{1, \overline{a}, \overline{b}\}$, $sn(G/Q) = \{\overline{a}, \overline{b}\}$, $sn(G/Q) = \{1, \overline{a}\}$ (resp. $sn(G/Q) = \{1, \overline{b}\}$) or $sn(G/Q) = \{1\}$, where $\overline{a}$ is the q'-part of a respectively $\overline{b}$ is the q'-part of b.

Proof of Proposition 2. Let $p \neq q$ and let $P \in \mathrm{Syl}_p(G)$. Let q^k be the highest power of q with $q^k|n_p(G)$.
Then

$$n_p(G) = n_p(G/Q)n_p(Q)n_p(\mathrm{N}_{PQ}(P \cap Q)/(P \cap Q)) = n_p(G/Q) \cdot n_p(PQ)$$

and $n_p(PQ) = q^k$ due to $q \nmid n_p(G/Q)$. Therefore $n_p(G/Q) \in \{1, \overline{a}, \overline{b}\}$.

Proof of part b). We use induction on $|G|$: If $|G| = 2$, then $|G|$ is solvable. Consider $|G| \geq 3$ and $|\pi(G)| \geq 3$. There exist a normal Sylow q-subgroup Q of G with $n_q(G) = 1$. Q is as $q-$group solvable. With Proposition 2 G/Q has 3 or less Sylow numbers. G/Q is solvable for $|n_p(G/Q)| \leq 2$. If $n_p(G/Q) = \{1, \overline{a}, \overline{b}\}$ we use the induction hypothesis for G/Q.

Literatur

[Chi98] Naoki Chigira: *Number of Sylow subgroups and p-Nilpotence of finite groups*. Journal of Algebra 201, 71-85 (1998).

[Guo96] Wenbin Guo: *Finite Groups with given indices of normalizers of Sylow subgroups*. Siberian Math. Journal 37 No. 2, 253-257 (1996).

[Hal67] Marshall Hall, Jr.: *On the Number of Sylow Subgroups in a Finite Group*. Journal of Algebra 7, 363-371 (1967).

[Koe13] Iris Köster: *Sylowzahlen*. Master Thesis, University of Stuttgart (2013).

[Luc98] Florian Luca: *Groups with two Sylow numbers are solvable*. Arch. Math. 71, 95-96 (1998).

[Mor12] Alexander Moretó: *Groups with two Sylow numbers are the product of two nilpotent Hall subgroups*. Arch. Math 99, 301-304 (2012)

[Zha95] Jiping Zhang: *Sylow Numbers of Finite Groups*. Journal of Algebra 176 111-123 (1995).

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