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OPTIMAL GRADING OF THE NEWEST VERTEX BISECTION AND H^1-STABILITY OF THE L_2-PROJECTION

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Abstract. We show for adaptive triangulations in 2d, which are generated by the Newest Vertex Bisection, an optimal grading estimate. Roughly speaking, we construct from the piecewise constant mesh-size function a regularized one with the following two properties. First, the two functions are equivalent, and second, the regularized mesh-size function differs at most by a factor of 2 on neighboring elements. In combination with [1] this optimal grading estimate enables us to show that the L_2-orthogonal projections onto the space of continuous Lagrange finite elements up to order twelve is H^1-stable. We extend these results to a modified Red-Green-Refinement.

1. Introduction

The L_2-projection onto discrete spaces plays an essential role in the analysis of finite element discretizations. Tantardini has shown that the implicit Euler discretization in time of the heat equation leads to a discretely inf-sup stable bilinear form, provided that the L_2-projection onto the finite element space for the spacial discretization is H^1-stable. On top of that, it is shown that for the semi-discretization in space the H^1-stability is *necessary and sufficient* for the discrete inf-sup condition [14]. Heine, Gaspoz, and Siebert have developed a variational formulation for Dirichlet boundary data suitable for optimal control problems with Dirichlet boundary control. The ensuing bilinear form is discretely inf-sup stable *if and only if* the L_2-projection onto the trace space on the Dirichlet boundary is $H^{1/2}$-stable [8]. Finally we would like to mention the prominent role of H^1-stability of the L_2-projection in the analysis of multigrid methods [16, 17].

On uniform grids, H^1-stability of the L_2-projection can easily be deduced by an inverse estimate, using its definition and employing an H^1-stable interpolation operator. This simple proof hinges on the fact that the minimal mesh-size is comparable to the maximal mesh-size, i. e., $h_{\max} \eqsim h_{\min}$. It thus cannot be transfered to adaptively generated meshes, where $h_{\max}$ and $h_{\min}$ are in general entirely unrelated. On top of this, the example in [1, §7] suggests that the L_2-projection is not H^1-stable if the local mesh-size changes to fast.

Since adaptive grids have become an important tool in science and engineering there has been an increase of interest in proving H^1-stability of the L_2-projection on graded meshes. By now, there are mainly three proofs in higher space dimension. Crouzeix and Thomée decompose a triangulation in 2d into rings of elements satisfying a suitable grading condition to show stability [6]. Bramble, Pasciak, Steinbach give in any dimension a condition on a disturbed element mass matrix, which is the basis of the stability proof [4]. This condition reduces for lowest order finite elements to a grading condition of a regularized mesh-size function; compare with (6.6) in [4]. The most recent result of Bank and Yserentant in 2d and 3d assumes

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a suitable decomposition of the grid induced by the level of elements [1]. This decomposition then in turn yields a suitable grading of the mesh and in combination with local eigenvalue problems they show stability for Lagrange finite elements up to order 12 in 2d and 7 in 3d.

Summarizing, all of these results rely on an assumption connected with a suitable grading of the grid. All of these requirements can be checked aposteriori for a given grid. However, it is apriori unclear that any of these assumptions is fulfilled for a sequence of locally refined grids generated by standard refinement algorithms. In fact, the decomposition assumption (1.2) in [1] is not valid in the stated form for grids generated by the Newest Vertex Bisection; compare with Remark 3.7. By now there is only the result of Carstensen, who shows that in two space dimensions the condition (6.6) in [4] is fulfilled for lowest order finite elements and Red-Blue-Green-Refinement [5].

In this article we prove an optimal grading estimate for conforming two dimensional triangulations that are generated by bisectional refinement from a given conforming initial triangulation $\mathcal{T}_0$. We introduce in §2 the Newest Vertex Bisection (NVB) and recall its most important properties for this venture.

Let $\mathcal{T}$ be any refinement of $\mathcal{T}_0$ by NVB. Denote by $\mathcal{V}$ the set of its vertices and by $\mathcal{V}(T) := \mathcal{V} \cap T$ the set of vertices of a triangle $T \in \mathcal{T}$. We then show in §3 the first main result, assuming a compatible labeling of the refinement edges on $\mathcal{T}_0$; compare with Assumption 2.2.

Main Result 1. *There exists a continuous, piecewise linear function H with nodal values $\{h_z = H(z)\}_{z\in\mathcal{V}}$ such that*

$$\max_{z,z'\in\mathcal{V}(T)} \frac{h_z}{h_{z'}} \le \gamma = 2 \quad \textit{and} \quad c_0 \max_{z\in\mathcal{V}(T)} h_z \le \operatorname{diam}(T) \le C_0 \min_{z\in\mathcal{V}(T)} h_z \qquad \forall\, T \in \mathcal{T}.$$

The constants $0 < c_0 \le C_0$ solely depend on $\mathcal{T}_0$ and the grading constant $\gamma = 2$ is optimal.

This optimal grading estimate implies condition (6.6) in [4]. But even better, the regularized mesh-size function H can be used to define a regularized element generation, which allows us to construct a decomposition complying with assumption (1.2) in [1]. Therefore, applying the techniques from [1] we derive in §4 the second main result.

Main Result 2. *The L_2-orthogonal projection onto the space of continuous Lagrange finite elements up to order twelve is H^1-stable.*

We conclude in §5 with some extensions. For an arbitrary labeling of the refinement edges on $\mathcal{T}_0$ not complying with Assumption 2.2 we are able to prove a non-optimal grading estimate with grading constant $\gamma = \frac{3}{2}$. This still entails H^1-stability of the L_2-projection for all polynomials degrees up to order nine except for quadratic finite elements. The same grading estimate has been shown in [5] to show stability for lowest order. In this respect, we can generalize [5] to all polynomial degrees up to order nine except for two.

We then show that standard Red-Green-Refinement leads to the grading constant $\gamma = 4$, which is, in general, optimal. First, this shows that the decomposition assumption (1.2) in [1] for the element level is not valid for standard Red-Green-Refinement. Second, such a grading is not covered by the theory in §4. We then pose some more restrictive rules for the green closure. This reduces the grading constant to the optimal value $\gamma = 2$ and we can also prove both Main Results 1 and 2 for this modified Red-Green-Refinement.

Below we track the important constants explicitly. For less important constants we use the notation $a \lesssim b$ for $a \leq Cb$ with some generic constant C that solely depends on $\mathcal{T}_0$ and write $a \approx b$ whenever $a \lesssim b \lesssim a$.

2. Newest Vertex Bisection

In this section we shortly introduce NVB for conforming triangulations in two space dimensions. It was introduced by Mitchell in 1988 together with a recursive refinement algorithm [10]; compare also with [2, 9, 13, 15].

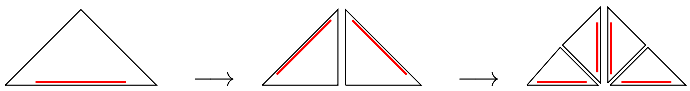

Figure 2.1. NVB: a triangle with its two children and four grandchildren. The refinement edges are indicated in red.

2.1. **Recurrent bisection of a triangle.** In order to easily describe NVB we identify a triangle T with its set of *ordered* vertices

$$T = [z_0, z_1, z_2].$$

The edge between the first and last vertex we call *refinement edge*. NVB refines T by inserting a new vertex in the midpoint $\bar{z} = \frac{1}{2}(z_0 + z_2)$ of the refinement edge $\overline{z_0 z_2}$ and

$$T_1 = [z_0, \bar{z}, z_1] \qquad \text{and} \qquad T_2 = [z_2, \bar{z}, z_1]$$

are the two children of T. This procedure automatically presets the children's refinement edges by the local ordering of their vertices. NVB thereby determines the refinement edge of any descendant produced by recurrent bisection of a given initial element T_0 from the vertex order of T_0; see Figure 2.1.

Recurrent bisection induces the structure of an *infinite binary tree* $\mathcal{F}(T_0)$: Any node T inside the tree is an element generated by recurrent application of NVB. The two successors of a node T are the children T_1, T_2 created by a applying NVB to T.

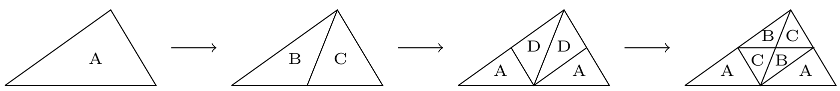

Figure 2.2. NVB: The four similarity classes for an initial element T_0.

Finally, NVB produces shape regular descendants since all $T \in \mathcal{F}(T_0)$ belong to at most four similarity classes; compare with Figure 2.2. This is a consequence of the fact that NVB always bisects the angle at the newest vertex. In the end, any angle of any triangle is bisected at most once.

2.2. **Recurrent refinement of triangulations with NVB.** Let $\mathcal{T}_0$ be a conforming and exact triangulation of a bounded polygon $\Omega \subset \mathbb{R}^2$. We can refine $\mathcal{T}_0$ or a refinement $\mathcal{T}$ of $\mathcal{T}_0$ by a applying the NVB to selected triangles. More than the selected elements have to be refined when striking for conforming triangulations. Here we refer to [10] for a recursive refinement algorithm and to [2] for an iterative one.

We next introduce notations related to triangulations. The *master forest*

$$\mathcal{F} := \mathcal{F}(\mathcal{T}_0) = \bigcup_{T_0 \in \mathcal{T}_0} \mathcal{F}(T_0)$$

holds full information about all possible refinements of $\mathcal{T}_0$. We denote by $\mathbb{T} = \mathbb{T}(\mathcal{T}_0)$ the class of all *conforming* refinements of $\mathcal{T}_0$.

For $\mathcal{T} \in \mathbb{T}$ the sets of all its vertices and edges are $\mathcal{V}$ and $\mathcal{E}$, respectively. For $T \in \mathcal{T}$ we set $\mathcal{V}(T) := \mathcal{V} \cap T$ and for $z \in \mathcal{V}$ we define $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}$. The finite element star at a vertex z is then $\Omega_z := \bigcup\{T : T \in \mathcal{T}(z)\}$. We let $h_{\mathcal{T}} \in L_\infty(\Omega)$ be the piecewise constant mesh-size function with $h_{\mathcal{T}|T} = h_T := |T|^{1/2} \approx \operatorname{diam}(T)$ for $T \in \mathcal{T}$. We use $h_{\min,\max}(\mathcal{T})$ for the smallest and largest element size of $\mathcal{T}$. We say $T, T' \in \mathcal{T}$ are *direct neighbors* iff $T \cap T' \in \mathcal{E}$.

Important in the course of this article is the *generation* of an element. For each $T \in \mathcal{T}$ there is a $T_0 \in \mathcal{T}_0$ such that $T \in \mathcal{F}(T_0)$. The generation $\operatorname{gen}(T)$ is the number of its ancestors in the tree $\mathcal{F}(T_0)$, or, equivalently, the number of bisections needed to create T from T_0.

The following simple properties are useful.

Lemma 2.1. (1) *For $T \in \mathcal{F}(T_0)$ with $T_0 \in \mathcal{T}_0$ we have*

$$h_T = 2^{-\operatorname{gen}(T)/2} h_{T_0}.$$

(2) *Defining $\alpha_0 := \max\{\#\mathcal{T}(z_0) \mid z_0 \in \mathcal{V}_0\}$ we have for $z \in \mathcal{V}$ the bound*

$$\#\mathcal{T}(z) \le \begin{cases} 8 & \text{if } z \in \mathcal{V} \setminus \mathcal{V}_0, \\ 2\alpha_0 & \text{if } z \in \mathcal{V}_0. \end{cases}$$

Proof. Bisection halves the volume of a triangle. The definition of $\operatorname{gen}(T)$ then gives the first claim. During refinement any angle is bisected at most once, which yields the second assertion. □

The following assumption on a compatible distribution of refinement edges in $\mathcal{T}_0$ is instrumental in the analysis of NVB, like the complexity estimates in [3, 13]. It is a vital assumption in §3.

Assumption 2.2 (Initial grid)**.** Suppose $T, T' \in \mathcal{T}_0$ are direct neighbors with common edge $T \cap T' = E \in \mathcal{E}_0$. Then either E is the common refinement edge of both T and T', or E is neither the refinement edge of T nor of T'.

Mitchell has shown that such a distribution of refinement edges can be found for any initial triangulation $\mathcal{T}_0$ [10, Theorem 2.9]; compare also with [3, Lemma 2.1]. The assumption particularly implies that any uniform refinement of $\mathcal{T}_0$ is conforming, i. e., for any $g \in \mathbb{N}_0$ we find that $\{T \in \mathcal{F}(\mathcal{T}_0) \mid \operatorname{gen}(T) = g\} \in \mathbb{T}$. The proof of this property is a combination of [15, §4] and [13, Theorem 4.3]. It is the key to show the following property of NVB; compare with [13, Corollary 4.6].

Proposition 2.3 (Characteristics of NVB)**.** *Suppose that the initial triangulation $\mathcal{T}_0$ satisfies Assumption 2.2. Let $\mathcal{T} \in \mathbb{T}$ be given and suppose that $T, T' \in \mathcal{T}$ are direct neighbors such that the common edge $E = T \cap T'$ is the refinement edge of T. Then we either have* $\operatorname{gen}(T') = \operatorname{gen}(T)$ *and E is also the refinement edge of T', or* $\operatorname{gen}(T') = \operatorname{gen}(T) - 1$.

A simple consequence is $|\operatorname{gen}(T) - \operatorname{gen}(T')| \le 1$ for direct neighbors $T, T' \in \mathcal{T}$.

3. Optimal Mesh Grading Induced by Bisection

In this section we prove the first main result. We construct a regularized mesh-size function H adopting ideas from Carstensen for Red-Blue-Green-Refinement [5]. Using a more refined analysis we are able to prove an optimal grading estimate.

Throughout the section we suppose that $\mathcal{T}_0$ is an initial triangulation satisfying Assumption 2.2. We come back to initial grids violating this assumption in §5.

Theorem 3.1 (Optimal mesh grading)**.** *For any $\mathcal{T} \in \mathbb{T}$ there exists a piecewise linear mesh-size function $H \in C^0(\bar{\Omega}) \to \mathbb{R}_{>0}$ with nodal values $\{h_z = H(z)\}_{z\in\mathcal{V}}$ having the following properties: With $\gamma = 2$ and constants $0 < c_0 \leq C_0$, which solely depend on $\mathcal{T}_0$, we have for all $T \in \mathcal{T}$ the estimates*

$$\textit{grading:} \qquad \max_{z,z'\in\mathcal{V}(T)} \frac{h_z}{h_{z'}} \leq \gamma, \tag{3.1a}$$

$$\textit{lower bound:} \qquad \max_{z\in\mathcal{V}(T)} h_z \leq C_0 h_T, \tag{3.1b}$$

$$\textit{upper bound:} \qquad c_0 h_T \leq \min_{z\in\mathcal{V}(T)} h_z. \tag{3.1c}$$

On top of this, the grading constant $\gamma = 2$ is optimal.

The whole proof of this theorem is divided into several steps, where an arbitrary triangulation $\mathcal{T} \in \mathbb{T}$ is fixed.

3.1. **Definition of the regularized mesh-size function.** We define H on $\mathcal{T}$ and prove two directly accessible properties. We start with the following notion of connecting two vertices of $\mathcal{T}$. We say that the set of vertices

$$\mathrm{CE}(z, z') = \{z = z_0, z_1, \ldots, z_{M-1}, z_M = z'\} \subset \mathcal{V}$$

is a *chain of connecting edges* of z, z' iff $\overline{z_{m-1}z_m} \in \mathcal{E}$ for $m = 1, \ldots, M$. For $z \in \mathcal{V}$ we write $\mathrm{CE}(z, z) = \{z\}$. For any two distinct $z, z' \in \mathcal{V}$ there exists at least one $\mathrm{CE}(z, z') \neq \emptyset$. These preparations enable us to define the distance of nodes and the distance of nodes to elements.

Definition 3.2 (Distance)**.** Given $z, z' \in \mathcal{V}$ we define their *distance* as

$$\mathrm{dist}(z, z') := \min\{\#\,\mathrm{CE}(z, z') - 1 \mid \mathrm{CE}(z, z') \text{ connects } z \text{ and } z' \text{ by edges}\}.$$

The distance of $z \in \mathcal{V}$ to a simplex $T \in \mathcal{T}$ is

$$\mathrm{dist}(z, T) := \min\{\mathrm{dist}(z, z') \mid z' \in \mathcal{V}(T)\}.$$

This notion of distance is the basis for the definition of the regularized mesh-size function.

Definition 3.3 (Regularized mesh-size function)**.** We define the auxiliary function $v\colon \mathcal{V} \times \mathcal{T} \to \mathbb{R}_+$ as

$$v(z, T) := 2^{2\,\mathrm{dist}(z,T) - \mathrm{gen}(T)}.$$

The regularized, continuous and piecewise linear mesh-size function H is then uniquely defined by its nodal values

$$H(z) = h_z := \min\{v(z, T)^{1/2} \mid T \in \mathcal{T}\}, \qquad \forall\, z \in \mathcal{V}.$$

We observe $0 < H \leq 1$ in Ω since for any $z \in T$ we deduce for the nodal value

$$h_z \leq 2^{\mathrm{dist}(z,T) - \mathrm{gen}(T)/2} = 2^{-\,\mathrm{gen}(T)/2} \leq 1. \tag{3.2}$$

Besides that, the definition of H directly implies the following two basic properties.

Lemma 3.4 (Grading)**.** *The grading estimate* (3.1a) *is valid with $\gamma = 2$.*

Proof. Pick up any $z, z' \in \mathcal{V}(T)$. By definition of $h_{z'}$ there exists $T_* \in \mathcal{T}$ such that $h_{z'}^2 = v(z', T_*)$. Since $z, z' \in \mathcal{V}(T)$ we have $\mathrm{dist}(z, z') \leq 1$, which yields $\mathrm{dist}(z, T_*) \leq \mathrm{dist}(z', T_*) + 1$. Observing $h_z^2 \leq v(z, T_*)$ we therefore obtain

$$\frac{h_z^2}{h_{z'}^2} \leq \frac{v(z, T_*)}{v(z', T_*)} = 2^{2(\mathrm{dist}(z,T_*) - \mathrm{dist}(z',T_*))} \leq 4. \qquad \square$$

Lemma 3.5 (Lower bound)**.** *The lower bound* (3.1b) *applies with $C_0 := h_{\min}^{-1}(\mathcal{T}_0)$.*

Proof. For T there is $T_0 \in \mathcal{T}_0$ such that $T \in \mathcal{F}(T_0)$. For any $z \in \mathcal{V}(T)$ this gives

$$h_z^2 \leq v(z, T) = 2^{-\,\mathrm{gen}(T)} = |T| \,/\, |T_0| \leq C_0\, |T|. \qquad \square$$

3.2. **Upper bound.** We next strive for the upper bound (3.1c). This is the complicated part in the proof of Theorem 3.1. Before starting this endeavor we give a short motivation. For the sake of clarity we hereby assume $|T_0| = 1$ for all $T_0 \in \mathcal{T}_0$. For $z \in \mathcal{V}$ let $T_* \in \mathcal{T}$ be an element such that $h_z^2 = v(z, T_*)$. The definition of h_z yields

$$2^{2\,\mathrm{dist}(z,T_*)}\,|T_*| = 2^{2\,\mathrm{dist}(z,T)-\mathrm{gen}(T_*)} = h_z^2 \le 2^{-\,\mathrm{gen}(T)} = |T| \qquad \forall\, T \in \mathcal{T}(z).$$

If $T_* \notin \mathcal{T}_z$ we have $\mathrm{dist}(z, T_*) > 1$ and we learn $\mathrm{gen}(T_*) > \mathrm{gen}(T)$ for all $T \in \mathcal{T}(z)$. This means, we can only reduce the nodal value h_z below the local mesh-size by elements with a higher refinement level. A possible gain from $\mathrm{gen}(T_*) > \mathrm{gen}(T)$ is balanced by the distance $\mathrm{dist}(z, T_*)$. This balance with the factor 2 leads to the desired upper bound $h_T \lesssim h_z$ for all $T \in \mathcal{T}(z)$. To show this, the plan is as follows. We let $z_* \in T_*$ be the vertex with $\mathrm{dist}(z, z_*) = \mathrm{dist}(z, T_*)$. We then precisely track how the element generation increases when traversing from $T \ni z$ to T_* along a chain $\mathrm{CE}(z, z_*)$ connecting z, z_*.

Throughout this section we support the arguments with several images depicting reference situations. We use isosceles rectangular triangles for the ease of presentation. For element patches with general triangles these reference situations can be transformed by employing piecewise affine mappings.

We start this part with bounding the maximal difference of element generations in a finite element star Ω_z.

Lemma 3.6 (Generation in stars)**.** *For any $z \in \mathcal{V}$ and all $T, T' \in \mathcal{T}(z)$ we have*

$$\mathrm{gen}(T') - \mathrm{gen}(T) \le \alpha = \begin{cases} 3 & \text{if } z \in \mathcal{V} \setminus \mathcal{V}_0, \\ \alpha_0 & \text{if } z \in \mathcal{V}_0, \end{cases}$$

where α_0 is the constant from Lemma 2.1.

Proof. Let $T_{\min}, T_{\max} \in \mathcal{T}(z)$ be the elements with smallest and largest generation. We can traverse from $T_{\min}$ to $T_{\max}$ in $\mathcal{T}(z)$ by crossing at most $\lfloor \#\mathcal{T}(z)/2 \rfloor$ edges. Recalling Proposition 2.3, the generation of two neighboring elements differs at most by one. This yields the claim for $z \in \mathcal{V}_0$ and for $z \in \mathcal{V} \setminus \mathcal{V}_0$ if $\#\mathcal{T}(z) < 8$.

For $z \in \mathcal{V} \setminus \mathcal{V}_0$ and $\#\mathcal{T}(z) = 8$ we observe that $T_{\max}$ was generated last in $\mathcal{T}(z)$. In that refinement step the angle at the center vertex z is not bisected. Its twin $T'_{\max}$ therefore also belongs to $\mathcal{T}(z)$. The only two scenarios are depicted in the left images of Figures 3.4 and 3.5, where the vertex created last is denoted by x_1. We can traverse from $T_{\min}$ either to $T_{\max}$ or $T'_{\max}$ by crossing at most three edges. Since $\mathrm{gen}(T'_{\max}) = \mathrm{gen}(T_{\max})$ this constitutes $\alpha = 3$ and finishes the proof. □

Remark 3.7. We learn from this lemma that assumption (1.2) in [1] is in that form not valid for triangulations in $\mathcal{T} \in \mathbb{T}$. In the notation of [1] we have to use the level $k(T) = \mathrm{gen}(T)/2$. Lemma 3.6 then yields the optimal bound $|k(T) - k(T')| \le \frac{3}{2}$ and not $|k(T) - k(T')| \le 1$ as requested below (1.2) in [1].

We turn to the evolution of the element generation between two vertices $z, z' \in \mathcal{V}$ when traversing along a chain $\mathrm{CE}(z, z')$. We start with the simplest scenario in case $\mathrm{CE}(z, z')$ does not contain any vertex from the initial grid. Such a chain we call *simple*. Lemma 3.6 then implies that for any $\tilde{z} \in \mathrm{CE}(z, z')$ the element generation in $\Omega_{\tilde{z}}$ differs at most by 3. The stars of two subsequent vertices of the chain overlap, which readily yields

$$\mathrm{gen}(T') - \mathrm{gen}(T) \le 3\#\,\mathrm{CE}(z, z') \qquad \forall\, T \in \mathcal{T}(z),\, T' \in \mathcal{T}(z').$$

This bound can be improved as shown in the following proposition.

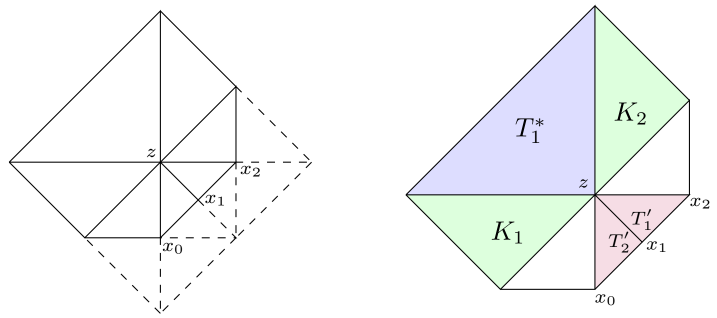

FIGURE 3.1. Generation in simple chains, Case 1: $\text{gen}(T'_j) - \text{gen}(T^*_1) = 3$, for $j = 1, 2$, and $\text{gen}(T'_j) - \text{gen}(K_i) = 3$, for $i, j = 1, 2$.

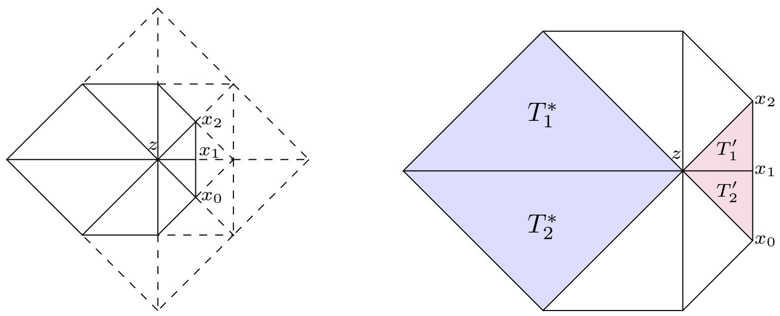

FIGURE 3.2. Generation in simple chains, Case 2: $\text{gen}(T'_j) - \text{gen}(T^*_i) = 3$, for $i, j = 1, 2$.

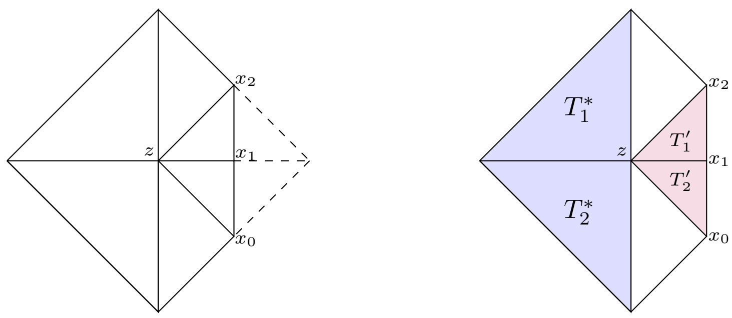

FIGURE 3.3. Generation in simple chains, Case 3: $\text{gen}(T'_j) - \text{gen}(T^*_i) = 2$.

Proposition 3.8 (Generation in simple chains)**.** *Suppose* $\text{CE}(z, z')$ *is simple, this is* $\text{CE}(z, z') \cap \mathcal{V}_0 = \emptyset$*. Then*

$$\text{gen}(T') - \text{gen}(T) \leq 2\#\,\text{CE}(z, z') + 1 \qquad \forall\, T \in \mathcal{T}_z,\, T' \in \mathcal{T}_{z'}.$$

Proof. $\boxed{1}$ We write $\text{CE}(z, z') = \{z = z_0, \ldots, z_M = z'\}$, set $T_0 = T$ and $T_{M+1} = T'$. For $m = 1, \ldots, M$ we choose $T_m = \arg\max\{\text{gen}(K) \mid K \in \mathcal{T} \text{ with } \overline{z_{m-1} z_m} \subset K\}$. We are interested in the changes of $\text{gen}(T_m)$, where we only have to consider the case $\text{gen}(T_{m+1}) > \text{gen}(T_m)$. Since $T_{m+1}, T_m \in \mathcal{T}(z_m)$ we know by Proposition 3.6 that $\text{gen}(T_{m+1}) \leq \text{gen}(T_m) + 3$. For $i = 2, 3$ we define the index sets

$$I_i := \{m \in \{0, \ldots, M\} \mid \text{gen}(T_{m+1}) = \text{gen}(T_m) + i\},$$

set $I := \{0, \dots, M\} \setminus (I_2 \cup I_3)$, and decompose $\{0, \dots, M\} = I \cup I_2 \cup I_3$. We then claim the following for a pair of indices $m, n \in \{0, \dots, M\}$.

$$\text{If } m < n \text{ fulfill } m \in I_3,\ \{m, \dots, n\} \subset I_2 \cup I_3 \text{ then } \{m+1, \dots, n\} \subset I_2. \tag{3.3}$$

This means, if $m_1, m_2 \in I_3$ are two subsequent indices then there is an $\tilde{m}$ with $m_1 < \tilde{m} < m_2$ and

$$\begin{aligned} &\operatorname{gen}(T_{m+1}) \le \operatorname{gen}(T_m) + 2 && m = m_1 + 1, \dots, m_2 - 1, \\ &\operatorname{gen}(T_{m+1}) \le \operatorname{gen}(T_m) + 1 && m = \tilde{m}. \end{aligned}$$

Utilizing the telescopic sum we therefore deduce with $\#\operatorname{CE}(z, z') = M + 1$

$$\begin{aligned} \operatorname{gen}(T') - \operatorname{gen}(T) &= \sum_{m=0}^{M} \operatorname{gen}(T_{m+1}) - \operatorname{gen}(T_m) \\ &\le 2(M+1) + \#I_3 - (\#I_3 - 1) = 1 + 2\#\operatorname{CE}(z, z'). \end{aligned}$$

It thus remains to show (3.3).

$\boxed{2}$ In Figures 3.1 and 3.2 we show the only two situations (up to a rotation or reflection) where for the difference of generations in a finite element star Ω_z the maximum value 3 is attained. The elements of the dashed area may be refined further, but any single bisection of an element in Ω_z leads to a refinement of the coarsest element T_1^*.

We realize that the elements T_1' and T_2' are of locally highest generation in Ω_z. They are generated at the same time when creating the new vertex x_1. Since $m \in I_3$ we have $T_m \in \{T_1^*, T_2^*\}$, $T_{m+1} \in \{T_1', T_2'\}$, $z_m = z$, and $z_{m+1} \in \{x_0, x_1, x_2\}$. We call the situation $z_{m+1} = x_1$ the *central case* and $z_{m+1} \in \{x_0, x_2\}$ a *non-central case*. In the latter case the neighbor of either T_1' or T_2' at the edge $\overline{z_m z_{m+1}}$ is one generation less.

$\boxed{3}$ The assumption $m + 1 \in I_2 \cup I_3$ implies $m + 1 \in I_2$ since only two cases are possible in the second step:

(1) The *central case* presented in Figure 3.3 with $z = z_{m+1}$, $T_{m+1} \in \{T_1^*, T_2^*\}$, $T_{m+2} \in \{T_1', T_2'\}$, and $z_{m+2} \in \{x_0, x_1, x_2\}$. The maximal difference of generations in Ω_z cannot exceed 2 and we conclude $m + 1 \in I_2$.
(2) The *non-central case* depicted in Figure 3.1 with $z = z_{m+1}$, $T_{m+1} \in \{K_1, K_2\}$, $T_{m+2} \in \{T_1', T_2'\}$, and $z_{m+2} \in \{x_0, x_1, x_2\}$. Consequently, $m + 1 \in I_2$.

Likewise, the assumption $m + 2 \in I_2 \cup I_3$ then leads to the same cases that we have considered for $m + 1$. We therefore inductively conclude $m + \ell \in I_2$ for all $\ell \ge 1$ until $m + \ell \in I$, i. e., $\operatorname{gen}(T_{m+\ell+1}) \le \operatorname{gen}(T_{m+\ell}) + 1$. This shows (3.3) and finishes the proof. □

Lemma 3.9 (Distance and generation)**.** *With $C_1 := 3 + (\alpha_0 + 1)\#\mathcal{V}_0$ we have for all $z, z' \in \mathcal{V}$ the bound*

$$2\operatorname{dist}(z, z') - \operatorname{gen}(T') \ge -C_1 - \operatorname{gen}(T) \qquad \forall\, T \in \mathcal{T}(z),\, T' \in \mathcal{T}(z').$$

Proof. Let $\operatorname{CE}(z, z') = \{z = z_0, \dots, z_M = z'\}$ be a *minimal chain* connecting z and z'. We set $T_0 = T$ and $T_{M+1} = T'$ and for $m = 1, \dots, M$ we choose

$$T_m = \arg\max\{\operatorname{gen}(K) \mid K \in \mathcal{T} \text{ with } \overline{z_{m-1} z_m} \subset K\}.$$

We next split up $\operatorname{CE}(z, z')$ into simple chains. Let $\{z_{m_j}\}_{j=1}^{J} = \operatorname{CE}(z, z') \cap \mathcal{V}_0$ be the ordered set of vertices of the initial triangulation that belong to the chain. Let $\{\operatorname{CE}(j)\}_{j=0}^{J}$ be the list of simple chains (possibly empty) such that

$$\operatorname{CE}(z, z') = \operatorname{CE}(0) \cup \{z_{m_1}\} \cup \operatorname{CE}(1) \cup \{z_{m_2}\} \cup \dots \cup \{z_{m_J}\} \cup \operatorname{CE}(J);$$

i.e., for $1 \le j \le J-1$ we have $\mathrm{CE}(j) = \{z_{m_j+1}, \dots, z_{m_{j+1}-1}\}$ or $\mathrm{CE}(j) = \emptyset$ if $\overline{z_{m_j} z_{m_{j+1}}} \in \mathcal{E}$. Moreover, $\mathrm{CE}(0) = \{z = z_0, \dots, z_{m_1-1}\}$ or $\mathrm{CE}(0) = \emptyset$ if $z_0 \in \mathcal{V}_0$, and $\mathrm{CE}(J) = \{z_{m_J+1,\dots,z_M} = z'\}$ or $\mathrm{CE}(J) = \emptyset$ if $z_J \in \mathcal{V}_0$.

Let $K_0 = T_0 = T$ and $K'_J = T'_{M+1} = T'$. We then define for $j = 1, \dots, J$ the elements $K_j = T_{m_j-1}$ and for $j = 0, \dots, J-1$ the elements $K'_j = T_{m_{j+1}}$. Applying Proposition 3.8 we conclude

$$\mathrm{gen}(K'_j) - \mathrm{gen}(K_j) \le 2\#\,\mathrm{CE}(j) + 1.$$

Since $K'_{j-1}, K_j \in \mathcal{T}(z_{m_j})$ and $z_{m_j} \in \mathcal{V}_0$ we deduce from Lemma 3.6

$$\mathrm{gen}(K_j) - \mathrm{gen}(K_{j-1}) \le \alpha_0.$$

Utilizing a telescopic sum we therefore obtain

$$\begin{aligned}\mathrm{gen}(T') - \mathrm{gen}(T) &= \sum_{j=0}^{J} \underbrace{\mathrm{gen}(K'_j) - \mathrm{gen}(K_j)}_{\le 2\#\,\mathrm{CE}(j)+1} + \sum_{j=1}^{J} \underbrace{\mathrm{gen}(K_j) - \mathrm{gen}(K'_{j-1})}_{\le \alpha_0} \\ &\le 2\#\,\mathrm{CE}(z, z') + (J+1) + J\alpha_0.\end{aligned}$$

Minimality of $\mathrm{CE}(z, z')$ yields $J \le \#\mathcal{V}_0$ and $\#\,\mathrm{CE}(z, z') = \mathrm{dist}(z, z') + 1$. Recalling the definition $C_1 = 3 + (\alpha_0 + 1)\#\mathcal{V}_0$ we finally conclude

$$2\,\mathrm{dist}(z, z') - \mathrm{gen}(T') \ge -C_1 - \mathrm{gen}(T). \qquad \square$$

This puts us in a position to prove the upper bound $c_0 h_T \le h_z$ for all $z \in \mathcal{V}(T)$.

Corollary 3.10 (Upper bound)**.** *The upper bound* (3.1c) *is valid with the constant* $c_0 := 2^{-C_1/2} h_{\max}^{-1}(\mathcal{T}_0) = 2^{-(3+(\alpha_0+1)\#\mathcal{V}_0)/2} h_{\max}^{-1}(\mathcal{T}_0) > 0$.

Proof. For T there is $T_0 \in \mathcal{T}_0$ such that $T \in \mathcal{F}(T_0)$. For $z \in \mathcal{V}(T)$ pick up a T' and $z' \in T'$ such that

$$h_z^2 = 2^{2\,\mathrm{dist}(z,T') - \mathrm{gen}(T')} = 2^{2\,\mathrm{dist}(z,z') - \mathrm{gen}(T')} \ge 2^{-C_1 - \mathrm{gen}(T)},$$

utilizing Lemma 3.9 in the last step. Recalling that $|T| = 2^{-\mathrm{gen}(T)}\,|T_0|$ we finish by

$$h_z^2 \ge 2^{-C_1} 2^{-\mathrm{gen}(T)} = 2^{-C_1}\,|T|\,/\,|T_0| \ge 2^{-C_1} h_{\max}^{-2}(\mathcal{T}_0)\, h_T^2. \qquad \square$$

3.3. **Optimal grading.** We prove the last statement of Theorem 3.1, namely that the grading constant $\gamma = 2$ cannot be improved. In the interest of simplification we assume that there exists at least one refinement edge in the interior of Ω.

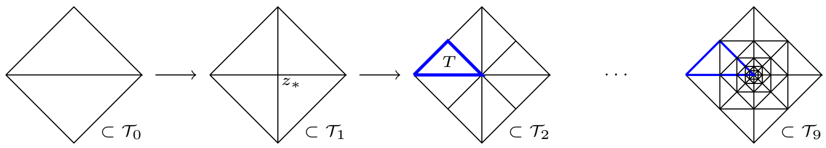

FIGURE 3.4. Optimal grading: A two element patch in $\mathcal{T}_0$, two initial global refinements, a possible choice for T indicated by blue lines, and the refinement of the two element patch in $\mathcal{T}_9$.

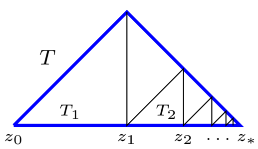

FIGURE 3.5. Optimal Grading: Refinement of the selected element $T \in \mathcal{T}_2$ in $\mathcal{T}_9$.

Lemma 3.11 (Optimal grading)**.** *The grading parameter* $\gamma = 2$ *is optimal in the following sense. If* H *is* any continuous function *with nodal values* $\{h_z = H(z)\}_{z\in\mathcal{V}}$ *satisfying* (3.1) *then*

$$\gamma = \sup_{\mathcal{T}\in\mathbb{T}} \max_{T\in\mathcal{T}} \max_{z,z'\in\mathcal{V}(T)} \frac{h_z}{h_{z'}} \geq 2.$$

Proof. We construct a sequence $\{\mathcal{T}_k\}_{k\geq 0} \subset \mathbb{T}$ to show $\gamma \geq 2$. Refine $\mathcal{T}_0$ by bisecting all elements twice. Since all edges of $\mathcal{T}_0$ are bisected the ensuing triangulation $\mathcal{T}_2$ is conforming. Fix then an arbitrary vertex $z_* \in (\mathcal{V}_1 \setminus \mathcal{V}_0) \cap \Omega$ and choose $T \in \mathcal{T}_2$ with refinement edge $\overline{z_0 z_*}$; compare with Figure 3.4.

We then iteratively construct the sequence $\mathcal{T}_k$ for $k \geq 2$ as follows. Define the subset $\mathcal{M}_k = \{T \in \mathcal{T}_k \mid z_* \in T\}$ and let $\mathcal{T}_{k+1}$ be the triangulation, which results from bisecting all triangles in $\mathcal{M}_k$ exactly once. We realize $\#\mathcal{M}_k = 8$, and $\mathcal{M}_k$ contains four pairs of elements, each pair sharing a common refinement edge; see Figure 3.4. Consequently, $\mathcal{T}_{k+1}$ is conforming and therefore $\{\mathcal{T}_k\}_{k\geq 0} \subset \mathbb{T}$.

We then ascertain the following for $k \geq 1$. Let T_* be the descendent of T in $\mathcal{T}_{2k+1}$ with $z_* \in T_*$. We unravel $\operatorname{gen}(T_*) = \operatorname{gen}(T) + 2k - 1$, which yields $h_{T_*} = 2^{-k+1/2} h_T$. We observe that for $\ell = 1, \dots, k$ the vertices $z_\ell = 2^{-\ell} z_0 + (1 - 2^\ell) z_* \in \mathcal{V}_{2k+1}$ are created during refinement. Then $z_k \in T_*$, and for any pair $(z_{\ell-1} z_\ell)$ there is an element $T_\ell \in \mathcal{T}_{2k+1}$ with $\overline{z_{\ell-1} z_\ell} \subset T_\ell$; compare with Figure 3.5.

By assumption (3.1a) we have $h_{z_{\ell-1}} \leq \gamma h_{z_\ell}$ for $\ell = 1, \dots, k$. We therefore conclude with the constants $0 < c_0 \leq C_0$ from Corollary 3.10 and Lemma 3.5

$$c_0 h_{T_1} \leq h_{z_0} \leq \gamma h_{z_1} \leq \gamma^2 h_{z_2} \leq \cdots \leq \gamma^k h_{z_k} \leq \gamma^k\, C_0 h_{T_*} = C_0\, \gamma^k 2^{-k+1/2} h_T.$$

Since $h_{T_1} = 2^{-1/2} h_T$ we deduce

$$0 < \frac{c_0}{2C_0} \leq \left(\frac{\gamma}{2}\right)^k \qquad \forall\, k \geq 1,$$

which shows $\gamma \geq 2$. ☐

4. H^1-Stability of the L_2-projection

In this section we prove the second main result, namely the H^1-stability of the L_2-projection. Throughout this section we closely follow the arguments by Bank and Yserentant [1] with some modifications accounting for the grading estimate of §3 for bisectional refinement.

We suppose that the Dirichlet boundary $\partial_D\Omega \subset \partial\Omega$ is meshed exactly by $\mathcal{T}_0$, i. e., $\partial_D\Omega$ is the union of boundary edges of $\mathcal{T}_0$. We set

$$H^1_D(\Omega) := \{v \in H^1(\Omega) \mid v \equiv 0 \text{ on } \partial_D\Omega\},$$

and for fixed $p \in \mathbb{N}$ we use conforming Lagrange finite elements over $\mathcal{T} \in \mathbb{T}$ of degree $p \in \mathbb{N}$, i. e.,

$$\mathbb{V}(\mathcal{T}, p) = \{V \in C^0(\bar{\Omega}) \mid V_{|T} \in \mathbb{P}_p,\ T \in \mathcal{T},\ V \equiv 0 \text{ on } \partial_D\Omega\}.$$

Throughout this section we fix $\mathcal{T}, p$ and use the shorthand notation $\mathbb{V} = \mathbb{V}(\mathcal{T}, p)$.

4.1. **An orthogonal decomposition of Lagrange finite elements.**

We strive at a decomposition of $\mathbb{V}$ into a subspace $\mathbb{V}_L$, where Π is completely determined from *local* values, and its orthogonal complement $\mathbb{V}_G$ in $\mathbb{V}$. We denote by $\mathcal{N} = \mathcal{N}(\mathcal{T})$ the set of all Lagrange nodes of $\mathbb{V}$ and write $\{V_a \mid a \in \mathcal{N}\}$ for the Lagrange basis of $\mathbb{V}$. We split $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$, where $\mathcal{N}_L := \{a \in \mathcal{N} \mid a \in \operatorname{interior}(T) \text{ for some } T \in \mathcal{T}\}$ and $\mathcal{N}_G := \mathcal{N} \setminus \mathcal{N}_L$. The nodes on an element T are decomposed into the interior nodes $\mathcal{N}_L(T) := \mathcal{N}_L \cap T$ and boundary nodes $\mathcal{N}_G(T) := \mathcal{N}_G \cap T$.

We define the subspace $\mathbb{V}_L := \operatorname{span}\{V_a \mid a \in \mathcal{N}_L\}$ and observe that the L_2-projection $\Pi_L \colon L_2(\Omega) \to \mathbb{V}_L$ can be computed *locally* within the single elements.

We are next interested in the L_2-projection into the L_2-orthogonal complement $\mathbb{V}_G := \mathbb{V}_L^\perp = \{V \in \mathbb{V} \mid \langle V, V_L\rangle_\Omega = 0 \quad \forall V_L \in \mathbb{V}_L\}$, which is non-local and the L_2-projection $\Pi_G\colon L_2(\Omega) \to \mathbb{V}_G$ couples information of different element. We next investigate how local information is spread by Π_G. This turns out to be instrumental for the analysis of $\Pi = \Pi_L + \Pi_G$.

Before doing this we would like to comment on the structure of $\mathbb{V}_L$. For $a \in \mathcal{N}_L$ let V_a be the associated Lagrange basis function. On $T \in \mathcal{T}$ we then set

$$\tilde{V}_a := V_a - \sum_{b\in\mathcal{N}_L(T)} \beta_{ab} V_b \qquad \text{such that} \qquad \langle \tilde{V}_a, V_b\rangle_T = 0 \quad \forall\, b \in \mathcal{N}_L(T)$$

we see that $\{\tilde{V}_a \mid a \in \mathcal{N}_G\}$ is a basis of $\mathbb{V}_L$. This means, although the definition $\mathbb{V}_G = \mathbb{V}_L^\perp$ is global it still allows for a construction of a local nodal basis. We define an interpolant $I_G\colon C^0(\bar{\Omega}) \cap H^1_D(\Omega) \to \mathbb{V}_G$ by

$$I_G v = \sum_{a\in\mathcal{N}_G} v(a)\tilde{V}(a).$$

Let Σ be the the skeleton of $\mathcal{T}$, i. e., the union of all edges of $\mathcal{T}$. We then find that $(I_G v)(a) = v(a)$ for all nodes $a \in \mathcal{N}_G = \mathcal{N} \cap \Sigma$ and $I_G(V) = V$ on Σ for any $V \in \mathbb{V}$.

4.2. **A projection with localizing properties.** We next construct an operator $\hat{\Pi}\colon L_2(\Omega) \to \mathbb{V}_G$ having suitable localizing properties. Let $z \in \mathcal{V}$ be a vertex of $\mathcal{T}$ and denote by Φ_z the piecewise linear hat function located at z. We recall $\operatorname{supp}(\Phi_z) = \Omega_z$ and set $\mathbb{V}_z := \{V \in \mathbb{V}_G \mid \operatorname{supp}(V) \subset \Omega_z\}$. We let $\Pi_z\colon L_2(\Omega_z) \to \mathbb{V}_z$ be the local L_2-projection onto $\mathbb{V}_z$ and define

$$\hat{\Pi} = \sum_{z\in\mathcal{V}} \Pi_z. \tag{4.1}$$

Since $\mathbb{V}_G = \bigoplus_{z\in\mathcal{V}} \mathbb{V}_z$ we can write any $V \in \mathbb{V}$ as $V = \sum_{z\in\mathcal{V}} V_z$ with $V_z \in \mathbb{V}_z$. Note, that such a decomposition is in general not unique. One particular and unique decomposition is given by the piecewise linear hat functions $\{\Phi_z\}_{z\in\mathcal{V}}$, which are partition of unity:

$$V = I_G(V) = I_G\Big(\sum_{z\in\mathcal{V}} \Phi_z V\Big) = \sum_{z\in\mathcal{V}} I_G(\Phi_z V) \qquad \text{with } I_G(\Phi_z V) \in \mathbb{V}_z. \tag{4.2}$$

Suppose that there are constants $0 < \lambda_{\min} \le \lambda_{\max}$ such that for all $V \in \mathbb{V}_G$ we have

$$\lambda_{\min} \sum_{z\in\mathcal{V}} \|I_G(\Phi_z V)\|^2_\Omega \le \|V\|^2_\Omega \tag{4.3a}$$

and

$$\|V\|^2_\Omega \le \lambda_{\max} \sum_{z\in\mathcal{V}} \|V_z\|^2_\Omega \text{ for any decomposition } V = \sum_{z\in\mathcal{V}} V_z \text{ with } V_z \in \mathbb{V}_z. \tag{4.3b}$$

Obviously, it suffices to show (4.4) element-wise for $T \in \mathcal{T}$. Moreover, resorting to the transformation rule one only has to consider an arbitrary but fixed triangle T. Setting $\mathbb{V}_G(T) := \{V_{|T} \mid V \in \mathbb{V}_G\}$ and $\mathbb{V}_z(T) := \mathbb{V}_G(T) \cap \mathbb{V}_z$, $z \in \mathcal{V}(T)$ it remains to prove

$$\lambda_{\min} \sum_{z\in\mathcal{V}(T)} \|I_G(\Phi_z V)\|^2_T \le \|V\|^2_T \qquad \forall\, V \in \mathbb{V}_G(T) \tag{4.4a}$$

and

$$\|V\|^2_T \le \lambda_{\max} \sum_{z\in\mathcal{V}(T)} \|V_z\|^2_T \qquad \forall\, V = \sum_{z\in\mathcal{V}(T)} V_z \in \bigoplus_{z\in\mathcal{V}(T)} \mathbb{V}_z(T). \tag{4.4b}$$

p	$\lambda_{\min}^{-1}$	$\lambda_{\max}$	$q \le$
1	2.000000000000000	2.000000000000000	0.333333333333333
2	1.632455532033676	2.720759220056126	0.356393958692601
3	1.393486807238790	2.644675210593510	0.315002511332227
4	1.295003216312185	2.636962512818568	0.297737526545759
5	1.222972165878670	2.594459484027661	0.280906146388308
6	1.302765305805047	2.593404439622517	0.295302231149967
7	1.299140221691548	2.565323271153285	0.292178820394059
8	1.512553465736873	2.567071358864272	0.326710513903202
9	1.576386303728820	2.547084406182064	0.334175300880362
10	2.028522383753149	2.549982029590451	0.389192884071447
11	2.269932525316121	2.535127973622464	0.411568856476138
12	3.300684993647074	2.538378961496179	0.486461159099767
13	4.089468949259074	2.526997823375298	0.525466578179801

TABLE 1. Values of for $\lambda_{\min}^{-1}$, $\lambda_{\max}$, and $q \le \frac{\sqrt{\lambda_{\max}/\lambda_{\min}}-1}{\sqrt{\lambda_{\max}/\lambda_{\min}}+1}$ taken from [1, Table 1].

Estimates (4.4) can be transformed into low dimensional, generalized eigenvalue problems. The minimal and maximal eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ have been determined in [1, §2] up to polynomial degree $p = 13$; compare with Table 1. The reported values in this table have been obtained using symbolic computations of the involved matrices and a numerical determination of the eigenvalues with high precision. We have numerically double-checked these values for $p \le 4$ with our finite element toolbox ALBERTA [11].

The constants $\lambda_{\min,\max}$ are lower and upper bounds for the eigenvalues of $\hat{\Pi}$.

Lemma 4.1. *The operator* $\hat{\Pi}\colon \mathbb{V}_G \to \mathbb{V}_G$ *is self-adjoint with respect to the* L_2*-inner product with minimal and maximal eigenvalues* $0 < \lambda_{\min} \le \lambda_{\max}$.

Proof. The operator $\hat{\Pi} = \sum_{z\in\mathcal{V}} \Pi_z$ is self-adjoint since all Π_z are self-adjoint. Employing for $U \in \mathbb{V}_G$ the decomposition (4.2) with $U_z = I_G(\Phi_z U)$ we find

$$\begin{aligned}\langle U, U\rangle_\Omega &= \sum_{z\in\mathcal{V}} \langle U_z, U\rangle_\Omega = \sum_{z\in\mathcal{V}} \langle U_z, \Pi_z U\rangle_\Omega \le \Big(\sum_{z\in\mathcal{V}} \|U_z\|_\Omega^2\Big)^{1/2} \Big(\sum_{z\in\mathcal{V}} \|\Pi_z U\|_\Omega^2\Big)^{1/2} \\ &\le \lambda_{\min}^{-1/2} \|U\|_\Omega \Big(\sum_{z\in\mathcal{V}} \|\Pi_z U\|_\Omega^2\Big)^{1/2},\end{aligned}$$

where we have used (4.3a) in the last step. Consequently,

$$\lambda_{\min} \|U\|_\Omega^2 \le \sum_{z\in\mathcal{V}} \|\Pi_z U\|_\Omega^2 = \sum_{z\in\mathcal{V}} \langle \Pi_z U, \Pi_z U\rangle_\Omega = \sum_{z\in\mathcal{V}} \langle U, \Pi_z U\rangle_\Omega = \langle U, \hat{\Pi} U\rangle_\Omega,$$

i. e., $\hat{\Pi}\colon \mathbb{V}_G \to \mathbb{V}_G$ is positive definite with eigenvalues $\lambda \ge \lambda_{\min}$.

Utilizing (4.3b) for $V = \hat{\Pi} U$ with $V_z = \Pi_z U$ we derive with the same arguments as above

$$\|\hat{\Pi} U\|_\Omega^2 \le \lambda_{\max} \sum_{z\in\mathcal{V}} \|\Pi_z U\|_\Omega^2 = \lambda_{\max} \langle U, \hat{\Pi} U\rangle_\Omega \le \lambda_{\max} \|U\|_\Omega \|\hat{\Pi} U\|_\Omega.$$

This implies for all eigenvalues λ of $\hat{\Pi}$ the bound $\lambda \le \lambda_{\max}$. □

Information of a locally supported function u is spread only locally by $\hat{\Pi}$. For a subset $\mathcal{T}'$ we set $\Omega(\mathcal{T}') := \bigcup\{T\colon T \in \mathcal{T}'\}$. We let $\overline{\mathcal{T}}'$ be the triangulation of all elements in $\mathcal{T}$ having a non-empty intersection with some element in $\mathcal{T}'$, i. e., $\overline{\mathcal{T}}' = \{T \in \mathcal{T} \mid T \cap T' \ne \emptyset \text{ for some } T' \in \mathcal{T}'\}$. We then write $\overline{\Omega}(\mathcal{T}') = \Omega(\overline{\mathcal{T}}')$, i. e., the subset $\Omega(\mathcal{T}')$ is enlarged by one layer of adjacent elements.

Lemma 4.2. *Let $\mathcal{T}'$ be a subset of $\mathcal{T}$ and suppose $u \in L_2(\Omega)$ satisfies* $\operatorname{supp}(u) \subset \Omega(\mathcal{T}')$. *Then* $\operatorname{supp}(\hat{\Pi}u) \subset \overline{\Omega}(\mathcal{T}')$.

Proof. If $\mathcal{V}(\mathcal{T}')$ are the vertices of $\mathcal{T}'$ we find $\overline{\Omega}(\mathcal{T}') = \bigcup_{z\in\mathcal{V}(\mathcal{T}')} \Omega_z$. For $z \in \mathcal{V}\backslash\mathcal{V}(\mathcal{T}')$ we have $|\Omega_z \cap \operatorname{supp}(u)| = 0$, which yields $\Pi_z u \equiv 0$ and finishes the proof. □

4.3. **An additive subspace correction method.** For given $u \in L_2(\Omega)$ we next construct suitable approximations of $\Pi_G u$ by a polynomially accelerated additive subspace correction method.

Let $\lambda_{\min}, \lambda_{\max}$ be the minimal and maximal eigenvalues of $\hat{\Pi}$ from Lemma 4.1. For $k \in \mathbb{N}$ denote by $\hat{T}_k \in \mathbb{P}_k([-1,1])$ the k^{th} Chebyshev polynomial and by $T_k \in \mathbb{P}_k([\lambda_{\min}, \lambda_{\max}])$ the transformed and normalized Chebyshev polynomial with $T_k(0) = 1$, i. e.,

$$T_k(\lambda) = \hat{T}_k\left(\frac{\lambda_{\max} + \lambda_{\min} - 2\lambda}{\lambda_{\max} - \lambda_{\min}}\right)\hat{T}_k\left(\frac{\lambda_{\max} + \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}\right)^{-1};$$

compare for instance with [7, Lemma 7.21]. Finally, define $P_k(\lambda) := T_k(1-\lambda)$ and let $a_{k\ell}$, $\ell = 0, \dots, k$, the coefficients of $P_k \in \mathbb{P}_k$. The normalization $T_k(0) = 1$ then yields

$$1 = T_k(0) = P_k(1) = \sum_{\ell=0}^{k} a_{k\ell}.$$

Let $u \in L_2(\Omega)$ be given. We set $W^{(0)} = 0$, inductively define for $\ell \geq 0$

$$W^{(\ell+1)} := W^{(\ell)} - \hat{\Pi}(W^{(\ell)} - u),$$

and set

$$U^{(k)} := \sum_{\ell=0}^{k} a_{k\ell} W^{(\ell)}. \tag{4.5}$$

The quality of this approximation to $\Pi_G u$ is estimated in the following lemma.

Lemma 4.3 (Convergence rate)**.** *The subspace correction* (4.5) *converges linearly with rate*

$$q \leq \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}.$$

where $\kappa \leq \lambda_{\max}/\lambda_{\min}$ is the condition number of $\hat{\Pi}$. This is, for any $u \in L_2(\Omega)$ we have

$$\|U^{(k)} - \Pi_G u\|_\Omega \leq 2q^k \|\Pi_G u\|_\Omega \qquad \forall\, k \geq 0.$$

Proof. We observe that $\hat{\Pi}u = \hat{\Pi}\,\Pi_G u$ yields $W^{(\ell+1)} - \Pi_G u = (\mathbb{1} - \hat{\Pi})(W^{(\ell)} - \Pi_G u)$. The normalization $\sum_{\ell=0}^{k} a_{k\ell} = 1$, $W^{(0)} = 0$ and (4.5) then implies

$$U^{(k)} - \Pi_G u = -\sum_{\ell=0}^{k} a_{k\ell}(\mathbb{1} - \hat{\Pi})^\ell \Pi_G u =: -T_k(\mathbb{1} - \hat{\Pi})\Pi_G u.$$

Expanding $\Pi_G u$ in a set of L_2-orthonormal eigenfunctions of the self-adjoint operator $\hat{\Pi}\colon \mathbb{V}_G \to \mathbb{V}_G$ we obtain

$$\|U^{(k)} - \Pi_G u\| \leq \max_{\lambda\in[\lambda_{\min},\lambda_{\max}]} |T_k(\lambda)|\, \|\Pi_G u\| \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|\Pi_G u\| = 2q^k \|\Pi_G u\|,$$

where we have used a standard estimate for the normalized Chebyschev polynomial T_k; compare for instance with [7, Lemma 7.21]. □

4.4. **Propagation of information.** We next investigate how the L_2-projection propagates information of a locally supported function. We combine the optimal grading estimate of Theorem 3.1 with the convergence rate q of the subspace correction (4.5).

We start with a *regularized generation* to define a suitable decomposition of Ω.

Definition 4.4 (Regularized generation)**.** For $\mathcal{T} \in \mathbb{T}$ let H be the regularized mesh-size function of $\mathcal{T}$ from Definition 3.3 with nodal values $\{h_z = H(z)\}_{z\in\mathcal{V}}$. We define the regularized generation rgen: $\mathcal{T} \to \mathbb{N}_0$ by

$$\operatorname{rgen}(T) := -\log_2\left(\min_{z\in\mathcal{V}(T)} h_z^2\right) \qquad T \in \mathcal{T}.$$

Theorem 3.1 yields the following fundamental properties of rgen.

Lemma 4.5 (Properties of rgen)**.** *With the constants* $0 < c_0 \le C_0$ *from Theorem 3.1 we have for any* $T \in \mathcal{T}$

$$\operatorname{rgen}(T) \in \mathbb{N}_0 \qquad \textit{and} \qquad c_0 h_T \le 2^{-\operatorname{rgen}(T)/2} \le C_0 h_T$$

If $T_1, T_2 \in \mathcal{T}$ *share a common vertex, i. e.,* $T_1 \cap T_2 \cap \mathcal{V} \neq \emptyset$*, then*

$$|\operatorname{rgen}(T_1) - \operatorname{rgen}(T_2)| \le 2.$$

Proof. Let $z' = \arg\min_{z\in\mathcal{V}(T)} h_z^2$. Recalling Definition 3.3 of H we find a $T_* \in \mathcal{T}$ such that

$$2^{-\operatorname{gen}(T)} \ge h_{z'}^2 = v(z', T_*) = 2^{2\operatorname{dist}(z',T_*)-\operatorname{gen}(T_*)} \in \{2^{-\ell} \mid \ell \in \mathbb{N}_0\}$$

This yields $\operatorname{rgen}(T) \in \mathbb{N}_0$. Moreover, $2^{-\operatorname{rgen}(T)/2} = 2^{-\log_2(h_{z'})} = h_{z'}$ shows in combination with (3.1b) and (3.1c) the second claim.

To show the last proposition we let $z \in \mathcal{V}$ be a common vertex of T_1 and T_2. For $i = 1, 2$ choose $z_i \in T_i$ with $h_{z_i}^2 = \min_{z\in\mathcal{V}(T_i)} h_z^2$, whence $\operatorname{rgen}(T_i) = -\log_2(h_{z_i}^2)$. Without loss of generality we may assume $h_{z_1}^2 \le h_{z_2}^2$, and recalling (3.2) we find $h_{z_1}^2 \le h_{z_2}^2 \le h_z^2 \le 1$. The grading estimate (3.1a) then yields

$$\tfrac{1}{4}h_{z_2}^2 \le h_{z_1}^2 \le h_{z_2}^2 \qquad \Longrightarrow \qquad -\log_2(h_{z_1}^2) \in [-\log_2(h_{z_2}^2), 2 - \log_2(h_{z_2}^2)]$$

which concludes the proof. □

We use the regularized element generation to define the following decomposition of Ω. We set

$$\mathcal{T}_{\operatorname{rgen}=\ell} := \{T \in \mathcal{T} \mid \operatorname{rgen}(T) = \ell\} \quad \text{and} \quad \Omega_\ell := \Omega(\mathcal{T}_{\operatorname{rgen}=\ell}) = \bigcup\{T : T \in \mathcal{T}_{\operatorname{rgen}=\ell}\}.$$

This definition readily implies the following basic properties.

Lemma 4.6. *We have* $\bigcup_{\ell\ge 0} \Omega_\ell = \overline{\Omega}$ *and* $\Omega_\ell \cap \Omega_k \neq \emptyset$ *yields* $|\ell - k| \le 2$.

Proof. This is a direct consequence of the definition of $\mathcal{T}_{\operatorname{rgen}=\ell}$ and Lemma 4.5. □

We are now able to bound the propagation of information by the L_2-projection.

Lemma 4.7. *Let* $u_i \in L_2(\Omega)$ *with* $\operatorname{supp}(u_i) \subset \Omega_i$*. Then we have for all* ℓ *the bound*

$$\|\Pi u_i\|_{\Omega_\ell} \le \Gamma(|\ell - i|)\|u_i\|_\Omega$$

with $\Gamma(m) := \min\{1,\, 2q^{\frac{m}{2}-1}\}$.

Proof. For $\ell = i$ we have $\|\Pi u_i\|_{\Omega_\ell} \le \|\Pi u_i\|_\Omega \le \|u_i\|_\Omega$, which is the proposition in that case.

If $\ell \neq i$ we observe $\Pi u_i = \Pi_G u_i$ in Ω_ℓ. Let $U^{(k)}$ be the approximation of $\Pi_G u_i$ defined in (4.5). Thanks to Lemma 4.3 we obtain

$$\|\Pi_G u_i - U^{(k)}\|_\Omega \le 2q^k \|u_i\|_\Omega.$$

We recall the construction of $U^{(k)}$. Starting with $W^{(0)} = 0$ we define

$$W^{(m+1)} := W^{(m)} + \hat{\Pi}(W^{(m)} - u_i) \quad m \geq 0, \qquad U^{(k)} := \sum_{m=0}^{k} a_{km} W^{(m)}.$$

The operator $\hat{\Pi}$ defined in (4.1) is the sum of the local projections Π_z. Defining $\Omega_i^{(m)} := \bigcup_{|j-i|\leq 2m} \Omega_j$ and combining Lemmas 4.2 and 4.7 we inductively deduce $\operatorname{supp}(W^{(m)}) \subset \Omega_i^{(m)}$ for $m = 1, \ldots, k$. This yields $\operatorname{supp}(U^{(k)}) \subset \Omega_i^{(k)}$. In particular, for any k with $|\ell - i| \geq 2k$ we have $\operatorname{supp}(U^{(k)}) \cap \Omega_\ell = \emptyset$, which gives

$$\|\Pi u_i\|_{\Omega_\ell} = \|\Pi_G u_i\|_{\Omega_\ell} = \|\Pi_G u_i - U^{(k)}\|_{\Omega_\ell} \leq 2q^k \|u_i\|_\Omega.$$

The largest possible k is $k = \left\lceil \frac{|\ell - i|}{2} \right\rceil - 1$, which finishes the proof. □

We next deduce a bound for the L_2-projection involving a negative power of the piecewise constant mesh-size function $h_\mathcal{T}$.

Proposition 4.8. *Suppose that the subspace correction* (4.5) *converges linearly with rate $q < \frac{1}{2}$. Then there is a constant $C_q < \infty$ such that*

$$\|h_\mathcal{T}^{-1} \Pi u\|_\Omega \leq C_q \|h_\mathcal{T}^{-1} u\|_\Omega \qquad \forall\, u \in L_2(\Omega).$$

Proof. We introduce the weighted norm

$$|||u|||^2 := \sum_\ell 2^\ell \|u\|^2_{\Omega_\ell}.$$

Note, that this is a finite sum. From Theorem 3.1 and Definition 4.4 we find

$$\frac{c_0}{2} \|h_\mathcal{T}^{-1} u\|_\Omega \leq |||u||| \leq 2C_0 \|h_\mathcal{T}^{-1} u\|_\Omega \qquad \forall\, u \in L_2(\Omega).$$

Thus it suffices to prove the estimate

$$|||\Pi_G u||| \leq \bar{C}_q |||u||| \qquad \forall\, u \in L_2(\Omega), \tag{4.6}$$

which yields the claim with $C_q = 4\bar{C}_q\, C_0 / c_0$.

Let $\boldsymbol{A}$ be the symmetric matrix with entries

$$a_{ij} := \sum_\ell 2^{\frac{\ell - i}{2}} \Gamma(|\ell - i|)\, 2^{\frac{\ell - j}{2}} \Gamma(|\ell - j|)$$

and let $\Lambda_{\max}$ be its maximal eigenvalue. If we define $u_i = u|_{\Omega_i}$ we then obtain by Lemma 4.7

$$\begin{aligned}
|||\Pi_G u|||^2 &= \left|\left|\left| \sum_i \Pi_G u_i \right|\right|\right|^2 = \sum_\ell 2^\ell \sum_i \sum_j \langle \Pi_G u_i,\, \Pi_G u_j \rangle_{\Omega_\ell} \\
&\leq \sum_i \sum_j \sum_\ell 2^\ell\, \Gamma(|\ell - i|) \Gamma(|\ell - j|) \|u_i\|_\Omega \|u_j\|_\Omega \\
&= \sum_{i,j} a_{ij}\, |||u_i|||\, |||u_j||| \leq \Lambda_{\max} \sum_i |||u_i|||^2 = \Lambda_{\max} |||u|||^2.
\end{aligned}$$

The eigenvalue $\Lambda_{\max}$ is bounded by the maximum column sum

$$\begin{aligned}
\Lambda_{\max} &\leq \max_j \sum_i |a_{ij}| = \max_j \sum_i \sum_\ell 2^{\frac{\ell - i}{2}} \Gamma(|\ell - i|)\, 2^{\frac{\ell - j}{2}} \Gamma(|\ell - j|) \\
&\leq \left(\max_\ell \sum_i 2^{\frac{\ell - i}{2}} \Gamma(|\ell - i|) \right) \left(\max_j \sum_\ell 2^{\frac{\ell - j}{2}} \Gamma(|\ell - j|) \right).
\end{aligned}$$

Recalling the definition of Γ we continue by

$$\sum_i 2^{\frac{\ell-i}{2}}\Gamma(|\ell-i|) \le \frac{2}{q}\bigg(\sum_{i\le\ell}(2q)^{\frac{\ell-i}{2}} + \sum_{i>\ell}(q/2)^{\frac{i-\ell}{2}}\bigg) \le \bigg(\frac{2/q}{1-\sqrt{2q}} + \frac{2/q}{1-\sqrt{q/2}}\bigg),$$

$$\sum_\ell 2^{\frac{\ell-j}{2}}\Gamma(|\ell-j|) \le \frac{2}{q}\bigg(\sum_{\ell\le j}(q/2)^{\frac{\ell-j}{2}} + \sum_{\ell>j}(2q)^{\frac{\ell-j}{2}}\bigg) \le \bigg(\frac{2/q}{1-\sqrt{q/2}} + \frac{2/q}{1-\sqrt{2q}}\bigg).$$

This shows (4.6) with

$$\Lambda_{\max} \le \bar{C}_q = \frac{4}{q^2}\bigg(\frac{1}{1-\sqrt{q/2}} + \frac{1}{1-\sqrt{2q}}\bigg)^2 \tag{4.7}$$

and finishes the proof. □

4.5. $\boldsymbol{H^1}$**-stability.** It is worth noticing in (4.7) that $C_q = \frac{4C_0}{c_0}\bar{C}_q \to \infty$ as $q \nearrow \frac{1}{2}$. This means, that we can prove Proposition 4.8 if the subspace correction (4.5) converges linearly with rate $q < \frac{1}{2}$. Referring to Table 1 we see

$$q < \tfrac{1}{2} \qquad \text{for } p \le 12.$$

For those p's we can employ the previous result, which is the key to prove H^1-stability of the L_2-projection.

Theorem 4.9 (H^1-stability for NVB)**.** *Suppose that $\mathcal{T}_0$ satisfies Assumption 2.2. For any triangulation $\mathcal{T} \in \mathbb{T}$ and polynomial degree $p \le 12$ the L_2-projection $\Pi\colon L_2(\Omega) \to \mathbb{V}(\mathcal{T},p)$ is H^1-stable and satisfies*

$$\|\nabla \Pi u\|_\Omega + \|h_{\mathcal{T}}^{-1}(\Pi u - u)\|_\Omega \lesssim \|\nabla u\|_\Omega \qquad \forall\, u \in H^1_D(\Omega).$$

Proof. Let $I_{\mathsf{SZ}}\colon H^1_D(\Omega) \to \mathbb{V}(\mathcal{T},p)$ be the Scott-Zhang interpolant, which satisfies

$$\|h_{\mathcal{T}}^{-1}(I_{\mathsf{SZ}}u - u)\|_\Omega + \|\nabla I_{\mathsf{SZ}}u\|_\Omega \lesssim \|\nabla u\|_\Omega \qquad \forall\, u \in H^1_D(\Omega). \tag{4.8}$$

The hidden constant solely depends on the shape coefficient of $\mathcal{T}$ and therefore on $\mathcal{T}_0$; compare with [12]. We apply Proposition 4.8 and (4.8) to obtain

$$\|h_{\mathcal{T}}^{-1}\Pi(I_{\mathsf{SZ}}u - u)\|_\Omega \le C_q\|h_{\mathcal{T}}^{-1}(I_{\mathsf{SZ}}u - u)\|_\Omega \lesssim \|\nabla u\|_\Omega. \tag{4.9}$$

Since Π is a projection we have $\Pi I_{\mathsf{SZ}}u = I_{\mathsf{SZ}}u$. We therefore obtain by (4.9) and (4.8) the estimate

$$\|h_{\mathcal{T}}^{-1}(\Pi u - u)\|_\Omega \le \|h_{\mathcal{T}}^{-1}\Pi(u - I_{\mathsf{SZ}}u)\|_\Omega + \|h_{\mathcal{T}}^{-1}(I_{\mathsf{SZ}}u - u)\|_\Omega \lesssim \|\nabla u\|_\Omega.$$

To bound the gradient of Πu we proceed by

$$\|\nabla\Pi u\|_\Omega \le \|\nabla(\Pi u - I_{\mathsf{SZ}}u)\|_\Omega + \|\nabla I_{\mathsf{SZ}}u\|_\Omega = \|\nabla\Pi(u - I_{\mathsf{SZ}}u)\|_\Omega + \|\nabla I_{\mathsf{SZ}}u\|_\Omega.$$

The second term on the right hand side is bounded by $\|\nabla u\|_\Omega$, thanks to (4.8). We next resort to the inverse inequality

$$\|\nabla V\|_\Omega^2 = \sum_{T\in\mathcal{T}}\|\nabla V\|_T^2 \lesssim \sum_{T\in\mathcal{T}}\|h_T^{-1}V\|_T^2 = \|h_{\mathcal{T}}^{-1}V\|_\Omega^2 \qquad \forall\, V \in \mathbb{V}(\mathcal{T},p),$$

and employ in addition (4.9) to finally deduce

$$\|\nabla\Pi(u - I_{\mathsf{SZ}}u)\|_\Omega \lesssim \|h_{\mathcal{T}}^{-1}\Pi(u - I_{\mathsf{SZ}}u)\|_\Omega \lesssim \|\nabla u\|_\Omega.$$ □

5. Extensions

In this final section we extend the theory of the previous section allowing for a non-optimal grading constant γ. We then address NVB without Assumption 2.2 on the initial grid and Red-Green-Refinement (RGR).

5.1. **Non-optimal grading.** Suppose that we have for $z \in \mathcal{V} \setminus \mathcal{V}_0$ the estimate $\operatorname{gen}(T) - \operatorname{gen}(T') \leq \alpha$ for all $T, T' \in \mathcal{T}(z)$. Then we can easily deduce for simple chains with $\mu = \alpha$ the bound

$$\operatorname{gen}(T') - \operatorname{gen}(T) \leq \mu \# \operatorname{CE}(z, z') \qquad \forall\, T \in \mathcal{T}_z,\, T' \in \mathcal{T}_{z'}.$$

Most of the work invested in §3 was to reduce the constant μ from $\alpha = 3$ to the optimal value $\mu = 2$ at the price of an additive constant.

When not striking for the optimal grading constant γ we aim at the following replacement in Lemma 3.9: There is a $\mu > 0$ and $C_\mu > 0$ such that

$$\mu \operatorname{dist}(z, z') - \operatorname{gen}(T') \geq -C_\mu - \operatorname{gen}(T) \qquad \forall\, T \in \mathcal{T}(z),\, T' \in \mathcal{T}(z'). \tag{5.1}$$

We then define the piecewise linear function H by the nodal values

$$h_z^2 := \min \left\{ 2^{\mu \operatorname{dist}(z,T) - \operatorname{gen}(T)} \mid T \in \mathcal{T} \right\}.$$

One crucial property of $\operatorname{dist}\colon \mathcal{V} \times \mathcal{V} \to \mathbb{N}_0$ is that $z, z' \in \mathcal{V}(T)$ yields $\operatorname{dist}(z, z') \leq 1$. This in turn implies $\operatorname{dist}(z, T_*) - \operatorname{dist}(z', T_*) \leq 1$ for any element T_* and we deduced as in Lemma 3.4 the grading estimate (3.1a)

$$\frac{h_z^2}{h_{z'}^2} \leq 2^{\mu(\operatorname{dist}(z,T_*) - \operatorname{dist}(z'(T_*))} \leq 2^\mu =: \gamma^2. \tag{5.2}$$

The lower bound (3.1b) shown in Lemma 3.5 is independent of the parameter μ. The upper bound (3.1c) is a consequence of (5.1) with a constant $C_0 = C_0(\mathcal{T}_0, \mu)$; compare with Corollary 3.10. In summary, we can prove Theorem 3.1 with constant $\gamma = 2^{\mu/2}$ whenever we can show (5.1).

We claim next that we can prove H^1-stability of Π provided that $q < \gamma^{-1}$. This can be seen by adapting all results in §4 that depend on γ. For $\gamma \geq 1$ we generalize the regularized element generation rgen of Definition 4.4 by

$$\operatorname{rgen}(T) := \left\lceil -\log_\gamma \left(\min_{z \in \mathcal{V}(T)} h_z^2 \right) \right\rceil \qquad T \in \mathcal{T}.$$

We then obtain the following replacement of Lemma 4.5 by repeating its proof with the new definition of rgen.

Lemma 5.1. *For all $T \in \mathcal{T}$ we have*

$$\operatorname{rgen}(T) \in \mathbb{N}_0 \qquad \text{and} \qquad c_0 \gamma^{-1} h_T \leq \gamma^{-\operatorname{rgen}(T)/2} \leq C_0 \gamma h_T$$

If $T_1, T_2 \in \mathcal{T}$ share a common vertex, i. e., $T_1 \cap T_2 \cap \mathcal{V} \neq \emptyset$, then

$$|\operatorname{rgen}(T_1) - \operatorname{rgen}(T_2)| \leq 2.$$

Since rgen only differs by 2 on neighboring elements we directly conclude Lemmas 4.6 and 4.7. It thus remains to verify Proposition 4.8.

Proposition 5.2. *Suppose that the subspace correction* (4.5) *converges linearly with rate $q < \gamma^{-1}$. Then there is a constant $C_q < \infty$ such that*

$$\|h_{\mathcal{T}}^{-1} \Pi u\|_\Omega \leq C_q \|h_{\mathcal{T}}^{-1} u\|_\Omega \qquad \forall\, u \in L_2(\Omega).$$

The constant C_q blows up as $q \to \gamma^{-1}$.

Proof. Repeat the proof of Proposition 4.8 with norm

$$|||v|||^2 := \sum_\ell \gamma^\ell \|v\|_{\Omega_\ell}^2.$$

The symmetric matrix $\boldsymbol{A}$ associated with this norm is

$$a_{ij} := \sum_\ell \gamma^{\frac{\ell - i}{2}} \Gamma(|\ell - i|) \, \gamma^{\frac{\ell - j}{2}} \Gamma(|\ell - j|).$$

Its maximum column sum is bounded by

$$\bar{C}_q = \frac{4\gamma^2}{q^2}\left(\frac{1}{1-\sqrt{q/\gamma}} + \frac{1}{1-\sqrt{\gamma q}}\right)^2 \to \infty \qquad \text{as } q \to \gamma^{-1}. \qquad \square$$

This shows the key ingredient of the proof of Theorem 4.9 and we conclude H^1-stability whenever $q < \gamma^{-1}$.

5.2. **Bisection without Assumption 2.2.** We can use NVB (in the iterative variant) even though $\mathcal{T}_0$ does not satisfy Assumption 2.2. If some $E \in \mathcal{E}_0$ is the refinement edge of some element $T \in \mathcal{T}_0$ but not of its neighbor $T' \in \mathcal{T}_0$ with $T \cap T' = E$ we say that E belongs to set of *non-compatible edges* $\mathcal{E}_0^{\mathsf{nc}} \subset \mathcal{E}_0$.

When comparing the generation of two direct neighbors $T, T' \in \mathcal{T}$ we observe the following. If $T \cap T' \subset E_0$ for some $E_0 \in \mathcal{E}_0^{\mathsf{nc}}$ the generation of T and T' can differ by 2. In all other cases we can apply Proposition 2.3 to conclude that the generation of direct neighbors differs at most by 1. This gives the following replacement of Lemma 3.6.

Lemma 5.3. *For any $z \in \mathcal{V}$ and all $T, T' \in \mathcal{T}(z)$ we have*

$$\operatorname{gen}(T') - \operatorname{gen}(T) \le \alpha = \begin{cases} 2\alpha_0 & \textit{if } z \in \mathcal{V}_0, \\ 4 & \textit{if } z \in \mathcal{V}(\mathcal{E}_0^{\mathsf{nc}}), \\ 3 & \textit{else.} \end{cases}$$

where α_0 is the constant from Lemma 2.1 and $\mathcal{V}(\mathcal{E}_0^{\mathsf{nc}}) \subset \mathcal{V}$ is the subset of all vertices that belong to the interior of some $E_0 \in \mathcal{E}_0^{\mathsf{nc}}$.

We next proceed as in §3.2, where we have in addition to account for crossing non-compatible macro edges $E_0 \in \mathcal{E}_0^{\mathsf{nc}}$. Therefore, repeating the arguments of Proposition 3.8 and Lemma 3.9 one arrives at the optimal estimate

$$\operatorname{gen}(T') - \operatorname{gen}(T) \le 2\operatorname{dist}(z, z') + 3 + (2\alpha_0 + 1)\#(\mathcal{V}_0 \cap \mathrm{CE}) + \sharp(\mathcal{E}_0^{\mathsf{nc}} \cap \mathrm{CE}),$$

where $\#(\mathcal{V}_0 \cap \mathrm{CE})$ is number of macro vertices in CE, and $\sharp(\mathcal{E}_0^{\mathsf{nc}} \cap \mathrm{CE})$ counts the number of instances that CE crosses an edge $E_0 \in \mathcal{E}_0$.

Any minimal chain connecting two vertices crosses a macro vertex z only once. This is used in Lemma 3.9 to bound $\#(\mathcal{V}_0 \cap \mathrm{CE}) \le \#\mathcal{V}_0$. However, a minimal chain may cross a macro edge several times. This prevents us from proving a simple bound for $\sharp(\mathcal{V}(\mathcal{E}_0^{\mathsf{nc}}) \cap \mathrm{CE})$.

Not insisting on an optimal estimate we use $\alpha = 4$ for all $z \in \mathcal{V} \setminus \mathcal{V}_0$ in §3.2. Repeating the arguments of Proposition 3.8 and Lemma 3.9 we then deduce (5.1) with $\mu = 3$ and $C_\mu = 4 + (2\alpha_0 + 1)\#\mathcal{V}_0$. Recalling (5.2) this yields $\gamma = 2^{\frac{3}{2}}$.

In combination with §5.1 this entails H^1-stability of the L_2-projection if the convergence rate q of the subspace correction (4.5) satisfies $q < \gamma^{-1} = 2^{-\frac{3}{2}}$. Checking the values for q in Table 1 we obtain the following result.

Theorem 5.4 (H^1-stability for NVB without Assumption 2.2)**.** *Suppose an arbitrary labeling of refinement edges on $\mathcal{T}_0$. For any refinement $\mathcal{T}$ of $\mathcal{T}_0$ produced by NVB the L_2-projection $\Pi\colon L_2(\Omega) \to \mathbb{V}(\mathcal{T}, p)$ is H^1-stable for all polynomial degrees $p \le 9$ except for $p = 2$.*

It is worth realizing that the non-optimal grading not only reduces the maximal polynomial degree but also excludes the highly important case of quadratic finite elements.

Remark 5.5 (Red-Blue-Green-Refinement). Carstensen has shown in [5] estimate (5.1) with $\mu = 3$ for Red-Blue-Green-Refinement. The criterion (6.6) in [4] is fulfilled with the ensuing grading constant $\gamma = \frac{3}{2}$. This is used in [5] to deduce

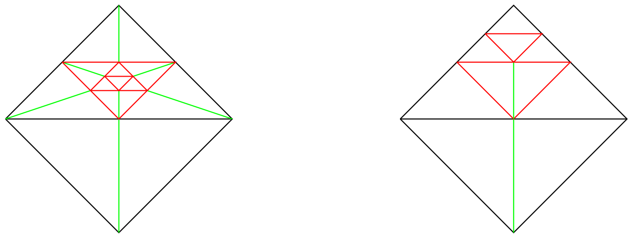

FIGURE 5.1. Generation in stars: The maximal difference of generations in a star is 4 with RGR (left) and 2 with MRGR (right).

H^1-stability of the L_2-projection for linear finite elements. The theory presented above extends this result to polynomial degrees $p = 3, \dots, 9$.

5.3. **Red-Green-Refinement.** Mesh adaptation using regular refinement is based on two refinement rules. The red refinement regularly decomposes a single triangle into four similar triangles. The green refinement bisects elements to remove single irregular vertices. The green refinement is necessary to obtain locally refined and conforming meshes.

The refinement loop may be summarized as follows.

Algorithm 5.1: Red-Green-Refinement (RGR)

```
undo all green refinements;
repeat
    decompose all marked elements using red refinement;
    mark all elements hosting two irregular vertices;
until no element is marked;
decompose all elements hosting an irregular vertex using green refinement;
```

We use the following notation. On $\mathcal{T}_0$ all elements are *red elements*. The four children of a red refinement we call *red elements*. The two children of a green refinement we call *green elements*. Two red elements sharing a common edge are *red neighbors*. We differ between two sort of green elements sharing a common edge. *Green twins* have the same parent and *green neighbors* have different parents.

For RGR we recursively define the following generation of elements.

(1) $\operatorname{gen}(T') = \operatorname{gen}(T) + 2$ for the four red children T' of T,
(2) $\operatorname{gen}(T') = \operatorname{gen}(T) + 1$ for the two green twins T' of T.

This definition yields the important property $h_T = 2^{-\operatorname{gen}(T)/2} h_{T_0}$ for all $T \in \mathcal{F}(T_0)$ and $T_0 \in \mathcal{T}_0$. Besides that, we have the following simple rules for the generation difference of direct neighbors in a star Ω_z with $z \in \mathcal{V} \setminus \mathcal{V}_0$. Two red neighbors and green twins have the same generation. The generation of a neighboring pair of a red and green element differs exactly by one, and that of two green neighbors differs at most by two.

Applying these rules, it is straightforward to construct a refinement of a star yielding the maximal difference of generations. This situation is depicted in the left image of Figure 5.1. We can see $\alpha = 4$, which implies $\gamma \le 2^{\frac{\alpha}{2}} = 4$. We also realize that a scaled version of the star can be inserted into the center triangle of the four smallest red triangles, yielding a conforming triangulation. This procedure can be repeated as often as desired, which shows that the optimal grading constant for RGR is $\gamma = 4$. Recalling §5.1 we need $q < \gamma^{-1} = \frac{1}{4}$ to prove H^1-stability. Looking into Table 1 we see $q > \frac{1}{4}$ for all p. Moreover, $\alpha = 4$ entails that (1.2) in [1] is not valid for standard RGR. Finally, the criterion (6.6) in [4] for linear finite elements is also not fulfilled with the grading constant $\gamma = 4$.

In summary, there is – to our best knowledge – not a single proof of H^1-stability for the L_2-projection for standard RGR.

5.4. **Modified-Red-Green-Refinement.** Analyzing standard RGR we see that an accumulation of green neighbors allows for very rapid changes of the local mesh-size. Asking for more moderate changes we have to avoid situations of green neighbors. We therefore should not apply green refinement to elements satisfying

T hosts a single irregular vertex that is created by some other element also hosting a single irregular vertex. (5.3)

This means, that on locally finest level we still use green refinement for the conforming closure but may refine elements with a single irregular vertex on locally coarser levels using red refinement. This gives rise to the following algorithm, which terminates latest when all initial elements are decomposed using red refinement.

Algorithm 5.2: Modified-Red-Green-Refinement (MRGR)

undo all green refinements;
repeat
    repeat
        decompose all marked elements using red refinement;
        mark all elements hosting two irregular vertices;
    until *no element is marked*;
    mark all elements with (5.3);
until *no element is marked*;
decompose all elements hosting an irregular vertex using green refinement;

This simple modification rules out the existence of green neighbors in any star Ω_z with $z \in \mathcal{V} \setminus \mathcal{V}_0$. We recall that two red neighbors have the same generation and the generation difference of a red and green neighbor is 1. Suppose a star of red triangles only. Then all elements are of the same generation. In a star with red and green triangles there are at most 5 elements with two pairs of green twins. Such a situation is shown in the right image of Figure 5.1 and we conclude for MRGR, that the maximal difference of two elements in Ω_z is bounded by $\alpha = 2$.

Applying then the techniques of §3.2 with $\alpha = 2$ allows us to show $\gamma \leq 2$. On top of that, with similar arguments as in §3.3 one can show that $\gamma = 2$ is the optimal grading constant. This entails Theorem 3.1 for MRGR and we can apply §4 to obtain the following result.

Theorem 5.6 (H^1-stability for MRGR)**.** *For any refinement $\mathcal{T}$ of $\mathcal{T}_0$ produced by MRGR the L_2-projection $\Pi\colon L_2(\Omega) \to \mathbb{V}(\mathcal{T}, p)$ is H^1-stable for all polynomial degrees $p \leq 12$.*

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