Universität Stuttgart

Fachbereich Mathematik

The growth optimal investment strategy is secure, too.

László Györfi, György Ottucsák, Harro Walk

Preprint 2014/012

Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik.uni-stuttgart.de/preprints

ISSN **1613-8309**

The growth optimal investment strategy is secure, too.

László Györfi, György Ottucsák, and Harro Walk

This paper is a revisit of discrete time, multi period and sequential investment strategies for financial markets showing that the log-optimal strategies are secure, too. Using exponential inequality of large deviation type, the rate of convergence of the average growth rate is bounded both for memoryless and for Markov market processes. A kind of security indicator of an investment strategy can be the market time achieving a target wealth. It is shown that the log-optimal principle is optimal in this respect.

1 Introduction

This paper gives some additional features of the investment strategies in financial stock markets inspired by the results of information theory, non-parametric statistics and machine learning. Investment strategies are allowed to use information collected from the past of the market and determine, at the beginning of a trading period, a portfolio, that is, a way to distribute their current capital among the available assets. The goal of the investor is to maximize his wealth in the long run without knowing the underlying distribution generating the stock prices. Under this assumption the asymptotic

---

László Györfi and György Ottucsák
Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Stoczek u.2, 1521 Budapest, Hungary e-mail: gyorfi@cs.bme.hu, oti@cs.bme.hu. This work was partially supported by the European Union and the European Social Fund through project FuturICT.hu (grant no.: TAMOP-4.2.2.C-11/1/KONV-2012-0013).

Harro Walk
Department of Mathematics, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart e-mail: walk@mathematik.uni-stuttgart.de.

rate of growth has a well-defined maximum which can be achieved in full knowledge of the underlying distribution generated by the stock prices.

In Section 2, under memoryless assumption on the underlying process generating the asset prices, the log-optimal portfolio achieves the maximal asymptotic average growth rate, that is, the expected value of the logarithm of the return for the best fix portfolio vector. Using exponential inequality of large deviation type, the rate of convergence of the average growth rate to the optimum growth rate is bounded. Consider a security indicator of an investment strategy, which is the market time achieving a target wealth. The log-optimal principle is optimal in this respect, too.

In Section 3, for generalized dynamic portfolio selection, when asset prices are generated by a stationary and ergodic process, there are universally consistent (empirical) methods that achieve the maximal possible growth rate. If the market process is a first order Markov process, then the rate of convergence of the average growth rate is obtained more generally.

Consider a market consisting of d assets. The evolution of the market in time is represented by a sequence of price vectors $\mathbf{S}_1, \mathbf{S}_2, \ldots \in \mathbb{R}_+^d$, where

$$\mathbf{S}_n = (S_n^{(1)}, \ldots, S_n^{(d)})$$

such that the j-th component $S_n^{(j)}$ of $\mathbf{S}_n$ denotes the price of the j-th asset on the n-th trading period.

Let us transform the sequence of price vectors $\{\mathbf{S}_n\}$ into the sequence of return (relative price) vectors $\{\mathbf{X}_n\}$ as follows:

$$\mathbf{X}_n = (X_n^{(1)}, \ldots, X_n^{(d)})$$

such that

$$X_n^{(j)} = \frac{S_n^{(j)}}{S_{n-1}^{(j)}}.$$

Thus, the j-th component $X_n^{(j)}$ of the return vector $\mathbf{X}_n$ denotes the amount obtained after investing a unit capital in the j-th asset on the n-th trading period.

2 Constantly rebalanced portfolio selection

The dynamic portfolio selection is a multi-period investment strategy, where at the beginning of each trading period the investor rearranges the wealth among the assets. A representative example of the dynamic portfolio selection is the constantly rebalanced portfolio (CRP). The investor is allowed to diversify his capital at the beginning of each trading period according to a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$. The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in asset j. Throughout the paper it is assumed that the portfolio vector $\mathbf{b}$ has nonnegative components with $\sum_{j=1}^{d} b^{(j)} = 1$. The fact that $\sum_{j=1}^{d} b^{(j)} = 1$ means that the investment strategy is self financing and consumption of capital is excluded. The non-negativity of the components of $\mathbf{b}$ means that short selling and buying stocks on margin are not permitted. The simplex of possible portfolio vectors is denoted by Δ_d.

Let S_0 denote the investor's initial capital. Then at the beginning of the first trading period $S_0 b^{(j)}$ is invested into asset j, and it results in return $S_0 b^{(j)} x_1^{(j)}$, therefore at the end of the first trading period the investor's wealth becomes

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} X_1^{(j)} = S_0 \langle \mathbf{b}, \mathbf{X}_1 \rangle ,$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. For the second trading period, S_1 is the new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b}, \mathbf{X}_2 \rangle = S_0 \cdot \langle \mathbf{b}, \mathbf{X}_1 \rangle \cdot \langle \mathbf{b}, \mathbf{X}_2 \rangle .$$

By induction, for the trading period n the initial capital is S_{n-1}, therefore

$$S_n = S_{n-1} \langle \mathbf{b}, \mathbf{X}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b}, \mathbf{X}_i \rangle .$$

The asymptotic average growth rate of this portfolio selection is

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \ln S_n &= \lim_{n\to\infty} \left(\frac{1}{n} \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}, \mathbf{X}_i \rangle \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}, \mathbf{X}_i \rangle , \end{aligned}$$

therefore without loss of generality one can assume in the sequel that the initial capital $S_0 = 1$.

If the market process $\{\mathbf{X}_i\}$ is memoryless, i.e., it is a sequence of independent and identically distributed (i.i.d.) random return vectors then we show that the best constantly rebalanced portfolio (BCRP) is the log-optimal portfolio:

$$\mathbf{b}^* := \arg\max_{\mathbf{b}\in\Delta_d} \mathbb{E}\{\ln \langle \mathbf{b}, \mathbf{X}_1 \rangle\}.$$

This optimality was formulated as follows:

Proposition 1. *(Kelly [30], Latané [32], Breiman [11], Finkelstein and Whitley [19], Barron and Cover [8].) If $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after day n achieved by a log-optimal portfolio strategy $\mathbf{b}^*$, then for any portfolio strategy $\mathbf{b}$ with finite $\mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_1\rangle\}$ and with capital $S_n = S_n(\mathbf{b})$ and for any memoryless market process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,*

$$\lim_{n\to\infty} \frac{1}{n}\ln S_n \le \lim_{n\to\infty}\frac{1}{n}\ln S_n^* \quad \textit{almost surely (a.s.)} \tag{1}$$

and maximal asymptotic average growth rate is

$$\lim_{n\to\infty}\frac{1}{n}\ln S_n^* = W^* := \mathbb{E}\{\ln\langle \mathbf{b}^*, \mathbf{X}_1\rangle\} \quad \textit{a.s.}$$

Proof. This optimality is a simple consequence of the strong law of large numbers. Introduce the notation

$$W(\mathbf{b}) = \mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_1\rangle\}.$$

Then

$$\begin{aligned}\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n}\ln\langle \mathbf{b}, \mathbf{X}_i\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_i\rangle\} + \frac{1}{n}\sum_{i=1}^{n}\left(\ln\langle \mathbf{b}, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_i\rangle\}\right) \\ &= W(\mathbf{b}) + \frac{1}{n}\sum_{i=1}^{n}\left(\ln\langle \mathbf{b}, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_i\rangle\}\right).\end{aligned}$$

Kolmogorov's strong law of large numbers implies that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\ln\langle \mathbf{b}, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_i\rangle\}\right) \to 0 \quad \text{a.s.},$$

therefore

$$\lim_{n\to\infty}\frac{1}{n}\ln S_n = W(\mathbf{b}) = \mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_1\rangle\} \quad \text{a.s.}$$

Similarly,

$$\lim_{n\to\infty}\frac{1}{n}\ln S_n^* = W^* := W(\mathbf{b}^*) = \max_{\mathbf{b}} W(\mathbf{b}) \quad \text{a.s.}$$

■

In Kuhn and Luenberger [31] the log-optimal portfolio selection was studied for a continuous time model, where the main question of interest is the choice of sampling frequency such that the rebalancing is done at sampling time instances. They assumed that the assets' prices are cross-correlated geometric motions and therefore the return vectors of sampled price processes are memoryless. For high sampling frequency, the log-optimal strategy is a special case of mean-variance rule, called semi-log-optimal strategy (cf. Györfi, Urbán, Vajda [23], Pulley [36], Roll [37]).

There is an obvious question here: how secure a growth optimal portfolio strategy is? The strong law of large numbers has another interpretation. Put

$$R_n := \inf_{n \le m} \frac{1}{m} \ln S_m^*,$$

then e^{nR_n} is a lower exponential envelope for S_n^*, i.e.,

$$e^{nR_n} \le S_n^*.$$

Moreover,

$$R_n \uparrow W^* \quad \text{a.s.},$$

which means that for an arbitrary $R < W^*$, we have that

$$e^{nR} \le S_n^*$$

for all n after a random time N large enough.

In the sequel we bound N, i.e., derive a rate of convergence of the strong law of large numbers. Assume that there exist $0 < a_1 < 1 < a_2 < \infty$ such that

$$a_1 \le X^{(j)} \le a_2 \tag{2}$$

for all $j = 1, \dots, d$. For the New York Stock Exchange (NYSE) daily data, this condition is satisfied with $a_1 = 0.7$ and with $a_2 = 1.2$. $a_1 = 0.7$ means that the worst that happened in a single day was 30% drop, while $a_2 = 1.2$ corresponds to 20% increase within a day. (Cf. Fernholz [18], Horváth and Urbán [28].) Figure 1 shows the histogram of Coca Cola's daily logarithmic relative prices such that most of the days the relative prices are in the interval $[0.95, 1.05]$ from 1962 to 2006. Here are some statistical data:

$$\begin{aligned} \text{minimum} &= -0.2836 \\ \text{1st qu.} &= -0.0074 \\ \text{median} &= 0.0000 \\ \text{mean} &= 0.00053 \\ \text{3rd qu.} &= 0.0083 \\ \text{maximum} &= 0.1796. \end{aligned}$$

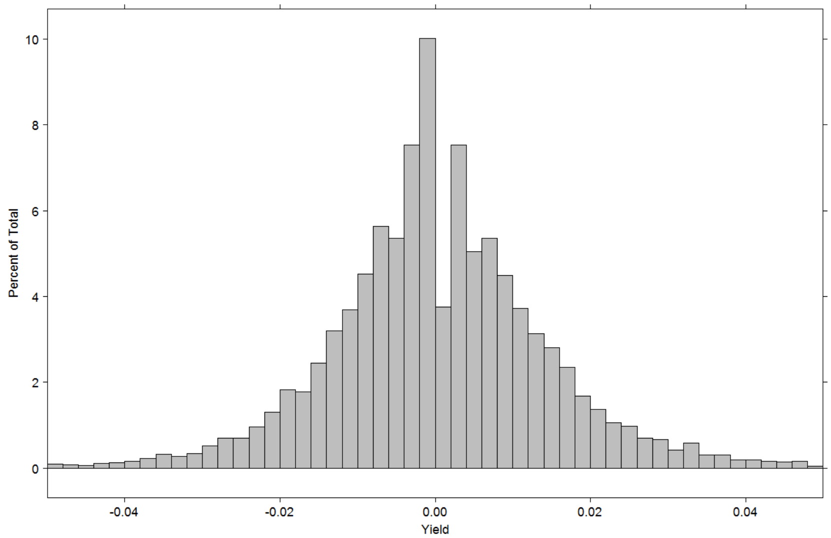

Fig. 1 The histogram of log-returns for Coca Cola

Theorem 1. *If the market process $\{\mathbf{X}_i\}$ is memoryless and the condition (2) is satisfied, then for an arbitrary $R < W^*$, we have that*

$$\mathbb{P}\left\{e^{nR} > S_n^*\right\} \leq e^{-2n\frac{(W^*-R)^2}{(\ln a_2 - \ln a_1)^2}}.$$

Proof. We have that

$$\mathbb{P}\left\{e^{nR} > S_n^*\right\} = \mathbb{P}\left\{R > \frac{1}{n}\ln S_n^*\right\}$$
$$= \mathbb{P}\left\{R - W^* > \frac{1}{n}\sum_{i=1}^{n}(\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle\})\right\}.$$

Apply the Hoeffding [27] inequality: Let $X_1, \ldots, X_n$ be independent random variables with $X_i \in [c, c+K]$ with probability one. Then, for all $\epsilon > 0$,

$$\mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}\{X_i\}) < -\epsilon\right\} \leq e^{-2n\frac{\epsilon^2}{K^2}}.$$

Because of the condition,

$$\ln a_1 \leq \ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle \leq \ln a_2,$$

therefore the theorem follows from the Hoeffding inequality for the correspondences

$$\epsilon = W^* - R$$

and

$$X_i = \ln \langle \mathbf{b}^* , \mathbf{X}_i \rangle$$

and

$$K = \ln a_2 - \ln a_1 .$$

■

Using Theorem 1, we can bound the probability that after n there is a time instant m such that $e^{mR} > S^*_m$:

Corollary 1. *If the market process $\{\mathbf{X}_i\}$ is memoryless and the condition (2) is satisfied, then for an arbitrary $R < W^*$, we have that*

$$\mathbb{P}\left\{\cup_{m=n}^{\infty}\{e^{mR} > S^*_m\}\right\} \le e^{-2n\frac{(W^*-R)^2}{K^2}} \frac{e^{2\frac{(W^*-R)^2}{K^2}}}{e^{2\frac{(W^*-R)^2}{K^2}} - 1}. \tag{3}$$

Proof. From Theorem 1 we get that

$$\begin{aligned}
\mathbb{P}\left\{\cup_{m=n}^{\infty}\{e^{mR} > S^*_m\}\right\} &\le \sum_{m=n}^{\infty} \mathbb{P}\left\{e^{mR} > S^*_m\right\} \\
&\le \sum_{m=n}^{\infty} e^{-2m\frac{(W^*-R)^2}{(\ln a_2 - \ln a_1)^2}} \\
&= e^{-2n\frac{(W^*-R)^2}{(\ln a_2 - \ln a_1)^2}} \frac{1}{1 - e^{-2\frac{(W^*-R)^2}{(\ln a_2 - \ln a_1)^2}}}.
\end{aligned}$$

■

Theorem 1 and Corollary 1 are about the probability of underperformance depending on a_1 and a_2. Using central limit theorem (CLT), one can derive modifications of Theorem 1 and Corollary 1. The advantage of the CLT is that the resulted formula does not depend on a_1 and a_2, it depends only of the variance of the log-returns. However, in contrast to large deviation bounds, the CLT is only an approximation.

An additional hard open problem is how to construct empirical strategies taking into account proportional transaction cost (see, for example, Györfi and Walk [24], [25]).

When it comes to security, the small-sample behavior should be more interesting. Consider the relative amount of times j between 1 and n, for which S_j^* is below e^{jR} for $R < W^*$ near to W^*, say $R = R_n = W^* - \frac{m}{\sqrt{n}}\sigma$ for fixed $m > 0$ with $\sigma^2 = \mathbf{Var}(\ln\langle \mathbf{b}^*, \mathbf{X}_1\rangle)$ assumed to be positive and finite. For $0 \le x \le 1$ we have

$$\begin{aligned}
&\mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n}\mathbb{I}_{\{S_j^* < e^{jR}\}} \le x\right\}\\
&= \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n}\mathbb{I}_{\{\frac{1}{j}\sum_{i=1}^{j}(\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle\}) < R - W^*\}} \le x\right\}\\
&= \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n}\mathbb{I}_{\{\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{j}(\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle\}) + m\frac{j}{n} < 0\}} \le x\right\}\\
&\to \mathbb{P}\left\{\int_0^1 \mathbb{I}_{\{W(u)+mu\le 0\}}du \le x\right\}
\end{aligned}$$

with standard Brownian motion W, by Donsker's functional central limit theorem (see Billingsley [9]) for the functional $f \to \int_0^1 \mathbb{I}_{\{f(u)+mu\le 0\}}du$.

By the generalized arc-sine law of Takács [41] the right hand side equals

$$\begin{aligned}
&F_m(x)\\
&:= 2\int_0^x \left[\frac{\varphi(m\sqrt{1-u})}{\sqrt{1-u}} + m\Phi(m\sqrt{1-u})\right]\left[\frac{\varphi\left(-m\sqrt{u}\right)}{\sqrt{u}} - m\Phi\left(-m\sqrt{u}\right)\right]du
\end{aligned}$$

for $0 \le x \le 1$, where $F_m(1) = 1$, and φ and Φ are the standard normal density and distribution functions, respectively. We have a non-degenerate limit distribution. Here for $m \to \infty$ and also for the case $R = R_n'$ with $(W^* - R)\sqrt{n} \to \infty$, especially a constant $R_n' < W^*$, we have degeneration to the Dirac distribution concentrated at 0. The proof of these assertions can be as follows: For each $0 < \epsilon < 1/2$, on $[\epsilon, 1-\epsilon]$ the uniformly bounded integrand uniformly converges to 0 for $m \to \infty$, thus $F_m(1-\epsilon) - F(\epsilon) \to 0$. Further $F_m(0) = 0$ and $F_m(1) = 1$ for each m, and $F_m(x)$ is non-decreasing for each $0 \le x \le 1$. Thus, $F_m(x) \to 1$ for each $0 < x \le 1$. Finally one notices that $m < \sqrt{n}(W^* - R_n') \to \infty$ $(n \to \infty)$ implies

$$\liminf_n \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n}\mathbb{I}_{\{\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{j}(\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle\}) + \sqrt{n}(W^* - R_n')\frac{j}{n} < 0\}} \le x\right\}$$

$$\geq \lim_n \mathbb{P}\left\{ \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}_{\{\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{j}(\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle - \mathbb{E}\{\ln\langle \mathbf{b}^*, \mathbf{X}_i\rangle\}) + m\frac{j}{n} < 0\}} \leq x \right\},$$

for each m. It should be mentioned that under the assumption (2) the latter of the assertions is also a consequence of Theorem 1 for $R = R_n'$.

In the literature there is a discussion on good and bad properties of log-optimal investment (see MacLean, Thorp and Ziemba [34], sections 30 and 39, with references). Beside

$$\limsup \frac{1}{n}\log(S_n/S_n^*) \leq 0$$

almost surely (see (1) and (4) below, good long-run performance) one has

$$\mathbb{E}\{S_n/S_n^*\} \leq 1$$

for all n (good short-term performance). Both properties were established by Algoet and Cover [3] in the much more general context of a stationary and ergodic process of daily returns $\mathbf{X}_n$ and conditionally log-optimal investment (here regarding past returns, but nothing more: myopic policy). Leaving the concept of a logarithmic utility function induced by the multiplicative structure of investment, Samuelson [38] in his critics pointed out that maximizing the expected return $\mathbb{E}\{\langle \mathbf{b}, \mathbf{X}_i\rangle\}$ instead of expected logarithmic return, with in this sense optimal portfolio choice $\mathbf{b}^{**}$ and corresponding wealth S_n^{**}, leads to $\mathbb{E}\{S_n^{**}\}/\mathbb{E}\{S_n^*\} \to \infty$, see also the comments of Markowitz [35]. But under the risk aspect of the deviation of a random variable from its expectation, use of logarithm is more advantageous. The log transform is a special case of the Box-Cox [10] transforms introduced in view of stabilization and widely used in science, e.g., in medical science. Nevertheless there is the question whether the risk aversion of log utility is big enough to save an investor with very high probability from large terminal losses for medium time horizon. Simulation studies discussed by MacLean, Thorp, Zhao and Ziemba in MacLean, Thorp and Ziemba [34], section 38, show that in a minority of scenarios such events occur. These effects depend on time horizon and distribution of the daily return, which allows a "proper use in the short and medium run" provided one has a good knowledge of the distribution. Corollary 1 allows for small $\epsilon > 0$ to obtain a lower bound N for the time horizon having a probability $\geq 1 - \epsilon$ that after this time the investor's wealth is for ever at least the unit starting capital: on the right-hand side of (3) set $R = 0$ and then choose N as the lowest integer n such that the right-hand side is at most ϵ. Here as in the following, $W^* > 0$ is assumed.

The good long-run and short-run performance of the various strategies are discussed in the literature, but usually the corresponding results concern only the expectation. Both in financial theory and practice, people care about the distribution as well. For the log-optimal strategy, there are almost sure statements, too (cf. Proposition 1).

Besides the growth rate of an investment strategy, one may consider the market time achieving a target wealth. We consider only strategies $\mathbf{b}$ with $\mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_1\rangle\} > 0$. Again, $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after day n applying log-optimum portfolio strategy $\mathbf{b}^*$, and $S_n = S_n(\mathbf{b})$ the capital using the portfolio strategy $\mathbf{b}$. For a target wealth $\bar{s}$, introduce the market times

$$\tau(\bar{s}) := \min\{m; S_m \geq \bar{s}\}$$

and similarly

$$\tau^*(\bar{s}) := \min\{m; S_m^* \geq \bar{s}\}.$$

There are some studies how to minimize the expected market time $\mathbb{E}\{\tau(\bar{s})\}$ for large $\bar{s}$ (Aucamp [5], [6], Breiman [11], Hayes [26], Kadaras and Platen [29]), where Ethier [16] established an asymptotic median log-optimality of the (mean) log-optimal investment strategy. Breiman [11] conjectured that, for large $\bar{s}$, the asymptotically best strategy is the growth optimal one such that we apply the growth optimal strategy until we reach a neighborhood of $\bar{s}$.

Using the representation

$$\{S_m \geq \bar{s}\} = \left\{\sum_{i=1}^{m} \ln\langle \mathbf{b}, \mathbf{X}_i\rangle \geq \ln \bar{s}\right\}$$

the renewal theory for extended renewal processes, i.e., random walks with drift (see, for instance, Breiman [12] and Feller [17]), yields

Proposition 2. *(Breiman [11].) One has that*

$$\frac{\tau(\bar{s})}{\ln \bar{s}} \to \frac{1}{\mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_1\rangle\}}$$

a.s.,

$$\frac{\mathbb{E}\{\tau(\bar{s})\}}{\ln \bar{s}} \to \frac{1}{\mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_1\rangle\}},$$

especially

$$\frac{\tau^*(\bar{s})}{\ln \bar{s}} \to \frac{1}{W^*}$$

a.s.,

$$\frac{\mathbb{E}\{\tau^*(\bar{s})\}}{\ln \bar{s}} \to \frac{1}{W^*}$$

($\bar{s} \to \infty$).

In this sense the growth optimal strategy has another optimality property. This result has been refined by Breiman [11] and can be extended to

$$\begin{aligned}
&\frac{\ln \bar{s}}{\mathbb{E}\{\ln \langle \mathbf{b}^*, \mathbf{X}_1\rangle\}} - \frac{\ln \bar{s}}{\mathbb{E}\{\ln \langle \mathbf{b}, \mathbf{X}_1\rangle\}} + \frac{\mathbb{E}\{((\ln \langle \mathbf{b}^*, \mathbf{X}_1\rangle)_+)^2\}}{(\mathbb{E}\{\ln \langle \mathbf{b}^*, \mathbf{X}_1\rangle\})^2} \\
&\geq \mathbb{E}\{\tau^*(\bar{s})\} - \mathbb{E}\{\tau(\bar{s})\} \\
&\geq \frac{\ln \bar{s}}{\mathbb{E}\{\ln \langle \mathbf{b}^*, \mathbf{X}_1\rangle\}} - \frac{\ln \bar{s}}{\mathbb{E}\{\ln \langle \mathbf{b}, \mathbf{X}_1\rangle\}} - \frac{\mathbb{E}\{((\ln \langle \mathbf{b}, \mathbf{X}_1\rangle)_+)^2\}}{(\mathbb{E}\{\ln \langle \mathbf{b}, \mathbf{X}_1\rangle\})^2}
\end{aligned}$$

by Lorden's [33] upper bound for excess result.

Next we bound the tail distribution of $\tau^*(\bar{s})$ in case of large $\bar{s} = e^{nR}$, where $R < W^*$. We get that

$$\mathbb{P}\{\tau^*(e^{nR}) > n\} = \mathbb{P}\left\{\cap_{m=1}^{n}\{S_m^* < e^{nR}\}\right\} \leq \mathbb{P}\left\{S_n^* < e^{nR}\right\},$$

therefore Theorem 1 implies that

$$\mathbb{P}\{\tau^*(e^{nR}) > n\} \leq e^{-2n\frac{(W^*-R)^2}{(\ln a_2 - \ln a_1)^2}}.$$

3 Time varying portfolio selection

For a general dynamic portfolio selection, the portfolio vector may depend on the past data. As before, $\mathbf{X}_i = (X_i^{(1)}, \ldots X_i^{(d)})$ denotes the return vector on trading period i. Moreover, denote the segment $\mathbf{X}_1, \ldots, \mathbf{X}_i$ by $\mathbf{X}_1^i$. Let $\mathbf{b} = \mathbf{b}_1$ be the portfolio vector for the first trading period. For initial capital S_0, we get that

$$S_1 = S_0 \cdot \langle \mathbf{b}_1, \mathbf{X}_1\rangle.$$

For the second trading period, S_1 is new initial capital, the portfolio vector is $\mathbf{b}_2 = \mathbf{b}(\mathbf{X}_1)$, and

$$S_2 = S_0 \cdot \langle \mathbf{b}_1, \mathbf{X}_1\rangle \cdot \langle \mathbf{b}(\mathbf{X}_1), \mathbf{X}_2\rangle.$$

For the nth trading period, a portfolio vector is $\mathbf{b}_n = \mathbf{b}(\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}) = \mathbf{b}(\mathbf{X}_1^{n-1})$ and

$$S_n = S_0 \prod_{i=1}^{n} \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle = S_0 e^{nW_n(\mathbf{B})}$$

with the average growth rate

$$W_n(\mathbf{B}) = \frac{1}{n}\sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle.$$

The fundamental limits, determined in Algoet and Cover [3], and in Algoet [1, 2], reveal that the so-called *log-optimum portfolio* $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$ is the best possible choice.

Proposition 3. *(Algoet and Cover [3].) On trading period n let $\mathbf{b}^*(\cdot)$ be such that*

$$\mathbb{E}\left\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n\right\rangle \middle| \mathbf{X}_1^{n-1}\right\} = \max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n\right\rangle \middle| \mathbf{X}_1^{n-1}\right\}.$$

If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, after n trading periods, then for any other investment strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and with

$$\sup_n \mathbb{E}\left\{(\ln\left\langle \mathbf{b}_n(\mathbf{X}_1^{n-1}), \mathbf{X}_n\right\rangle)^2\right\} < \infty,$$

and for any stationary and ergodic process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{S_n}{S_n^*} \le 0 \quad a.s. \tag{4}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* \quad a.s., \tag{5}$$

where

$$W^* := \mathbb{E}\left\{\max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{\ln\left\langle \mathbf{b}(\mathbf{X}_{-\infty}^{-1}), \mathbf{X}_0\right\rangle \middle| \mathbf{X}_{-\infty}^{-1}\right\}\right\}$$

is the maximal possible growth rate of any investment strategy.

Note that for memoryless markets $W^* = \max_{\mathbf{b}} \mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_0\rangle\}$ which shows that in this case the log-optimal portfolio is a constantly rebalanced portfolio.

Proof. For martingale difference sequences, there is a strong law of large numbers: If $\{Z_n\}$ is a martingale difference sequence with respect to $\{X_n\}$ and

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\{Z_n^2\}}{n^2} < \infty$$

then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Z_i = 0 \text{ a.s.}$$

(cf. Chow [13], see also Stout [40, Theorem 3.3.1]). Introduce the decomposition

$$\begin{aligned}\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n}\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \\ &+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle - \mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\right).\end{aligned}$$

The last average is an average of martingale differences, so it tends to zero a.s. Similarly,

$$\begin{aligned}\frac{1}{n}\ln S_n^* &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \\ &+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle - \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\right).\end{aligned}$$

Because of the definition of the log-optimal portfolio we have that

$$\mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \le \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\},$$

and the proof of (4) is finished. In order to prove (5) we have to show that

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \to W^*$$

a.s. Introduce the notations

$$\mathbf{b}^*_{-k}(\mathbf{X}_{n-k}^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{\ln\left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}),\mathbf{X}_n\right\rangle \mid \mathbf{X}_{n-k}^{n-1}\right\}$$

($1 \le k < n$) and

$$\mathbf{b}^*_{-\infty}(\mathbf{X}_{-\infty}^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{\ln\left\langle \mathbf{b}(\mathbf{X}_{-\infty}^{n-1}),\mathbf{X}_n\right\rangle \mid \mathbf{X}_{-\infty}^{n-1}\right\}.$$

Obviously,

$$\mathbb{E}\{\ln\left\langle \mathbf{b}^*_{-k}(\mathbf{X}_{i-k}^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_{i-k}^{i-1}\} \le \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}$$

($i > k$) and

$$\mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\} \le \mathbb{E}\{\ln\left\langle \mathbf{b}^*_{-\infty}(\mathbf{X}_{-\infty}^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_{-\infty}^{i-1}\}.$$

Thus, the ergodic theorem implies that

$$\begin{aligned} W^*_{-k} &:= \mathbb{E}\left\{\max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{ \ln\left\langle \mathbf{b}(\mathbf{X}_{-k}^{-1}), \mathbf{X}_0 \right\rangle \middle| \mathbf{X}_{-k}^{-1}\right\}\right\} \\ &= \lim_n \frac{1}{n}\sum_{i=1}^n \mathbb{E}\{\ln\left\langle \mathbf{b}^*_{-k}(\mathbf{X}_{i-k}^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_{i-k}^{i-1}\} \\ &\le \liminf_n \frac{1}{n}\sum_{i=1}^n \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\} \end{aligned}$$

a.s. and

$$\begin{aligned} &\limsup_n \frac{1}{n}\sum_{i=1}^n \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\} \\ &\le \lim_n \frac{1}{n}\sum_{i=1}^n \mathbb{E}\{\ln\left\langle \mathbf{b}^*_{-\infty}(\mathbf{X}_{-\infty}^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_{-\infty}^{i-1}\} = W^*. \end{aligned}$$

a.s. Using martingale argument one can check that

$$W^*_{-k} \uparrow W^*,$$

and so (5) is proved. ∎

Put

$$\epsilon = \frac{W^* - R}{2}. \tag{6}$$

Concerning the rate of convergence we have that

Theorem 2. *If the market process $\{\mathbf{X}_i\}$ is stationary, ergodic and the condition (2) is satisfied, then for an arbitrary $R < W^*$, we have that*

$$\mathbb{P}\left\{e^{nR} > S_n^*\right\} \le e^{-n\frac{(W^*-R)^2}{2(\ln a_2 - \ln a_1)^2}} + \mathbb{P}\left\{R + \epsilon > \frac{1}{n}\sum_{i=1}^n \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\}\right\}.$$

Proof. Apply the previous decomposition:

$$\begin{aligned} &\mathbb{P}\left\{e^{nR} > S_n^*\right\} \\ &= \mathbb{P}\left\{R > \frac{1}{n}\ln S_n^*\right\} \\ &= \mathbb{P}\left\{R + \epsilon - \epsilon > \frac{1}{n}\sum_{i=1}^n \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\}\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n}\sum_{i=1}^{n}\Big(\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle - \mathbb{E}\{\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle \mid \mathbf{X}_1^{i-1}\}\Big)\Big\}\\
&\le \mathbb{P}\Big\{R+\epsilon > \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle \mid \mathbf{X}_1^{i-1}\}\Big\}\\
&+\mathbb{P}\Big\{-\epsilon > \frac{1}{n}\sum_{i=1}^{n}\Big(\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle - \mathbb{E}\{\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle \mid \mathbf{X}_1^{i-1}\}\Big)\Big\}
\end{aligned}
$$

For the second term of the right hand side, we apply the Hoeffding [27], Azuma [7] inequality: Let $X_1, X_2, \dots$ be a sequence of random variables, and assume that $V_1, V_2, \dots$ is a martingale difference sequence with respect to $X_1, X_2, \dots$. Assume, furthermore, that there exist random variables $Z_1, Z_2, \dots$ and nonnegative constants $c_1, c_2, \dots$ such that for every $i > 0$, Z_i is a function of $X_1, \dots, X_{i-1}$, and

$$Z_i \le V_i \le Z_i + c_i \quad \text{a.s.}$$

Then, for any $\epsilon > 0$ and n,

$$\mathbb{P}\left\{\sum_{i=1}^{n} V_i \ge \epsilon\right\} \le e^{-2\epsilon^2/\sum_{i=1}^{n} c_i^2}$$

and

$$\mathbb{P}\left\{\sum_{i=1}^{n} V_i \le -\epsilon\right\} \le e^{-2\epsilon^2/\sum_{i=1}^{n} c_i^2}.$$

Thus

$$
\begin{aligned}
&\mathbb{P}\Big\{-\epsilon > \frac{1}{n}\sum_{i=1}^{n}\Big(\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle - \mathbb{E}\{\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle \mid \mathbf{X}_1^{i-1}\}\Big)\Big\}\\
&\le e^{-2n\frac{\epsilon^2}{(\ln a_2 - \ln a_1)^2}}\\
&= e^{-n\frac{(W^*-R)^2}{2(\ln a_2 - \ln a_1)^2}}.
\end{aligned}
$$

■

If the market process is just stationary and ergodic, then it is impossible to get rate of convergence of the term

$$\mathbb{P}\Big\{R+\epsilon > \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\Big\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\Big\rangle \mid \mathbf{X}_1^{i-1}\}\Big\}.$$

In order to find conditions, for which a rate can be derived, one possibility is that for $i > k$

$$\begin{aligned}\mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} &= \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\\ &= \max_{\mathbf{b}} \mathbb{E}\{\ln\left\langle \mathbf{b}, \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\\ &\geq \max_{\mathbf{b}} \mathbb{E}\{\ln\left\langle \mathbf{b}, \mathbf{X}_i\right\rangle \mid \mathbf{X}_{i-k}^{i-1}\},\end{aligned}$$

and so we may increase the above probability. We expected that the density of

$$\max_{\mathbf{b}} \mathbb{E}\{\ln\left\langle \mathbf{b}, \mathbf{X}_{k+1}\right\rangle \mid \mathbf{X}_1^{k}\}$$

has a small support, which moves to the right, when k increases.

We made an experiment verifying this conjecture empirically. At the web page `http://www.szit.bme.hu/~oti/portfolio` there are two benchmark data sets from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years (5651 trading days ending in 1985). More precisely, the data set contains the daily price relatives, that was calculated from the nominal values of the *closing prices* corrected by the dividends and the splits for all trading day. This data set has been used for testing portfolio selection in Cover [15], Singer [39], Györfi, Lugosi, Udina [20], Györfi, Ottucsák, Urbán [21], Györfi, Udina, Walk [22] and Györfi, Urbán, Vajda [23].
- The second data set is an extended version of the first one. It was augmented by 22 years and covers 44 years period from 1962 to 2006 containing 11178 trading days. As opposed to the first data set it contains only 19 stocks out of the 36 stocks due to the fact that 4 illiquid and 13 bankrupted stocks were left out. In the analysis of financial time series there often happens a censoring, which means that the time series is terminated (bankrupt, merging, withdraw from the stock market, etc.). If one takes into account only the non-censored time series, then the survivals cause a bias in the statistical inference, called survival bias. Thus, the leaving out the bankrupted stocks adds survival bias to the simulation. However in case of actively managed portfolio strategies as rebalancing or online portfolio selection the effect of the survival bias is less important than the liquidity of the traded stocks. For example, if IROQU and KINAR (a bankrupted and a small capitalization stock) were not left out then the achieved wealth would be unrealistically high (cf. [20]). Based on the above argument the following 4 illiquid stocks were excluded from the data set: SHERW, KODAK, COMME and KINAR. Further benchmark data sources are available at `http://www.cais.ntu.edu.sg/~`

chhoi/olps/datasets.html. Clearly, the distributions of the market process were not the same over the past 44 years. The empirical strategies applied are not sensitive with respect to the changes of the distributions.

As in Györfi, Ottucsák, Urbán [21], we considered the kernel based portfolio strategies $\mathbf{B}^{(k)} = \{\mathbf{b}^{(k)}(\cdot)\}$, where the window size $k = 1, \dots, 5$ and the corresponding radius is

$$r_k^2 = 0.00035 \cdot d \cdot k.$$

According to the kernel based rule, the portfolio vector for day n is selected such that one searches for similar patterns to the near past segment $\mathbf{X}_{n-k}^{n-1}$ and design a portfolio to the subsequence of return vectors followed the similarities. For $n > k+1$, define the random variable $Z_{n,k}$ by

$$Z_{n,k} = \frac{\max_{\mathbf{b} \in \Delta_d} \sum_{\left\{k<i<n: \|\mathbf{X}_{i-k}^{i-1} - \mathbf{X}_{n-k}^{n-1}\| \le r_k\right\}} \ln \langle \mathbf{b}, \mathbf{X}_i \rangle}{\left| \left\{ k < i < n : \|\mathbf{X}_{i-k}^{i-1} - \mathbf{X}_{n-k}^{n-1}\| \le r_k \right\} \right|},$$

if the sum is non-void. Then the histogram of $\{Z_{n,k}, n = k+1, \dots N\}$ can be an approximation of the density of $\max_{\mathbf{b}} \mathbb{E}\{\ln \langle \mathbf{b}, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k\}$.

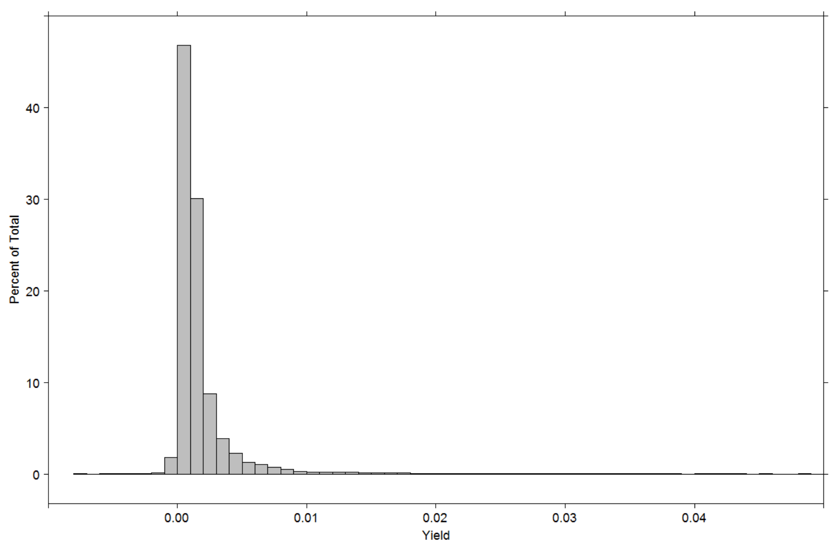

Fig. 2 The histogram of the maximum of the conditional expectations for $k = 1$

For $k = 1, \dots, 5$, we generated the five histograms of the maximum of these empirical conditional expectations. The main observation was that these histograms do not depend on k, therefore one can assume that the market process is a first order Markov process. Figure 2 shows a histogram out of the five, which corresponds to $k=1$. Surprisingly, this histogram has a small support. Here are some data:

$$\begin{aligned}
\text{minimum} &= -0.008\\
\text{1st qu.} &= 0.00061\\
\text{median} &= 0.0010\\
\text{mean} &= 0.0019\\
\text{3rd qu.} &= 0.0018\\
\text{maximum} &= 0.1092.
\end{aligned}$$

An important feature of this histogram is that it has a positive skewness, which means that the right hand side tail is larger than the left hand side one. The reason of this property is that $\max_{\mathbf{b}} \mathbb{E}\{\ln\langle \mathbf{b}, \mathbf{X}_{k+1}\rangle \mid \mathbf{X}_1^k\}$ is the maximum of (dependent) random variables.

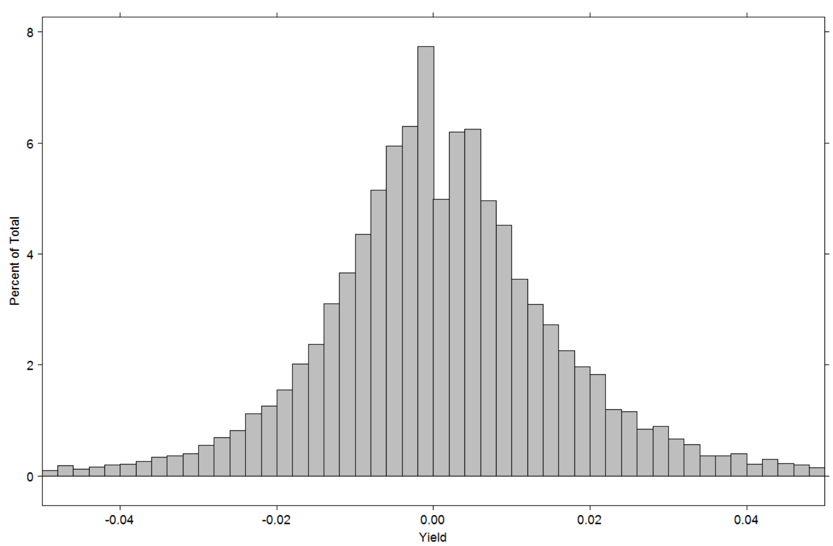

Fig. 3 The histogram of the log-returns for an empirical portfolio strategy

For the kernel based portfolio we generated the histogram of the log-return, too. The elementary portfolio is defined by

$$\mathbf{b}^{(k)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}\in\Delta_d} \sum_{\left\{k<i<n:\|\mathbf{x}_{i-k}^{i-1}-\mathbf{x}_{n-k}^{n-1}\|\le r_k\right\}} \ln\langle \mathbf{b}, \mathbf{x}_i\rangle \ ,$$

if the sum is non-void, and $\mathbf{b}_0 = (1/d, \ldots, 1/d)$ otherwise. Define the random variable $Z'_{n,k}$ by

$$Z'_{n,k} = \ln \left\langle \mathbf{b}^{(k)}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle,$$

which is the daily log-return for day n. For $k = 1$, we generated the histogram of $\{Z'_{n,k}, n = k+1, \ldots N\}$. Figure 3 shows the histogram of the log-return for the empirical portfolio strategy $\mathbf{B}^{(1)}$. Here are the corresponding data:

$$\begin{aligned} \text{minimum} &= -0.1535 \\ \text{1st qu.} &= -0.0077 \\ \text{median} &= 0.0003 \\ \text{mean} &= 0.00118 \\ \text{3rd qu.} &= 0.0093 \\ \text{maximum} &= 0.1522. \end{aligned}$$

Comparing the Figures 1 and 3, one can observe that the shape and the quantiles of the histograms are almost the same. The main difference is in the mean. Since these data sets contains the relative prices for trading days only, and one year consists of 250 trading days, therefore in terms of average annual yields (AAY) the mean= 0.00118 in Figure 3 corresponds to AAY 34%, while the mean= 0.00118 for the Coca Cola corresponds to AAY 14%.

Based on these empirical observations, in the following we assume that the market process $\{\mathbf{X}_i\}$ is a first-order stationary Markov process. In this case the log-optimum portfolio choice has the form $\mathbf{b}^*(\mathbf{X}_{n-1})$ (instead of $\mathbf{b}^*(\mathbf{X}_1^{n-1})$) maximizing $\mathbb{E}\{\ln \langle \mathbf{b}, \mathbf{X}_n \rangle \mid \mathbf{X}_{n-1}\}$ such that

$$\mathbb{E}\{\ln \langle \mathbf{b}^*(\mathbf{X}_{n-1}), \mathbf{X}_n \rangle\} = W^*.$$

We assume that $\mathbf{X}_i$ has a denumerable state space $S \subset [a_1, a_2]^d$, which is realistic because the values of the components of $\mathbf{X}_i$ are quotients of integer valued prices. Further we assume that the Markov process is irreducible and aperiodic. Finally, suppose that the Markov kernel $\nu(H \mid \mathbf{x})$ defined by

$$\nu(H \mid \mathbf{x}) := \mathbb{P}\{\mathbf{X}_2 \in H \mid \mathbf{X}_1 = \mathbf{x}\}$$

($\mathbf{x} \in S$, $H \subset S$) is continuous in total variation, i.e.,

$$V(\mathbf{x}, \mathbf{x}') := \sup_{H \subset S} |\nu(H \mid \mathbf{x}) - \nu(H \mid \mathbf{x}')| \to 0 \tag{7}$$

if $\mathbf{x}' \to \mathbf{x}$. Notice that by Scheffé's theorem

$$V(\mathbf{x},\mathbf{x}') := \frac{1}{2}\sum_{\mathbf{x}^*\in S} |\nu(\{\mathbf{x}^*\}\mid \mathbf{x}) - \nu(\{\mathbf{x}^*\}\mid \mathbf{x}')|.$$

The following theorem with $R < W^*$ gives exponential bounds for the probability that $e^{nR} > S_n^*$ and for the probability that after n there is a time instant m such that $e^{mR} > S_m^*$.

Theorem 3. *Let the market process $\{\mathbf{X}_i\}$ be a first-order stationary denumerable Markov chain, which is irreducible and aperiodic, satisfies (2) and (7). Then for arbitrary $R < W^*$, there exist $c, C, c^*, C^* \in (0,\infty)$ depending on $W^* - R$, $\ln a_2 - \ln a_1$ and the ergodic behavior of $\{\mathbf{X}_i\}$ such that for all n*

$$\mathbb{P}\left\{e^{nR} > S_n^*\right\} \le e^{-n\frac{(W^*-R)^2}{2(\ln a_2 - \ln a_1)^2}} + Ce^{-cn}, \tag{8}$$

and

$$\mathbb{P}\left\{\cup_{m=n}^{\infty}\{e^{mR} > S_m^*\}\right\} \le C^* e^{-c^* n}. \tag{9}$$

Proof. With the notation (6), Theorem 2 implies that

$$\mathbb{P}\left\{e^{nR} > S_n^*\right\} \le e^{-n\frac{(W^*-R)^2}{2(\ln a_2-\ln a_1)^2}} + \mathbb{P}\left\{R+\epsilon > \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\langle \mathbf{b}^*(\mathbf{X}_{i-1}), \mathbf{X}_i\rangle \mid \mathbf{X}_{i-1}\}\right\}.$$

By stationarity, the distribution μ of $\mathbf{X}_i$ does not depend on i and satisfies

$$\int \nu(\cdot\mid \mathbf{x})\mu(d\mathbf{x}) = \mu,$$

i.e.,

$$\sum_{\mathbf{x}\in S}\nu(\{\mathbf{x}^*\}\mid \mathbf{x})\mu(\{\mathbf{x}\}) = \mu(\{\mathbf{x}^*\}). \tag{10}$$

It is well known from the theory of denumerable Markov chains (see, e.g., Feller [17]), that (10) together with irreducibility and aperiodicity of $\{\mathbf{X}_i\}$ implies that $\{\mathbf{X}_i\}$ is positive recurrent with mean recurrence time $1/\mu(\{\mathbf{x}\}) < \infty$ and weak convergence of $P_{\mathbf{X}_n\mid \mathbf{X}_1=\mathbf{x}}$ to μ. Thus, by Scheffé and Riesz-Vitali theorems, even

$$\begin{aligned}
&\sup_{H\subset S}|\mathbb{P}\{\mathbf{X}_n\in H\mid \mathbf{X}_1=\mathbf{x}\} - \mu(H)| \\
&= \frac{1}{2}\sum_{\mathbf{x}^*\in S}|\mathbb{P}\{\mathbf{X}_n = \mathbf{x}^*\mid \mathbf{X}_1=\mathbf{x}\} - \mu(\{\mathbf{x}^*\})| \\
&\to 0
\end{aligned}$$

$(n\to\infty)$ for each $\mathbf{x}\in S$. Further for each integer n

$$
\begin{aligned}
&\sup_{H\subset S}|\mathbb{P}\{\mathbf{X}_n\in H\mid \mathbf{X}_1=\mathbf{x}\}-\mathbb{P}\{\mathbf{X}_n\in H\mid \mathbf{X}_1=\mathbf{x}'\}|\\
&=\frac{1}{2}\sum_{\mathbf{x}^*\in S}|\mathbb{P}\{\mathbf{X}_n=\mathbf{x}^*\mid \mathbf{X}_1=\mathbf{x}\}-\mathbb{P}\{\mathbf{X}_n=\mathbf{x}^*\mid \mathbf{X}_1=\mathbf{x}'\}|\\
&=\frac{1}{2}\sum_{\mathbf{x}^*\in S}|\sum_{\mathbf{y}\in S}\mathbb{P}\{\mathbf{X}_n=\mathbf{x}^*\mid \mathbf{X}_2=\mathbf{y}\}(\mathbb{P}\{\mathbf{X}_2=\mathbf{y}\mid \mathbf{X}_1=\mathbf{x}\}-\mathbb{P}\{\mathbf{X}_2=\mathbf{y}\mid \mathbf{X}_1=\mathbf{x}'\})|\\
&\le\frac{1}{2}\sum_{\mathbf{x}^*\in S}\sum_{\mathbf{y}\in S}\mathbb{P}\{\mathbf{X}_n=\mathbf{x}^*\mid \mathbf{X}_2=\mathbf{y}\}|\mathbb{P}\{\mathbf{X}_2=\mathbf{y}\mid \mathbf{X}_1=\mathbf{x}\}-\mathbb{P}\{\mathbf{X}_2=\mathbf{y}\mid \mathbf{X}_1=\mathbf{x}'\}|\\
&=\frac{1}{2}\sum_{\mathbf{y}\in S}|\mathbb{P}\{\mathbf{X}_2=\mathbf{y}\mid \mathbf{X}_1=\mathbf{x}\}-\mathbb{P}\{\mathbf{X}_2=\mathbf{y}\mid \mathbf{X}_1=\mathbf{x}'\}|\\
&=\sup_{H\subset S}|\mathbb{P}\{\mathbf{X}_2\in H\mid \mathbf{X}_1=\mathbf{x}\}-\mathbb{P}\{\mathbf{X}_2\in H\mid \mathbf{X}_1=\mathbf{x}'\}|\\
&\to 0
\end{aligned}
$$

$(\mathbf{x}'\to\mathbf{x})$ by (7). Therefore even

$$
\sup_{H\subset S,\mathbf{x}\in S}|\mathbb{P}\{\mathbf{X}_n\in H\mid \mathbf{X}_1=\mathbf{x}\}-\mu(H)|\to 0.
$$

Thus, the process $\{\mathbf{X}_i\}$ is φ-mixing. Also the sequence

$$
\{\mathbb{E}\{\ln\langle\mathbf{b}^*(\mathbf{X}_{i-1}),\mathbf{X}_i\rangle\mid\mathbf{X}_{i-1}\}\}
$$

is φ-mixing with mixing coefficients $\varphi_m\to 0$. Now we can apply Collomb's exponential inequality (p. 449 in [14]) with $d=\delta=\sqrt{D}=\frac{1}{n}(\ln a_2-\ln a_1)$. For $m\in\{1,\dots,n\}$ we obtain

$$
\begin{aligned}
&\mathbb{P}\Big\{R+\epsilon>\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\langle\mathbf{b}^*(\mathbf{X}_{i-1}),\mathbf{X}_i\rangle\mid\mathbf{X}_{i-1}\}\Big\}\\
&\le\exp\left\{\frac{n}{m}\left(3\sqrt{e}\varphi_m+\frac{3}{8}\frac{1+4\sum_{i=1}^{m}\varphi_i}{m}-\frac{\epsilon}{4(\ln a_2-\ln a_1)}\right)\right\}.
\end{aligned}
$$

Suitable choice of $m=M(\epsilon)$ with $n\ge N(\epsilon)$ leads to the second term on the right hand side of (8) as a bound for all n. Finally, from (8) we obtain (9) as in the proof of Corollary 1. ∎

Remark. Theorem 3 can be extended to the case of a Harris-recurrent, strongly aperiodic Markov chain, not necessarily being stationary or having denumerable state space; compare in a somewhat other context Theorem 2 in Györfi and Walk [25], where Theorem 4.1 (i) of Athreya and Ney [4] and Collomb's inequality are used.

References

1. Algoet, P.: Universal schemes for prediction, gambling, and portfolio selection. *Annals of Probability*, **20**, 901–941 (1992)
2. Algoet, P.: The strong law of large numbers for sequential decisions under uncertainty. *IEEE Transactions on Information Theory*, **40**, 609–634 (1994)
3. Algoet, P. and Cover, T. M.: Asymptotic optimality asymptotic equipartition properties of log-optimum investments. *Annals of Probability*, **16**, 876–898 (1988)
4. Athreya, K. B. and Ney, P.: A new approach to the limit theory of recurrent Markov chains, *Transactions of the American Mathematical Society*, **245**, 493–501 (1978)
5. Aucamp, D. C.: An investment strategy with overshoot rebates which minimizes the time to attain a specified goal. *Management Science*, **23**, 1234–1241 (1977)
6. Aucamp, D. C.: On the extensive number of plays to achieve superior performance with the geometric mean strategy. *Management Science*, **39**, 1163–1172 (1993)
7. Azuma, K.: Weighted sums of certain dependent random variables. *Tohoku Mathematical Journal*, **68**, 357–367 (1967)
8. Barron, A. R. and Cover, T. M.: A bound on the financial value of information. *IEEE Transactions on Information Theory*, **34**, 1097–1100 (1988)
9. Billingsley, R.: *Convergence of Probability Measures (2nd ed.),* New York: Wiley (1999)
10. Box, G. E. P. and Cox, D. R.: An analysis of transformations. *Journal of the Royal Statistical Society Series B*, **26**, 211–252 (1964)
11. Breiman, L.: Optimal gambling systems for favorable games. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.*, **1**, 65–78, Univ. California Press, Berkeley (1961)
12. Breiman, L.: *Probability,* Reading, Ma: Addison-Wesley (1968)
13. Chow, Y. S.: Local convergence of martingales and the law of large numbers. *Annals of Mathematical Statistics*, **36**, 552–558 (1965)
14. Collomb, G.: Propriétés de convergence presque complète du prédicteur à noyau. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **66**, 441–460 (1984)
15. Cover, T.: Universal portfolios. *Mathematical Finance*, **1**, 1–29 (1991)
16. Ethier, S. N.: The Kelly system maximizes median fortune. *J. Appl. Probab.* **41**, 1230–1236 (2004)
17. Feller, W.: *An Introduction to Probability Theory and its Applications, vol. 1 (3rd ed.), vol. 2 (2nd ed.)* New York: Wiley (1968, 1971)
18. Fernholz, E. R.: *Stochastic Portfolio Theory,* New York: Springer (2000)
19. Finkelstein, M. and Whitley, R.: Optimal strategies for repeated games. *Advances in Applied Probability*, **13**, 415–428 (1981)
20. Györfi, L., Lugosi, G. and Udina, F.: Nonparametric kernel based sequential investment strategies. *Mathematical Finance*, **16**, 337–357 (2006)
21. Györfi, L., Ottucsák, G. and Urbán, A.: Empirical log-optimal portfolio selections: a survey, in *Machine Learning for Financial Engineering*, eds. L. Györfi, G. Ottucsák, H. Walk, pp. 81–118, Imperial College Press, London (2012)
22. Györfi, L., Udina, F. and Walk, H.: Nonparametric nearest-neighbor-based sequential investment strategies, *Statistics and Decisions*, **26**, 145–157 (2008)
23. Györfi, L., Urbán, A. and Vajda, I.: Kernel-based semi-log-optimal empirical portfolio selection strategies. *International Journal of Theoretical and Applied Finance*, **10**, 505–516 (2007)
24. Györfi, L. and Walk, H.: Log-optimal portfolio selection strategies with proportional transaction costs, in *Machine Learning for Financial Engineering*, eds. L. Györfi, G. Ottucsák, H. Walk, pp. 119–152, Imperial College Press, London (2012)
25. Györfi, L. and Walk, H.: Empirical portfolio selection strategies with proportional transaction cost, *IEEE Trans. Information Theory*, **58**, 6320–6331 (2012)
26. Hayes, T. P.: How Not to Win a Million Dollars: A Counterexample to a Conjecture of L. Breiman. 2011.
http://arxiv.org/pdf/1112.0829v1.pdf

27. Hoeffding, W.: Probability inequalities for sums of bounded random variables. *Journal of American Statistical Association*, **58**, 13–30 (1963)
28. Horváth, M. and Urbán, A.: Growth optimal portfolio selection with short selling and leverage, in *Machine Learning for Financial Engineering*, eds. L. Györfi, G. Ottucsák, H. Walk, pp. 153–178, Imperial College Press, London (2012)
29. Kardaras, C. and Platen, E.: Minimizing the expected market time to reach a certain wealth level. *SIAM Journal of Financial Mathematics*, **1**, 16–29 (2010)
30. Kelly, J. L.: A new interpretation of information rate. *Bell System Technical Journal*, **35**, 917–926 (1956)
31. Kuhn, D.. and Luenberger, D. G.: Analysis of the rebalancing frequency in the log-optimal portfolio selection, *Quantitativ Finance*, **10**, 221–234 (2010)
32. Latané, H. A.: Criteria for choice among risky ventures. *Journal of Political Economy*, **67**, 145–155 (1959)
33. Lorden, G.: On excess over the boundary. *Annals of Mathematical Statistics*, **41**, 520–527 (1970)
34. MacLean, L. C., Thorp, E. O. and Ziemba, W. T. (Eds.): *The Kelly Capital Growth Investment Criterion: Theory and Practice.* World Scientific, New Jersey (2011)
35. Markowitz, H. M.: Investment for the long run: new evidence for an old rule. *Journal of Finance*, **31**, 1273–1286 (1976)
36. Pulley, L. B.: Mean-variance approximation to expected logarithmic utility, *Operations Research*, **40**, 685–696 (1983)
37. Roll, R.: Evidence on the growth-optimum model, *The Journal of Finance*, **28**, 551–566 (1973)
38. Samuelson, P. A.: The "fallacy" of maximizing the geometric mean in long sequences of investment or gambling. *Proceedings of the National Academy of Sciences U.S. A.*, **68**, 2493–2496 (1971)
39. Singer, Y.: Switching portfolios. *International Journal of Neural Systems*, **8**, 445–455 (1997)
40. Stout. W. F.: *Almost sure convergence.* Academic Press, New York (1974)
41. Takács, L.: On a generalization of the arc-sine law. *Annals of Applied Probability*, **6**, 1035–1040 (1996)

László Györfi
Department of Computer Science and Information Theory
Budapest University of Technology and Economics
Stoczek u.2
1521 Budapest
Hungary
E-Mail: gyorfi@cs.bme.hu

György Ottucsák
Department of Computer Science and Information Theory
Budapest University of Technology and Economics
Stoczek u.2
1521 Budapest
Hungary
E-Mail: oti@cs.bme.hu

Harro Walk
Universität Stuttgart
Fachbereich Mathematik
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: walk@mathematik.uni-stuttgart.de

Erschienene Preprints ab Nummer 2007/2007-001

Komplette Liste: http://www.mathematik.uni-stuttgart.de/preprints

2014-014 *Kohler, M.; Krzyżak, A.; Walk, H.:* Nonparametric recursive quantile estimation

2014-013 *Kohler, M.; Krzyżak, A.; Tent, R.; Walk, H.:* Nonparametric quantile estimation using importance sampling

2014-012 *Györfi, L.; Ottucsák, G.; Walk, H.:* The growth optimal investment strategy is secure, too.

2014-011 *Györfi, L.; Walk, H.:* Strongly consistent detection for nonparametric hypotheses

2014-010 *Köster, I.:* Finite Groups with Sylow numbers $\{q^x, a, b\}$

2014-009 *Kahnert, D.:* Hausdorff Dimension of Rings

2014-008 *Steinwart, I.:* Measuring the Capacity of Sets of Functions in the Analysis of ERM

2014-007 *Steinwart, I.:* Convergence Types and Rates in Generic Karhunen-Loève Expansions with Applications to Sample Path Properties

2014-006 *Steinwart, I.; Pasin, C.; Williamson, R.; Zhang, S.:* Elicitation and Identification of Properties

2014-005 *Schmid, J.; Griesemer, M.:* Integration of Non-Autonomous Linear Evolution Equations

2014-004 *Markhasin, L.:* L_2- and $S^r_{p,q}B$-discrepancy of (order 2) digital nets

2014-003 *Markhasin, L.:* Discrepancy and integration in function spaces with dominating mixed smoothness

2014-002 *Eberts, M.; Steinwart, I.:* Optimal Learning Rates for Localized SVMs

2014-001 *Giesselmann, J.:* A relative entropy approach to convergence of a low order approximation to a nonlinear elasticity model with viscosity and capillarity

2013-016 *Steinwart, I.:* Fully Adaptive Density-Based Clustering

2013-015 *Steinwart, I.:* Some Remarks on the Statistical Analysis of SVMs and Related Methods

2013-014 *Rohde, C.; Zeiler, C.:* A Relaxation Riemann Solver for Compressible Two-Phase Flow with Phase Transition and Surface Tension

2013-013 *Moroianu, A.; Semmelmann, U.:* Generalized Killling spinors on Einstein manifolds

2013-012 *Moroianu, A.; Semmelmann, U.:* Generalized Killing Spinors on Spheres

2013-011 *Kohls, K; Rösch, A.; Siebert, K.G.:* Convergence of Adaptive Finite Elements for Control Constrained Optimal Control Problems

2013-010 *Corli, A.; Rohde, C.; Schleper, V.:* Parabolic Approximations of Diffusive-Dispersive Equations

2013-009 *Nava-Yazdani, E.; Polthier, K.:* De Casteljau's Algorithm on Manifolds

2013-008 *Bächle, A.; Margolis, L.:* Rational conjugacy of torsion units in integral group rings of non-solvable groups

2013-007 *Knarr, N.; Stroppel, M.J.:* Heisenberg groups over composition algebras

2013-006 *Knarr, N.; Stroppel, M.J.:* Heisenberg groups, semifields, and translation planes

2013-005 *Eck, C.; Kutter, M.; Sändig, A.-M.; Rohde, C.:* A Two Scale Model for Liquid Phase Epitaxy with Elasticity: An Iterative Procedure

2013-004 *Griesemer, M.; Wellig, D.:* The Strong-Coupling Polaron in Electromagnetic Fields

2013-003 *Kabil, B.; Rohde, C.:* The Influence of Surface Tension and Configurational Forces on the Stability of Liquid-Vapor Interfaces

2013-002 *Devroye, L.; Ferrario, P.G.; Györfi, L.; Walk, H.:* Strong universal consistent estimate of the minimum mean squared error

2013-001 *Kohls, K.; Rösch, A.; Siebert, K.G.:* A Posteriori Error Analysis of Optimal Control Problems with Control Constraints

2012-018 *Kimmerle, W.; Konovalov, A.:* On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II

2012-017 *Stroppel, B.; Stroppel, M.:* Desargues, Doily, Dualities, and Exceptional Isomorphisms

2012-016 *Moroianu, A.; Pilca, M.; Semmelmann, U.:* Homogeneous almost quaternion-Hermitian manifolds

2012-015 *Steinke, G.F.; Stroppel, M.J.:* Simple groups acting two-transitively on the set of generators of a finite elation Laguerre plane

2012-014 *Steinke, G.F.; Stroppel, M.J.:* Finite elation Laguerre planes admitting a two-transitive group on their set of generators

2012-013 *Diaz Ramos, J.C.; Dominguez Vázquez, M.; Kollross, A.:* Polar actions on complex hyperbolic spaces

2012-012 *Moroianu; A.; Semmelmann, U.:* Weakly complex homogeneous spaces

2012-011 *Moroianu; A.; Semmelmann, U.:* Invariant four-forms and symmetric pairs

2012-010 *Hamilton, M.J.D.:* The closure of the symplectic cone of elliptic surfaces

2012-009 *Hamilton, M.J.D.:* Iterated fibre sums of algebraic Lefschetz fibrations

2012-008 *Hamilton, M.J.D.:* The minimal genus problem for elliptic surfaces

2012-007 *Ferrario, P.:* Partitioning estimation of local variance based on nearest neighbors under censoring

2012-006 *Stroppel, M.:* Buttons, Holes and Loops of String: Lacing the Doily

2012-005 *Hantsch, F.:* Existence of Minimizers in Restricted Hartree-Fock Theory

2012-004 *Grundhöfer, T.; Stroppel, M.; Van Maldeghem, H.:* Unitals admitting all translations

2012-003 *Hamilton, M.J.D.:* Representing homology classes by symplectic surfaces

2012-002 *Hamilton, M.J.D.:* On certain exotic 4-manifolds of Akhmedov and Park

2012-001 *Jentsch, T.:* Parallel submanifolds of the real 2-Grassmannian

2011-028 *Spreer, J.:* Combinatorial 3-manifolds with cyclic automorphism group

2011-027 *Griesemer, M.; Hantsch, F.; Wellig, D.:* On the Magnetic Pekar Functional and the Existence of Bipolarons

2011-026 *Müller, S.:* Bootstrapping for Bandwidth Selection in Functional Data Regression

2011-025 *Felber, T.; Jones, D.; Kohler, M.; Walk, H.:* Weakly universally consistent static forecasting of stationary and ergodic time series via local averaging and least squares estimates

2011-024 *Jones, D.; Kohler, M.; Walk, H.:* Weakly universally consistent forecasting of stationary and ergodic time series

2011-023 *Györfi, L.; Walk, H.:* Strongly consistent nonparametric tests of conditional independence

2011-022 *Ferrario, P.G.; Walk, H.:* Nonparametric partitioning estimation of residual and local variance based on first and second nearest neighbors

2011-021 *Eberts, M.; Steinwart, I.:* Optimal regression rates for SVMs using Gaussian kernels

2011-020 *Frank, R.L.; Geisinger, L.:* Refined Semiclassical Asymptotics for Fractional Powers of the Laplace Operator

2011-019 *Frank, R.L.; Geisinger, L.:* Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain

2011-018 *Hänel, A.; Schulz, C.; Wirth, J.:* Embedded eigenvalues for the elastic strip with cracks

2011-017 *Wirth, J.:* Thermo-elasticity for anisotropic media in higher dimensions

2011-016 *Höllig, K.; Hörner, J.:* Programming Multigrid Methods with B-Splines

2011-015 *Ferrario, P.:* Nonparametric Local Averaging Estimation of the Local Variance Function

2011-014 *Müller, S.; Dippon, J.:* k-NN Kernel Estimate for Nonparametric Functional Regression in Time Series Analysis

2011-013 *Knarr, N.; Stroppel, M.:* Unitals over composition algebras

2011-012 *Knarr, N.; Stroppel, M.:* Baer involutions and polarities in Moufang planes of characteristic two

2011-011 *Knarr, N.; Stroppel, M.:* Polarities and planar collineations of Moufang planes

2011-010 *Jentsch, T.; Moroianu, A.; Semmelmann, U.:* Extrinsic hyperspheres in manifolds with special holonomy

2011-009 *Wirth, J.:* Asymptotic Behaviour of Solutions to Hyperbolic Partial Differential Equations

2011-008 *Stroppel, M.:* Orthogonal polar spaces and unitals

2011-007 *Nagl, M.:* Charakterisierung der Symmetrischen Gruppen durch ihre komplexe Gruppenalgebra

2011-006 *Solanes, G.; Teufel, E.:* Horo-tightness and total (absolute) curvatures in hyperbolic spaces

2011-005 *Ginoux, N.; Semmelmann, U.:* Imaginary Kählerian Killing spinors I

2011-004 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part II — Gain-Scheduled Control

2011-003 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part I — Robust Control

2011-002 *Alexandrov, B.; Semmelmann, U.:* Deformations of nearly parallel G_2-structures

2011-001 *Geisinger, L.; Weidl, T.:* Sharp spectral estimates in domains of infinite volume

2010-018 *Kimmerle, W.; Konovalov, A.:* On integral-like units of modular group rings

2010-017 *Gauduchon, P.; Moroianu, A.; Semmelmann, U.:* Almost complex structures on quaternion-Kähler manifolds and inner symmetric spaces

2010-016 *Moroianu, A.; Semmelmann,U.:* Clifford structures on Riemannian manifolds

2010-015 *Grafarend, E.W.; Kühnel, W.:* A minimal atlas for the rotation group $SO(3)$

2010-014 *Weidl, T.:* Semiclassical Spectral Bounds and Beyond

2010-013 *Stroppel, M.:* Early explicit examples of non-desarguesian plane geometries

2010-012 *Effenberger, F.:* Stacked polytopes and tight triangulations of manifolds

2010-011 *Györfi, L.; Walk, H.:* Empirical portfolio selection strategies with proportional transaction costs

2010-010 *Kohler, M.; Krzyżak, A.; Walk, H.:* Estimation of the essential supremum of a regression function

2010-009 *Geisinger, L.; Laptev, A.; Weidl, T.:* Geometrical Versions of improved Berezin-Li-Yau Inequalities

2010-008 *Poppitz, S.; Stroppel, M.:* Polarities of Schellhammer Planes

2010-007 *Grundhöfer, T.; Krinn, B.; Stroppel, M.:* Non-existence of isomorphisms between certain unitals

2010-006 *Höllig, K.; Hörner, J.; Hoffacker, A.:* Finite Element Analysis with B-Splines: Weighted and Isogeometric Methods

2010-005 *Kaltenbacher, B.; Walk, H.:* On convergence of local averaging regression function estimates for the regularization of inverse problems

2010-004 *Kühnel, W.; Solanes, G.:* Tight surfaces with boundary

2010-003 *Kohler, M; Walk, H.:* On optimal exercising of American options in discrete time for stationary and ergodic data

2010-002 *Gulde, M.; Stroppel, M.:* Stabilizers of Subspaces under Similitudes of the Klein Quadric, and Automorphisms of Heisenberg Algebras

2010-001 *Leitner, F.:* Examples of almost Einstein structures on products and in cohomogeneity one

2009-008 *Griesemer, M.; Zenk, H.:* On the atomic photoeffect in non-relativistic QED

2009-007 *Griesemer, M.; Moeller, J.S.:* Bounds on the minimal energy of translation invariant n-polaron systems

2009-006 *Demirel, S.; Harrell II, E.M.:* On semiclassical and universal inequalities for eigenvalues of quantum graphs

2009-005 *Bächle, A, Kimmerle, W.:* Torsion subgroups in integral group rings of finite groups

2009-004 *Geisinger, L.; Weidl, T.:* Universal bounds for traces of the Dirichlet Laplace operator

2009-003 *Walk, H.:* Strong laws of large numbers and nonparametric estimation

2009-002 *Leitner, F.:* The collapsing sphere product of Poincaré-Einstein spaces

2009-001 *Brehm, U.; Kühnel, W.:* Lattice triangulations of E^3 and of the 3-torus

2008-006 *Kohler, M.; Krzyżak, A.; Walk, H.:* Upper bounds for Bermudan options on Markovian data using nonparametric regression and a reduced number of nested Monte Carlo steps

2008-005 *Kaltenbacher, B.; Schöpfer, F.; Schuster, T.:* Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems

2008-004 *Leitner, F.:* Conformally closed Poincaré-Einstein metrics with intersecting scale singularities

2008-003 *Effenberger, F.; Kühnel, W.:* Hamiltonian submanifolds of regular polytope

2008-002 *Hertweck, M.; Höfert, C.R.; Kimmerle, W.:* Finite groups of units and their composition factors in the integral group rings of the groups $PSL(2, q)$

2008-001 *Kovarik, H.; Vugalter, S.; Weidl, T.:* Two dimensional Berezin-Li-Yau inequalities with a correction term

2007-006 *Weidl, T.:* Improved Berezin-Li-Yau inequalities with a remainder term

2007-005 *Frank, R.L.; Loss, M.; Weidl, T.:* Polya's conjecture in the presence of a constant magnetic field

2007-004 *Ekholm, T.; Frank, R.L.; Kovarik, H.:* Eigenvalue estimates for Schrödinger operators on metric trees

2007-003 *Lesky, P.H.; Racke, R.:* Elastic and electro-magnetic waves in infinite waveguides

2007-002 *Teufel, E.:* Spherical transforms and Radon transforms in Moebius geometry

2007-001 *Meister, A.:* Deconvolution from Fourier-oscillating error densities under decay and smoothness restrictions