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whose commutators are scalar

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Well-posedness of non-autonomous linear evolution equations for generators whose commutators are scalar

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We prove the well-posedness of non-autonomous linear evolution equations for generators $A(t) : D(A(t)) \subset X \rightarrow X$ whose pairwise commutators are complex scalars and, in addition, we establish an explicit representation formula for the evolution. We also prove well-posedness in the more general case where instead of the 1-fold commutators only the p -fold commutators of the operators $A(t)$ are complex scalars. All these results are furnished with rather mild regularity assumptions: indeed, strong continuity conditions are sufficient. Additionally, we improve a well-posedness result of Kato for group generators $A(t)$ by showing that the original norm continuity condition can be relaxed to strong continuity. Applications include Segal field operators and Schrödinger operators for particles in external electric fields.

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1 Introduction

In this paper, we are concerned with non-autonomous linear evolution equations

$$x' = A(t)x \quad (t \in [s, 1]) \quad \text{and} \quad x(s) = y \quad (1.1)$$

for densely defined linear operators $A(t) : D(A(t)) \subset X \rightarrow X$ ($t \in [0, 1]$) and initial values $y \in Y \subset D(A(s))$ at initial times $s \in [0, 1]$. Well-posedness of such evolution equations has been studied by many authors in a large variety of situations. See, for instance, [25], [27], [18], [20] for an overview. In this paper, we are primarily interested in the special situation of semigroup generators $A(t)$ whose first (1-fold) or higher (p -fold) commutators at distinct times are complex scalars, in short:

$$[A(t_1), A(t_2)] = \mu(t_1, t_2) \in \mathbb{C} \quad (1.2)$$

or

$$[\dots [[A(t_1), A(t_2)], A(t_3)] \dots, A(t_{p+1})] = \mu(t_1, \dots, t_{p+1}) \in \mathbb{C} \quad (1.3)$$

in some sense to be made precise (see the commutation relations (2.1) and (2.12)). In this special situation we prove well-posedness for (1.1) on suitable dense subspaces Y of X and, moreover, in the case (1.2) we prove the representation formula

$$U(t, s) = e^{\overline{\int_s^t A(\tau) d\tau}} e^{1/2 \int_s^t \int_s^\tau \mu(\tau, \sigma) d\sigma d\tau} \quad (1.4)$$

for the evolution generated by the operators $A(t)$. We thereby generalize a well-posedness result of Goldstein and of Nickel and Schnaubelt from [9], [23] dealing with the special case of (1.2) where $\mu \equiv 0$: in [9] contraction semigroup generators are considered, while in [23] contraction semigroup generators are replaced by general semigroup generators and the formula (1.4) with $\mu \equiv 0$ is proved.

What one gains by restricting oneself to the special class of semigroup generators with (1.2) or (1.3) – instead of considering general semigroup generators as in [11], [12], [13], for instance – is that well-posedness follows under fairly weak regularity conditions: indeed, strong continuity conditions are sufficient – just like in the case of commuting operators from [9], [23] or in the elementary case of bounded operators. In general, by contrast, strong continuity is not sufficient as is well-known from [26]. Accordingly, in the well-posedness theorems from [12], [13] for general semigroup generators, for instance, a strong $W^{1,1}$ -regularity condition is imposed, and in the more special well-posedness result from [12] for a certain kind of group generators, there still is a norm continuity condition.

As is well-known from [17], [8], [31], in the case of bounded operators $A(t)$ one has representation formulas of Campbell–Baker–Hausdorff and Zassenhaus type for the evolution, which in the case (1.2) reduce to our representation formula (1.4). It should be noticed, however, that for bounded operators condition (1.2) can be satisfied only if $\mu \equiv 0$, so that (1.4) is independent of [17], [8], [31] (for non-zero μ). In view of the representation formulas from [31] it is desirable to prove representation formulas analogous to (1.4) also in the case (1.3), but this is left to future research.

All proofs in connection with the special situations (1.2) or (1.3) are, in essence, based upon the simple fact that in these situations the operators $A(r)$ can be commuted – up to controllable errors – through the exponential factors of the standard approximants $U_n(t, s)$ from [9], [12], [13], [23] for the sought evolution, which are of the form

$$U_n(t, s) = e^{A(r_m)\tau_m} \dots e^{A(r_1)\tau_1}$$

with partition points r_1, \dots, r_m of the interval $[s, t]$. See (2.4) and (2.13) respectively.

Apart from proving well-posedness for semigroup generators with (1.2) or (1.3) (which is our primary interest), we also improve the above-mentioned special well-posedness result from [12] for a certain kind of group generators: in the spirit of [16] we show that strong (instead of norm) continuity is sufficient in this result – just like in our other well-posedness results for the case (1.2) or (1.3). And in a certain special case involving quasicontraction group generators with time-independent domains in a uniformly convex

space, these other results can also be obtained by applying the improved well-posedness result for group generators.

In Section 2 we state and prove our abstract well-posedness results, Section 2.1 and Section 2.2 being devoted to the case (1.2) and (1.3) respectively and Section 2.3 being devoted to the improved well-posedness result for group generators. Section 2.4 discusses, among other things, the relation of our well-posedness results from Section 2.1 and 2.2 to the results from [12], [13], [16] and to the result from Section 2.3. In Section 3 we give some applications, namely to Segal field operators $\Phi(f_t)$ as well as to the related operators $H_\omega + \Phi(f_t)$ describing a classical particle coupled to a time-dependent quantized field of bosons (Section 3.1) and finally to Schrödinger operators describing a quantum particle coupled to a time-dependent spatially constant electric field (Section 3.2).

2 Abstract well-posedness results

We will use the notion of well-posedness and evolution systems from [7]. So, if $A(t) : D(A(t)) \subset X \rightarrow X$ for every $t \in I := [0, 1]$ is a linear operator and Y is a dense subspace of $\cap_{\tau \in I} D(A(\tau))$, then the initial value problems (1.1) for A are *well-posed on Y* if and only if there exists an *evolution system U solving (1.1) on Y* or, for short, an *evolution system U for A on Y* . Such an evolution system is necessarily unique (which follows by Corollary 2.1.2 of [25]). At some places we will also use the notion of evolution systems from [23], which is slightly weaker in that it does not require that $U(t, s)Y \subset Y$ for $s \leq t$. We will then speak of *evolution systems in the wide sense for A on Y* . Commutators of possibly unbounded operators are taken in the operator-theoretic sense,

$$D([A, B]) := D(AB - BA) = D(AB) \cap D(BA),$$

except in some formal heuristic computations (whose formal character will always be pointed out). As to the standard notions of (M, ω) -stability, of $A(t)$ -admissible subspaces, and of the part of an operator A in a subspace Y , we refer to [12] or [25]. X will always stand for a complex Banach space, $I = [0, 1]$ denotes the compact unit interval, and Δ the triangle $\{(s, t) \in I^2 : s \leq t\}$.

2.1 Scalar 1-fold commutators

We begin with a well-posedness result where, instead of the formal relation (1.2), the formally equivalent commutation relation (2.1) for the semigroups $e^{A(t)\cdot}$ with the generators $A(s)$ is imposed. Along with the well-posedness this theorem also yields a representation formula for the evolution. It is a generalization of a well-posedness result of Goldstein [9] (Theorem 1.1) and – after the slight modifications discussed in (2.27) and (2.28) below – of Nickel and Schnaubelt [23] (Theorem 2.3 and Proposition 2.5).

Theorem 2.1. *Suppose $A(t) : D(A(t)) \subset X \rightarrow X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M, ω) -stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$ and such that for some complex numbers $\mu(s, t) \in \mathbb{C}$*

$$A(s)e^{A(t)\tau} \supset e^{A(t)\tau}(A(s) + \mu(s, t)\tau) \tag{2.1}$$

for all $s, t \in I$ and $\tau \in [0, \infty)$. Suppose further that the maximal continuity subspace

$$Y^\circ := \{y \in \cap_{\tau \in I} D(A(\tau)) : t \mapsto A(t)y \text{ is continuous}\}$$

is dense in X and that $(s, t) \mapsto \mu(s, t)$ is continuous. Then there exists a unique evolution system U for A on Y° and it is given by

$$U(t, s) = e^{\overline{\int_s^t A(\tau) d\tau}^\circ} e^{1/2 \int_s^t \int_s^\tau \mu(\tau, \sigma) d\sigma d\tau} \quad ((s, t) \in \Delta),$$

where $(\int_s^t A(\tau) d\tau)^\circ$ is the (closable) operator defined by $y \mapsto \int_s^t A(\tau)y d\tau$ on Y° .

Proof. (i) We first show, in three steps, the existence of an evolution system U for A on Y° , which is then necessarily unique by the remark at the beginning of Section 2. In order to do so we approximate the sought evolution U by the standard approximants U_n from hyperbolic evolution equations theory, that is, we choose partitions

$$\pi_n = \{r_{ni} : i \in \{0, \dots, m_n\}\}$$

of I with $\text{mesh}(\pi_n) \rightarrow 0$ as $n \rightarrow \infty$ and, for any such partition, we evolve piecewise according to the values of $t \mapsto A(t)$ at the finitely many partition points of π_n . So,

$$U_n(t, s) := e^{A(r_n(t))(t-s)} \quad (2.2)$$

for $(s, t) \in \Delta$ with s, t lying in the same partition subinterval of π_n and

$$U_n(t, s) := e^{A(r_n(t))(t-r_n(t))} e^{A(r_n^-(t))(r_n(t)-r_n^-(t))} \dots e^{A(r_n(s))(r_n^+(s)-s)} \quad (2.3)$$

for $(s, t) \in \Delta$ with s, t lying in different partition subintervals of π_n . In the equations above, $r_n(u)$ for $u \in I$ denotes the largest partition point of π_n less than or equal to u and $r_n^-(u)$, $r_n^+(u)$ is the neighboring partition point below or above $r_n(u)$, respectively.

We then obtain, by repeatedly applying the assumed commutation relation, the following important commutation relation which allows us to take $A(r)$ from the left of $U_n(t, s)$ to the right and which is central to the entire proof:

$$A(r)U_n(t, s)y = U_n(t, s) \left(A(r) + \int_s^t \mu(r, r_n(\sigma)) d\sigma \right) y \quad (2.4)$$

for all $y \in D(A(r))$. As a first step, we observe that

$$U_n(t, s)U_n(s, r) = U_n(t, r) \quad \text{and} \quad \|U_n(t, s)\| \leq M e^{\omega(t-s)} \quad (2.5)$$

for all $(s, t), (r, s) \in \Delta$ and that $\Delta \ni (s, t) \mapsto U_n(t, s)$ is strongly continuous.

As a second step, we show that $(U_n(t, s)x)$ for every $x \in X$ is a Cauchy sequence in X uniformly in $(s, t) \in \Delta$. Since $\cap_{r' \in I} D(A(r'))$ is invariant under the semigroups $e^{A(r)}$ for all $r \in I$, it follows that $[s, t] \ni \tau \mapsto U_m(t, \tau)U_n(\tau, s)y$ for every $y \in \cap_{r' \in I} D(A(r'))$ is

piecewise continuously differentiable (with the partition points of $\pi_m \cup \pi_n$ as exceptional points) and therefore

$$\begin{aligned} U_n(t, s)y - U_m(t, s)y &= U_m(t, \tau)U_n(\tau, s)y \Big|_{\tau=s}^{\tau=t} \\ &= \int_s^t U_m(t, \tau)(A(r_n(\tau)) - A(r_m(\tau)))U_n(\tau, s)y \, d\tau = \int_s^t U_m(t, \tau)U_n(\tau, s) \\ &\quad \left(A(r_n(\tau)) - A(r_m(\tau)) + \int_s^\tau \mu(r_n(\tau), r_n(\sigma)) - \mu(r_m(\tau), r_n(\sigma)) \, d\sigma \right) y \, d\tau \end{aligned}$$

for every $y \in \cap_{r' \in I} D(A(r'))$ where, for the last equation, (2.4) has been used. So,

$$\begin{aligned} \sup_{(s,t) \in \Delta} \|U_n(t, s)y - U_m(t, s)y\| &\leq M^2 e^{w(b-a)} \left(\int_a^b \|A(r_n(\tau))y - A(r_m(\tau))y\| \, d\tau \right. \\ &\quad \left. + \int_a^b \int_a^b |\mu(r_n(\tau), r_n(\sigma)) - \mu(r_m(\tau), r_n(\sigma))| \|y\| \, d\sigma \, d\tau \right) \longrightarrow 0 \quad (m, n \rightarrow \infty) \end{aligned}$$

for every $y \in Y^\circ$ by the uniform continuity of $\tau \mapsto A(\tau)y$ and $(\tau, \sigma) \mapsto \mu(\tau, \sigma)$. And by (2.5) this uniform Cauchy property extends to all $y \in X$. Consequently,

$$U(t, s)x := \lim_{n \rightarrow \infty} U_n(t, s)x$$

for every $x \in X$ exists uniformly in $(s, t) \in \Delta$ and hence the properties observed in the first step carry over from U_n to U .

As a third step, we show that $t \mapsto U(t, s)y$ for every $y \in Y^\circ$ is a continuously differentiable solution to (1.1) with values in Y° . Since $\tau \mapsto U_n(\tau, s)y$ for $y \in \cap_{r' \in I} D(A(r'))$ is piecewise continuously differentiable, we have

$$\begin{aligned} U_n(t, s)y &= y + \int_s^t A(r_n(\tau))U_n(\tau, s)y \, d\tau \\ &= y + \int_s^t U_n(\tau, s) \left(A(r_n(\tau)) + \int_s^\tau \mu(r_n(\tau), r_n(\sigma)) \, d\sigma \right) y \, d\tau \end{aligned}$$

by virtue of (2.4) and therefore

$$U(t, s)y = y + \int_s^t U(\tau, s) \left(A(\tau) + \int_s^\tau \mu(\tau, \sigma) \, d\sigma \right) y \, d\tau$$

for all $y \in Y^\circ$. So, $t \mapsto U(t, s)y$ is continuously differentiable for every $y \in Y^\circ$ with derivative

$$t \mapsto U(t, s) \left(A(t) + \int_s^t \mu(t, \sigma) \, d\sigma \right) y = \lim_{n \rightarrow \infty} A(t)U_n(t, s)y = A(t)U(t, s)y,$$

where the last two equations hold by (2.4) and the closedness of $A(t)$. Also, since for all $y \in Y^\circ$ and $r \in I$

$$A(r)U_n(t, s)y \longrightarrow U(t, s) \left(A(r) + \int_s^t \mu(r, \sigma) \, d\sigma \right) y \quad (n \rightarrow \infty),$$

we see by the closedness of the operators $A(r)$ that $U(t, s)y \in Y^\circ$ for $y \in Y^\circ$. So, in summary, we have shown that U is an evolution system for A on Y° .

(ii) We now show, in three steps, that $(\int_s^t A(\tau) d\tau)^\circ$ for every fixed $(s, t) \in \Delta$ is closable and that its closure generates a strongly continuous semigroup in X with

$$e^{\overline{(\int_s^t A(\tau) d\tau)^\circ}} = U(t, s)e^{-1/2 \int_s^t \int_s^\tau \mu(\tau, \sigma) d\sigma d\tau}.$$

As a first step, we show a discrete version of the above representation formula: more precisely, we show that $B_n := \int_s^t A(r_n(\tau)) d\tau$ is closable and that B_n generates a strongly continuous semigroup with the following decomposition of Zassenhaus type:

$$e^{\overline{B_n}r} = U_n^r(t, s)e^{-1/2(\int_s^t \int_s^\tau \mu(r_n(\tau), r_n(\sigma)) d\sigma d\tau)r^2} \quad (r \in [0, \infty)), \quad (2.6)$$

where the operators $U_n^r(t, s)$ are defined in the same way as the operators $U_n(t, s)$ above with the only difference that now the generators $A(u)$ are all multiplied by the number r . Indeed, by the assumed commutation relations, we obtain the following commutation relations for semigroups,

$$e^{A_i\sigma} e^{A_j\tau} = e^{A_j\tau} e^{A_i\sigma} e^{\mu_{ij}\sigma\tau} \quad (\sigma, \tau \in [0, \infty)), \quad (2.7)$$

where $A_k := A(t_k)h_k$ and $\mu_{kl} := \mu(t_k, t_l)h_k h_l$ for arbitrary $t_k, t_l \in I$ and $h_k, h_l \in [0, \infty)$. (In fact, if $y \in D(A_i)$, then

$$e^{A_j\tau} e^{A_i\sigma} e^{\mu_{ij}\sigma\tau} y - e^{A_i\sigma} e^{A_j\tau} y = e^{A_i(\sigma-r)} e^{A_j\tau} e^{A_i r} e^{\mu_{ij}r\tau} y \Big|_{r=0}^{r=\sigma}$$

and $[0, \sigma] \ni r \mapsto e^{A_i(\sigma-r)} e^{A_j\tau} e^{A_i r} e^{\mu_{ij}r\tau} y$ is differentiable with derivative 0.) With the help of (2.7) one verifies that

$$[0, \infty) \ni r \mapsto e^{A_m r} \dots e^{A_1 r} e^{-1/2 \sum_{i \leq j} \mu_{ji} r^2} \quad (2.8)$$

is a strongly continuous semigroup in X . As this semigroup, by the assumed commutation relation, leaves the subspace $D(A_1) \cap \dots \cap D(A_m)$ invariant, its generator contains the operator $A_1 + \dots + A_m$, which is therefore closable with closure equal to the generator. Since B_n is of the form $A_1 + \dots + A_m$ and since the right-hand side of (2.6) is of the form (2.8), the assertion of the first step follows.

As a second step, we observe that the limit $T(r)x := \lim_{n \rightarrow \infty} e^{\overline{B_n}r} x$ exists locally uniformly in $r \in [0, \infty)$ for every $x \in X$ and that T is a strongly continuous semigroup in X . Indeed, with the same arguments as in (i), it follows that $(U_n^r(t, s)x)$ is convergent locally uniformly in r for every $x \in X$ and therefore the strongly continuous semigroups $e^{\overline{B_n} \cdot}$ by (2.6) are strongly convergent locally uniformly in r , so that

$$T(r)x := \lim_{n \rightarrow \infty} e^{\overline{B_n}r} x = U^r(t, s)e^{-1/2(\int_s^t \int_s^\tau \mu(\tau, \sigma) d\sigma d\tau)r^2} x \quad (x \in X) \quad (2.9)$$

defines a strongly continuous semigroup T on X .

As a third step, we show that the generator A_T of this semigroup is given by $\overline{B^\circ}$ where $B^\circ := (\int_s^t A(\tau) d\tau)^\circ$, from which the desired representation formula for U then follows by (2.9). Indeed, for all $y \in Y^\circ$,

$$\begin{aligned} \frac{T(h)y - y}{h} &= \lim_{n \rightarrow \infty} \frac{e^{\overline{B_n}h}y - y}{h} = \lim_{n \rightarrow \infty} \frac{1}{h} \int_0^h e^{\overline{B_n}r} \overline{B_n}y dr = \frac{1}{h} \int_0^h T(r)B^\circ y dr \\ &\longrightarrow B^\circ y \quad (h \searrow 0) \end{aligned}$$

by the dominated convergence theorem. So, B° is closable with $\overline{B^\circ} \subset A_T$. We now want to show that $D(\overline{B^\circ})$ is a core for A_T by verifying the invariance $T(r)D(\overline{B^\circ}) \subset D(\overline{B^\circ})$ for all $r \in [0, \infty)$. If $y \in Y^\circ$, then

$$B_m e^{\overline{B_n}r} y = e^{\overline{B_n}r} (B_m + \nu_{m,n}r) y \quad \text{with} \quad \nu_{m,n} := \int_s^t \int_s^t \mu(r_m(\tau), r_n(\sigma)) d\sigma d\tau$$

by the product decomposition of $e^{\overline{B_n}r}$ from (2.6) and by the central commutation relation (2.4). So,

$$B^\circ e^{\overline{B_n}r} y = e^{\overline{B_n}r} (B^\circ + \lim_{m \rightarrow \infty} \nu_{m,n}r) y$$

for all $y \in Y^\circ$, from which it further follows that

$$T(r)y \in D(\overline{B^\circ}) \quad \text{and} \quad \overline{B^\circ} T(r)y = T(r)(B^\circ + \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \nu_{m,n}r) y = T(r)B^\circ y$$

for all $y \in Y^\circ$. In the last equation, we used that $\mu(\tau, \sigma) = -\mu(\sigma, \tau)$ for all $\sigma, \tau \in I$ which can be seen from (2.7). It follows that $\overline{B^\circ} T(r) \supset T(r) \overline{B^\circ}$ and, in particular, $T(r)D(\overline{B^\circ}) \subset D(\overline{B^\circ})$ for all $r \in [0, \infty)$. So, $D(\overline{B^\circ})$ is a core for A_T and hence $A_T = \overline{B^\circ}$, as desired. \blacksquare

We also note the following variant of the above theorem where the form of the imposed commutation relation is closer to (1.2). In return, one has to require relatively strong invariance conditions.

Corollary 2.2. *Suppose $A(t) : D(A(t)) \subset X \rightarrow X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M, ω) -stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$. Suppose further that Y is an $A(t)$ -admissible subspace of X for every $t \in I$ such that*

$$Y \subset \bigcap_{\tau \in I} D(A(\tau)) \quad \text{and} \quad A(t)Y \subset \bigcap_{\tau \in I} D(A(\tau)),$$

$A(t)|_Y$ is a bounded operator from Y to X , and

$$[A(s), A(t)]|_Y \subset \mu(s, t) \in \mathbb{C}$$

for all $s, t \in I$. Suppose finally that $(s, t) \mapsto \mu(s, t)$ and $t \mapsto A(t)y$ are continuous for all $y \in Y$. Then the conclusions of the above theorem hold true.

Proof. We establish the commutation relations

$$e^{A_1\sigma}e^{A_2\tau} = e^{A_2\tau}e^{A_1\sigma}e^{\mu_{12}\tau\sigma} \quad (\sigma, \tau \in [0, \infty)), \quad (2.10)$$

where $A_k := A(t_k)$ and $\mu_{kl} := \mu(t_k, t_l)$ for arbitrary $t_1, t_2 \in I$. In order to do so, one shows that

$$A_1e^{A_2\tau}y = e^{A_2\tau}(A_1 + \mu_{12}\tau)y \quad (2.11)$$

for $y \in Y$ by differentiating $[0, \tau] \ni r \mapsto e^{A_2(\tau-r)}A_1e^{A_2r}y$ for vectors y in the domain of the part \tilde{A}_2 of A_2 in Y which by the A_2 -admissibility of Y is the generator of the strongly continuous semigroup $t \mapsto e^{A_2t}|_Y$ in Y (Proposition 2.3 of [12]). (In addition to the A_2 -admissibility, the boundedness of $A_1|_Y$ from Y to X and the invariance condition $A_1Y \subset D(A_2)$ come into play here.) Along the same lines as (2.7), the relation (2.10) then follows. And since (2.10) is equivalent to the commutation relations (2.1) and since $Y^\circ \supset Y$ is dense in X , the assumptions of Theorem 2.1 are satisfied, as desired. \blacksquare

2.2 Scalar p -fold commutators

We begin with a well-posedness result where, instead of the formal relation (1.3), the formally equivalent commutation relations (2.12) for the semigroups $e^{A(t)\cdot}$ with the generators $A(s_1) = C^{(0)}(s_1)$ and certain operators $C^{(k)}(s_1, \dots, s_{k+1})$ (which are formally given by k -fold commutators) are imposed.

Theorem 2.3. *Suppose $A(t) : D(A(t)) \subset X \rightarrow X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M, ω) -stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$ and such that for some operators $C^{(k)}(s_1, \dots, s_{k+1})$, where $k \in \{0, \dots, p-1\}$ and $C^{(0)}(s) := A(s)$, and for some complex numbers $\mu(t_1, \dots, t_{p+1}) \in \mathbb{C}$*

$$\begin{aligned} C^{(k)}(\underline{s})e^{A(t)\tau} \supset e^{A(t)\tau} & \left(C^{(k)}(\underline{s}) + C^{(k+1)}(\underline{s}, t)\tau + \dots + C^{(p-1)}(\underline{s}, t, \dots, t) \frac{\tau^{p-1-k}}{(p-1-k)!} + \right. \\ & \left. + \mu(\underline{s}, t, \dots, t) \frac{\tau^{p-k}}{(p-k)!} \right) \quad (\underline{s} := (s_1, \dots, s_{k+1})) \end{aligned} \quad (2.12)$$

for all $k \in \{0, \dots, p-1\}$ and $s_i, t \in I$ and $\tau \in [0, \infty)$. Suppose further that the maximal continuity subspace

$$\begin{aligned} Y^\circ & := \bigcap_{k=0}^{p-1} \{y \in D_k : (t_1, \dots, t_{k+1}) \mapsto C^{(k)}(t_1, \dots, t_{k+1})y \text{ is continuous}\} \\ D_k & := \bigcap_{\tau_1, \dots, \tau_{k+1} \in I} D(C^{(k)}(\tau_1, \dots, \tau_{k+1})) \end{aligned}$$

is dense in X and that $(t_1, \dots, t_{p+1}) \mapsto \mu(t_1, \dots, t_{p+1})$ is continuous. Then there exists a unique evolution system U for A on Y° .

Proof. We define U_n as in (2.2) and (2.3) and, for $u \in I$, we define i_{nu} to be the index $i \in \{0, \dots, m_n\}$ with $u \in [r_{ni}, r_{n(i+1)})$. We then obtain, by the assumed commutation relations, the following important commutation relation which allows us to take the operators $A(r)$ from the left of $U_n(t, s)$ to the right:

$$A(r)U_n(t, s)y = U_n(t, s)(A(r) + S_n^{(1)}(t, s, r) + \dots + S_n^{(p)}(t, s, r))y \quad (2.13)$$

$$S_n^{(k)}(t, s, r) := \int_s^t \int_s^{\tau_1} \dots \int_s^{\tau_{k-1}} C^{(k)}(r, r_{ni_{n\tau_1}}, \dots, r_{ni_{n\tau_k}}) / \alpha_{i_{n\tau_1}, \dots, i_{n\tau_k}} d\tau_k \dots d\tau_2 d\tau_1$$

for all $y \in Y^\circ$, $r \in I$, $(s, t) \in \Delta$, where α_{j_1, \dots, j_k} for a k -tuple (j_1, \dots, j_k) of natural numbers denotes the number of permutations σ leaving the k -tuple invariant, that is,

$$(j_{\sigma(1)}, \dots, j_{\sigma(k)}) = (j_1, \dots, j_k).$$

(In verifying (2.13), it is best to abbreviate $A(r) = A_0$ and $U_n(t, s) = e^{A_m} \dots e^{A_1}$ and to prove by induction over $m \in \mathbb{N}$, with the help of the assumed commutation relations, that

$$A_0 e^{A_m} \dots e^{A_1} y = e^{A_m} \dots e^{A_1} (A_0 + S^{(1)} + \dots + S^{(p)}) y$$

$$S^{(k)} := \sum_{1 \leq j_k \leq \dots \leq j_1 \leq m} C_{0, j_1, \dots, j_k}^{(k)} / \alpha_{j_1, \dots, j_k}$$

and finally to notice that the sums $S^{(k)}$ are nothing but the integrals $S_n^{(k)}(t, s, r)$ in (2.13).) With the commutation relation (2.13) at hand, we can now proceed in the same way as in the proof of Theorem 2.1 because the maps $(t_1, \dots, t_{k+1}) \mapsto C^{(k)}(t_1, \dots, t_{k+1})y$ are continuous for $y \in Y^\circ$ by assumption and because for every $(\tau_1, \dots, \tau_k) \in I^k$ with $\tau_1 > \dots > \tau_k$ one has $\alpha_{i_{n\tau_1}, \dots, i_{n\tau_k}} \rightarrow k!$ as $n \rightarrow \infty$. \blacksquare

We also note the following variant of the above theorem where the form of the imposed commutation relation is closer to (1.3). In return, one has to require relatively strong invariance conditions.

Proposition 2.4. *Suppose $A(t) : D(A(t)) \subset X \rightarrow X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M, ω) -stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$ and recursively define $C^{(0)}(t) := A(t)$ as well as $C^{(k)}(t_1, \dots, t_{k+1}) := [C^{(k-1)}(t_1, \dots, t_k), A(t_{k+1})]$ for $k \in \mathbb{N}$. Suppose further that Y is an $A(t)$ -admissible subspace of X for every $t \in I$, and $p \in \mathbb{N}$ a natural number such that for all $t_i \in I$*

$$Y \subset \bigcap_{\tau_1, \dots, \tau_p \in I} D(C^{(p-1)}(\tau_1, \dots, \tau_p)) \quad \text{and} \quad C^{(p-1)}(t_1, \dots, t_p)Y \subset \bigcap_{\tau \in I} D(C^{(0)}(\tau)),$$

$C^{(k)}(t_1, \dots, t_{k+1})|_Y$ is a bounded operator from Y to X for all $k \in \{0, \dots, p-1\}$, and

$$C^{(p)}(t_1, \dots, t_{p+1})|_{D(\tilde{A}(t_{p+1}))} \subset \mu(t_1, \dots, t_{p+1}) \in \mathbb{C},$$

where $\tilde{A}(t)$ is the part of $A(t)$ in Y . Suppose finally that $(t_1, \dots, t_{p+1}) \mapsto \mu(t_1, \dots, t_{p+1})$ and $(t_1, \dots, t_{k+1}) \mapsto C^{(k)}(t_1, \dots, t_{k+1})y$ are continuous for all $y \in Y$ and $k \in \{0, \dots, p-1\}$. Then there exists a unique evolution system U in the wide sense for A on Y .

Proof. With the same arguments as for (2.11) one verifies the commutation relations

$$C^{(k)}(\underline{s})e^{A(t)\tau}y = e^{A(t)\tau}\left(C^{(k)}(\underline{s}) + C^{(k+1)}(\underline{s}, t)\tau + \cdots + C^{(p-1)}(\underline{s}, t, \dots, t)\frac{\tau^{p-1-k}}{(p-1-k)!} + \mu(\underline{s}, t, \dots, t)\frac{\tau^{p-k}}{(p-k)!}\right)y \quad (\underline{s} := (s_1, \dots, s_{k+1}))$$

for all $y \in Y$ and $k \in \{0, \dots, p-1\}$ and from these, in turn, one obtains the existence of an evolution system U in the wide sense for A on Y in exactly the same way as in the proof of Theorem 2.3. In order to obtain uniqueness, one has only to observe that for any evolution system V in the wide sense for A on Y ,

$$U_n(t, s)y - V(t, s)y = V(t, \tau)U_n(\tau, s)y\Big|_{\tau=s}^{\tau=t} = \int_s^t V(t, \tau)(A(r_n(\tau)) - A(\tau))U_n(\tau, s)y d\tau$$

converges to 0 for every $y \in Y$ and $(s, t) \in \Delta$ by (2.13). \blacksquare

2.3 Well-posedness for group generators

After having proved well-posedness results for semigroup generators with (1.2) or (1.3), we now improve, inspired by [16], the special well-posedness result from [12] (Theorem 5.2 in conjunction with Remark 5.3) for a certain kind of group (instead of semigroup) generators $A(t)$ and certain uniformly convex subspaces Y of the domains $D(A(t))$: we show that this result is still valid if $t \mapsto A(t)|_Y$ is assumed to be only strongly continuous (instead of norm continuous as in [12]). In [16] the same is done for the general well-posedness theorem from [12] (Theorem 6.1). We point out that although several arguments from [16] can be used here as well, it is by no means obvious that the improvement made in [16] can be carried over to the special well-posedness result of [12]. In particular, the possibility of such an improvement is not mentioned in the literature – at least, not in [16], [32], [33], [14], [15], [29], [30].

Theorem 2.5. *Suppose $A(t) : D(A(t)) \subset X \rightarrow X$ for every $t \in I$ is the generator of a strongly continuous group on X such that $A^+ := A(\cdot)$ and $A^- := A(1 - \cdot)$ are (M, ω) -stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$. Suppose further that Y for every $t \in I$ is an $A^\pm(t)$ -admissible subspace of X contained in $\cap_{\tau \in I} D(A(\tau))$ and that $A(t)|_Y$ is a bounded operator from Y to X such that*

$$t \mapsto A(t)|_Y$$

is strongly continuous. And finally, suppose there exists for each $t \in I$ a norm $\|\cdot\|_t^\pm$ on Y equivalent to the original norm of Y such that $Y_t^\pm := (Y, \|\cdot\|_t^\pm)$ is uniformly convex and

$$\|y\|_t^\pm \leq e^{c^\pm|t-s|} \|y\|_s^\pm \quad (y \in Y \text{ and } s, t \in I) \quad (2.14)$$

for some constant $c^\pm \in (0, \infty)$ and such that the Y -part $\tilde{A}^\pm(t)$ of $A^\pm(t)$ generates a quasicontraction semigroup in Y_t^\pm , more precisely

$$\left\| e^{A^\pm(t)\tau} y \right\|_t^\pm \leq e^{\omega_0 \tau} \|y\|_t^\pm \quad (\tau \in [0, \infty), y \in Y, t \in I) \quad (2.15)$$

for some $\omega_0 \in \mathbb{R}$. Then there exists a unique evolution system U for A on Y .

Proof. We adopt from [16] the shorthand notation $U^\pm(t, s, \pi)$ for products of the semi-groups $e^{A^\pm(t)}$ associated with finite or infinite partitions π in I . Without further specification, convergence or continuity in X, Y will always mean convergence or continuity in the norm of X, Y . As a first step we show that for each $y \in Y$ and $s \in [0, 1]$ there exists a sequence $(\pi_n^\pm) = (\pi_{y,s,n}^\pm)$ of partitions of I such that $(U^\pm(t, s, \pi_{y,s,n}^\pm)y)$ is a Cauchy sequence in X for $t \in [s, 1]$. What we have to show here is that for every sequence $\pi = (t_k)$, strictly monotonically increasing in I , and arbitrary $t'_k \in [t_k, t_{k+1})$, the following assertions are satisfied (Lemma 1 of [16]):

- (i) $(U^\pm(t'_k, t_0, \pi)x)$ is a Cauchy sequence in X for every $x \in X$ whose limit will be denoted by $U^\pm(t_\infty, t_0, \pi)x$ where $t_\infty := \lim_{k \rightarrow \infty} t'_k$,
- (ii) $(U^\pm(t'_k, t_0, \pi)y)$ is a Cauchy sequence in Y for every $y \in Y$.

With the help of Lemma 2 and 3 of [16], whose proofs carry over without change to the present situation, the existence of sequences $(\pi_{y,s,n}^\pm)$ of partitions with the claimed properties then follows. Assertion (i) is simple and is proven in the same way as in [16], while assertion (ii) has to be proven in a completely different way because the proof of [16] essentially rests on the existence of certain isomorphisms $S(t)$ from Y onto X which are not available here. We show, using ideas from [12] (Section 5), that

$$U^\pm(t_\infty, t_0, \pi)y \in Y \quad \text{and} \quad U^\pm(t'_k, t_0, \pi)y \longrightarrow U^\pm(t_\infty, t_0, \pi)y \quad \text{weakly in } Y \quad (2.16)$$

for every $y \in Y$ and that

$$\limsup_{k \rightarrow \infty} \|U^\pm(t'_k, t_0, \pi)y\|_{\bar{t}_\infty}^\mp \leq \|U^\pm(t_\infty, t_0, \pi)y\|_{\bar{t}_\infty}^\mp \quad (\bar{t}_\infty := 1 - t_\infty) \quad (2.17)$$

for $y \in Y$, which two things by the uniform convexity of $Y_{\bar{t}_\infty}$ imply the convergence of $(U^\pm(t'_k, t_0, \pi)y)$ to $U^\pm(t_\infty, t_0, \pi)y$ in Y and in particular assertion (ii). In order to see (2.16) notice first that \tilde{A}^\pm is $(\tilde{M}, \tilde{\omega})$ -stable for some $\tilde{M} \in [1, \infty)$ and $\tilde{\omega} \in \mathbb{R}$ by (2.14) and (2.15) (Proposition 3.4 of [12]), so that the sequence $(U^\pm(t'_k, t_0, \pi)y)$ is bounded in the norm of Y . Since Y is reflexive (Milman's theorem), every subsequence of $(U^\pm(t'_k, t_0, \pi)y)$ has in turn a weakly convergent subsequence in Y whose weak limit must be equal to $U^\pm(t_\infty, t_0, \pi)y$ by assertion (i), and therefore (2.16) follows. In order to see (2.17) notice first that $(U^\pm(t'_k, t_n, \pi)x)_{n \in \mathbb{N}}$ is a Cauchy sequence in X for every $x \in X$ and $k \in \mathbb{N}$, where

$$U^\pm(t'_k, \tau, \pi) := U^\pm(\tau, t'_k, \pi)^{-1} = e^{-A^\pm(t_k)(t_{k+1}-t'_k)} \dots e^{-A^\pm(r_\pi(\tau))(\tau-r_\pi(\tau))}$$

for $\tau \in (t'_k, t_\infty)$ and where $r_\pi(\tau)$ denotes the largest point of π less than or equal to τ . Indeed, for every $x \in Y$,

$$U^\pm(t'_k, t_m, \pi)x - U^\pm(t'_k, t_n, \pi)x = - \int_{t_m}^{t_n} U^\pm(t'_k, \tau, \pi)A^\pm(r_\pi(\tau))x \, d\tau \longrightarrow 0 \quad (m, n \rightarrow \infty)$$

in X and by the (M, ω) -stability of A^\mp , this convergence extends to all $x \in X$. We denote the limit by $U^\pm(t'_k, t_\infty, \pi)x$ and note for later use that

$$U^\pm(t'_k, t_\infty, \pi)y \in Y \quad \text{and} \quad U^\pm(t'_k, t_n, \pi)y \longrightarrow U^\pm(t'_k, t_\infty, \pi)y \quad \text{weakly in } Y \quad (2.18)$$

by the same arguments as those for (2.16). Since $U^\pm(t'_k, t_0, \pi) = U^\pm(t'_k, t_n, \pi)U^\pm(t_n, t_0, \pi)$ for all $n \in \mathbb{N}$, it follows that

$$U^\pm(t'_k, t_0, \pi) = U^\pm(t'_k, t_\infty, \pi)U^\pm(t_\infty, t_0, \pi). \quad (2.19)$$

Also, since

$$U^\pm(t'_k, t_n, \pi) = e^{A^\mp(\bar{t}_k)(t_{k+1}-t'_k)} \dots e^{A^\mp(\bar{t}_{n-1})(t_n-t_{n-1})} \quad (\bar{t}_i := 1 - t_i)$$

for $n \geq k+1$, it follows by successively passing from $\|\cdot\|_{\bar{t}_\infty}^\mp$ to $\|\cdot\|_{\bar{t}_k}^\mp$ to ... to $\|\cdot\|_{\bar{t}_{n-1}}^\mp$ and back to $\|\cdot\|_{\bar{t}_\infty}^\mp$ with the help of (2.14) and by using (2.15) at each successive step, that

$$\|U^\pm(t'_k, t_n, \pi)z\|_{\bar{t}_\infty}^\mp \leq e^{2c(t_\infty-t_k)} e^{\omega_0(t_n-t'_k)} \|z\|_{\bar{t}_\infty}^\mp$$

for every $z \in Y$, and therefore

$$\|U^\pm(t'_k, t_\infty, \pi)z\|_{\bar{t}_\infty}^\mp \leq e^{2c(t_\infty-t_k)} e^{\omega_0(t_\infty-t'_k)} \|z\|_{\bar{t}_\infty}^\mp \quad (2.20)$$

for $z \in Y$ by virtue of (2.18). Combining now (2.19) and (2.20) we obtain (2.17), which concludes our first step.

As a second step we observe that $U_0^\pm(t, s)y := \lim_{n \rightarrow \infty} U^\pm(t, s, \pi_{y,s,n}^\pm)y$ for $y \in Y$ and $(s, t) \in \Delta$ defines a linear operator from Y to X extendable to a bounded operator $U^\pm(t, s)$ in X , and that U^\pm is an evolution system in X such that $t \mapsto U^\pm(t, s)y$ for every $y \in Y$ is right differentiable (in the norm of X) at s with right derivative $A^\pm(s)y$. All this follows in the same way as in [16] (Lemma 4 and 5). In particular, it follows from the right differentiability and evolution system properties just mentioned that $[0, t] \ni s \mapsto U^\pm(t, s)y$ is continuously differentiable (from both sides) for every $y \in Y$ with derivative $s \mapsto -U^\pm(t, s)A^\pm(s)y$ by Corollary 2.1.2 of [25].

As a third step we show that $U^\pm(t, s)$ leaves the subspace Y invariant for every $(s, t) \in \Delta$ and that $[s, 1] \ni t \mapsto U^\pm(t, s)y$ is right continuous in Y for every $y \in Y$. In order to see that $U^\pm(t, s)y$ lies in Y for $y \in Y$, notice that the sequence $(U^\pm(t, s, \pi_{y,s,n}^\pm)y)$ is bounded in the norm of Y , whence by the same argument as for (2.16)

$$U^\pm(t, s)y \in Y \quad \text{and} \quad U^\pm(t, s, \pi_{y,s,n}^\pm)y \longrightarrow U^\pm(t, s)y \quad \text{weakly in } Y. \quad (2.21)$$

In order to see that $[s, 1] \ni t \mapsto U^\pm(t, s)y$ is right continuous in Y for every $y \in Y$, we have only to show, by the invariance property just established, that $U^\pm(t+h, t)y \longrightarrow y$ in Y as $h \searrow 0$ for every $t \in [0, 1)$. And for this in turn it is sufficient to show, by the uniform convexity of Y_t , that

$$U^\pm(t+h, t)y \longrightarrow y \quad \text{weakly in } Y \quad \text{as } h \searrow 0 \quad (2.22)$$

and

$$\limsup_{h \searrow 0} \|U^\pm(t+h, t)y\|_t^\pm \leq \|y\|_t^\pm \quad (2.23)$$

Since this can be achieved in a way similar to the proof of (2.16) and (2.17), we may omit the details.

We can now show that $t \mapsto U^+(t, s)y$ is continuous in Y for every $y \in Y$ and then conclude the proof. Indeed, since $\tau \mapsto U^\mp(1 - s, 1 - \tau)z$ is differentiable for $z \in Y$ with derivative $\tau \mapsto U^\mp(1 - s, 1 - \tau)A^\mp(1 - \tau)z$ and $\tau \mapsto U^\pm(\tau, s)y$ is right differentiable for $y \in Y$ with right derivative $\tau \mapsto A^\pm(\tau)U^\pm(\tau, s)y$ by our second and third step, the map $[s, t] \ni \tau \mapsto U^\mp(1 - s, 1 - \tau)U^\pm(\tau, s)y$ is right differentiable for every $y \in Y$ with right derivative 0. So,

$$U^\mp(1 - s, 1 - t)U^\pm(t, s)y - y = U^\mp(1 - s, 1 - \tau)U^\pm(\tau, s)y \Big|_{\tau=s}^{\tau=t} = 0$$

for every $y \in Y$ by Corollary 2.1.2 of [25] and therefore

$$U^\mp(1 - s, 1 - t)U^\pm(t, s) = 1 = U^\mp(1 - \bar{t}, 1 - \bar{s})U^\pm(\bar{s}, \bar{t}) = U^\mp(t, s)U^\pm(1 - s, 1 - t)$$

for all $(s, t) \in \Delta$. It follows that

$$U^+(t - h, s)y = U^+(t, t - h)^{-1}U^+(t, s)y = U^-(1 - t + h, 1 - t)U^+(t, s)y \longrightarrow U^+(t, s)y$$

in Y as $h \searrow 0$ by our third step, whence $t \mapsto U^+(t, s)y$ right *and* left continuous and hence continuous in Y . Combining this with the previous steps, we see with the help of Corollary 2.1.2 of [25] that $t \mapsto U^+(t, s)y$ is continuously differentiable in X for every $y \in Y$ with derivative $t \mapsto A^+(t)U^+(t, s)y$ and therefore $U := U^+$ is an evolution system for $A = A^+$ on Y , as desired. \blacksquare

Incidentally, it is also possible to improve (a version of) the well-posedness theorem from [13] (Theorem 1) in the spirit of [16]: in this theorem strong continuity of $t \mapsto A(t)|_Y$ is sufficient as well, provided that A is (M, ω) -stable (instead of only quasistable) and that $t \mapsto \|B(t)\|$ is bounded (instead of only upper integrable). (We make this proviso in order to make sure that the boundedness condition (2.1) of [16] is still satisfied for arbitrary partitions π and that (2.2) of [16] is satisfied with the modified right hand side $C \|x\| \int_{t_i}^{t_k} \alpha(\tau) d\tau$, where α is a suitable integrable function. All other arguments from [16] carry over without formal change, a bit more care being necessary in the justification of assertion (c) of [16] because of the weaker regularity of $t \mapsto S(t)$ – see [6].)

2.4 Some remarks

We close this section about abstract well-posedness results with some remarks concerning, in particular, the relation of the results from Section 2.1 and 2.2 with the results from [12], [13], [16], [23] and the result from Section 2.3.

1. Clearly, the strong continuity conditions of the theorems from Section 2.1 and 2.2 are weaker than the regularity conditions of the general well-posedness results from [12] (Theorem 6.1), [16], and [13] (Theorem 1) for general semigroup generators $A(t)$ without commutator restrictions of the kind (1.2) or (1.3). Indeed, in these results a strong continuous differentiability condition or more generally a strong $W^{1,1}$ -condition is imposed on $t \mapsto A(t)$ in the case of time-independent domains $D(A(t)) = Y$ or on some

related operator function $t \mapsto S(t)$ in the case of time-dependent domains $D(A(t)) \supset Y$. In the special well-posedness result (Theorem 5.2 and Remark 5.3) from [12] for group generators $A(t)$, the imposed regularity condition is quite close to ours, namely a norm continuity condition on $t \mapsto A(t)|_Y$. As has been shown in Section 2.3, strong continuity of $t \mapsto A(t)|_Y$ is sufficient as well. And in fact, if in addition to the assumptions of Theorem 2.3 the following three conditions are satisfied, then the well-posedness assertion of this theorem (but no representation formula, of course) also follows from Theorem 2.5:

- $A(t)$ for every $t \in I$ is a quasicontraction group generator with time-independent domain $D(A(t)) = Y$ in the uniformly convex space X such that

$$\left\| e^{\pm A(t)\tau} \right\| \leq e^{\omega\tau} \quad (\tau \in [0, \infty)) \quad (2.24)$$

for some t -independent growth exponent $\omega \in \mathbb{R}$,

- $C^{(k)}(t_1, \dots, t_{k+1})$ is a bounded operator on X for every $(t_1, \dots, t_{k+1}) \in I^{k+1}$ and

$$\sup_{(t_1, \dots, t_{k+1}) \in I^{k+1}} \left\| C^{(k)}(t_1, \dots, t_{k+1}) \right\| < \infty \quad (2.25)$$

for every $k \in \{1, \dots, p-1\}$ (an empty condition for $p = 1!$),

- $t \mapsto A(t)y$ is continuous for every $y \in Y$.

Indeed, under these conditions the norms $\|\cdot\|_t^\pm$ appearing in Theorem 2.5 can be chosen to be $\|\cdot\|_* := \|(A(0) - \omega - 1) \cdot\|$ for every $t \in I$ (t -independent!): with this norm, Y becomes a uniformly convex subspace admissible for the group generators $\pm A(t)$ and

$$\left\| e^{\pm A(t)\tau} y \right\|_* \leq e^{\omega_0\tau} \|y\|_* \quad (y \in Y \text{ and } \tau \in [0, \infty)) \quad (2.26)$$

for a suitable $\omega_0 \in \mathbb{R}$, and finally $Y^\circ = Y$. (In order to see (2.26) and the $\pm A(t)$ -admissibility of Y one checks that (2.12) holds true for $\tau \in (-\infty, 0)$ as well, so that in particular

$$\begin{aligned} A(0)e^{\pm A(t)\tau} y &= e^{\pm A(t)\tau} (A(0) + C^{(1)}(0, t)(\pm\tau) + \dots + C^{(p-1)}(0, t, \dots, t)(\pm\tau)^{p-1}/(p-1)! \\ &\quad + \mu(0, t, \dots, t)(\pm\tau)^p/p!) y \end{aligned}$$

for all $y \in Y$ and $\tau \in [0, \infty)$. With the help of (2.24) and (2.25) the desired $\pm A(t)$ -admissibility and the quasicontraction group property (2.26) then readily follow.)

2. In the well-posedness theorems from [10] and [20] weaker notions of well-posedness are used than here [21], which in return allows for weaker regularity assumptions than those of [13] and [16]. In the second product representation theorem from [24] (Proposition 4.9) which also asserts well-posedness, there seems to be missing, in the hyperbolic case, an additional regularity assumption of the kind of condition (ii'') from [12]. At least, it is not clear [19] how the asserted well-posedness should be established and how the range condition from Chernoff's theorem (invoked in [24]) should be verified without

such an additional assumption. (In this respect, see in particular Theorem 4.19 of [22] and the remarks preceding it, which state that \mathcal{Y} is a core for \mathcal{G} only under the additional condition (ii') from [12].) As far as [5] is concerned, it should be remarked that the abstract well-posedness theorem of this paper is actually a corollary of the well-posedness theorem of [13]. (Indeed, if for every $y \in Y$ the map $t \mapsto S(t)y$ is differentiable at all except countably many points with an exceptional set N not depending on y and if $\sup_{t \in I \setminus N} \|S'(t)y\| < \infty$, then $t \mapsto S(t)y$ is already absolutely continuous (Theorem 6.3.11 of [4]) and

$$S(t)y = S(0)y + \int_0^t S'(\tau)y d\tau$$

(Proposition 1.2.3 of [1]) for every $y \in Y$, so that the strong $W^{1,1}$ -regularity condition for $t \mapsto S(t)$ from [13] is satisfied.)

3. It is clear from the proofs of Theorem 2.1 and Theorem 2.3 that the well-posedness statements remain valid if the (M, ω) -stability condition of these theorems is replaced by the condition from [23] that there exist a sequence (π_n) of partitions of I such that $\text{mesh}(\pi_n) \rightarrow 0$ and

$$\left\| e^{A(r_n(t))(t-r_n(t))} \dots e^{A(r_n(s))(r_n^+(s)-s)} \right\| \leq M e^{\omega(t-s)} \quad ((s, t) \in \Delta). \quad (2.27)$$

In [23] it is shown that this stability condition is strictly weaker than (M, ω) -stability. Also, it is clear from the proof of Theorem 2.1 that the representation formula for the evolution is still valid if (2.27) is sharpened to

$$\left\| e^{A(r_n(t))r(t-r_n(t))} \dots e^{A(r_n(s))r(r_n^+(s)-s)} \right\| \leq M e^{\omega r(t-s)} \quad ((s, t) \in \Delta, r \in [0, \infty)). \quad (2.28)$$

In particular, the method of proof of Theorem 2.1 yields an alternative and more elementary proof (without reference to the Trotter–Kato theorem) of Proposition 2.5 from [23] (or, rather, of a slightly corrected version of it: in order for the proof of [23] to work one has to choose as the domain of $\int_s^t A(\tau) d\tau$ the maximal continuity subspace Y° of A instead of the quite arbitrary subspace denoted by Y in [23] because such a subspace, in contrast to Y° , is not left invariant by $(\overline{B}_n - \lambda)^{-1}$ in general).

4. In the case (1.2), one might think that it should be possible to (more efficiently) obtain the well-posedness of the initial value problems (1.1) on Y° by first defining a candidate U for the sought evolution system through the representation formula

$$U(t, s) := \overline{e^{\int_s^t A(\tau) d\tau} }^\circ e^{1/2 \int_s^t \int_s^\tau \mu(\tau, \sigma) d\sigma d\tau},$$

and by then verifying that this candidate is indeed an evolution system for A on Y° . In order to prove that the closure of $(\int_s^t A(\tau) d\tau)^\circ$ exists and is a semigroup generator, one might want to employ the theorem of Trotter and Kato as in [23] – instead of exploiting the locally uniform convergence of the sequences $(U_n^r(t, s)x)$ as we did. And in order to verify the evolution system properties for U , one might want to make rigorous

the following formal differentiation rule for exponential operators (appearing in [31], for instance):

$$\begin{aligned} \frac{e^{B(t+h)} - e^{B(t)}}{h} &= \frac{e^{B(t+h)\tau} e^{B(t)(1-\tau)} \Big|_{\tau=0}^{\tau=1}}{h} = \int_0^1 e^{B(t+h)\tau} \frac{B(t+h) - B(t)}{h} e^{B(t)(1-\tau)} d\tau \\ &\longrightarrow \int_0^1 e^{B(t+h)\tau} B'(t) e^{B(t)(1-\tau)} d\tau \quad (h \rightarrow 0) \end{aligned} \quad (2.29)$$

with $B(t) := \overline{(\int_s^t A(\tau) d\tau)^\circ}$. Yet, this is possible only if $\operatorname{Re} \mu(\tau, \sigma) \geq 0$ for all $\sigma \leq \tau$ because only then can the right hand side of (2.6) be dominated by a bound $M' e^{\omega' r}$ for all $r \in [0, \infty)$ uniformly in $n \in \mathbb{N}$ (a first crucial assumption of the Trotter–Kato theorem). And moreover, the verification of the density of $\operatorname{ran}((\int_s^t A(\tau) d\tau)^\circ - \lambda)$ in X for $\lambda > \omega'$ (a second crucial assumption of the Trotter–Kato theorem) and the verifications of the evolution system properties for U with the help of (2.29) are more involved than the arguments in our approach.

3 Some applications of the abstract results

3.1 Segal field operators

In this subsection we apply the well-posedness result of Section 2.1 to Segal field operators $\Phi(f_t)$ in $\mathcal{F}_+(\mathfrak{h})$ (symmetric Fock space over a Hilbert space \mathfrak{h}) with $f_t \in \mathfrak{h}$. As for standard facts about such operators we refer to [2]. It is well-known by the Weyl form of the canonical commutation relations that

$$i\Phi(f_s) e^{i\Phi(f_t)\tau} = e^{i\Phi(f_t)\tau} (i\Phi(f_s) - i \operatorname{Im} \langle f_s, f_t \rangle \tau) \quad (3.1)$$

for all $s, t \in I$ and $\tau \in \mathbb{R}$, and therefore we obtain the following well-posedness result by means of Theorem 2.1.

Corollary 3.1. *Suppose $A(t) = i\Phi(f_t)$ in $X := \mathcal{F}_+(\mathfrak{h})$ and $t \mapsto f_t \in \mathfrak{h}$ is continuous. Then there exists a unique evolution system U for A on the maximal continuity subspace Y° for A and it is given by*

$$U(t, s) = e^{\overline{(\int_s^t i\Phi(f_\tau) d\tau)^\circ}} e^{-i/2 \int_s^t \int_s^\tau \operatorname{Im} \langle f_\tau, f_\sigma \rangle d\sigma d\tau} = W\left(\int_s^t f_\tau d\tau\right) e^{-i/2 \int_s^t \int_s^\tau \operatorname{Im} \langle f_\tau, f_\sigma \rangle d\sigma d\tau}$$

where $W(h) := e^{i\Phi(h)}$ denotes the Weyl operator for $h \in \mathfrak{h}$.

Proof. Since $t \mapsto f_t$ is continuous and since for all $f \in \mathfrak{h}$ and $\psi \in D(N^{1/2})$ (N the number operator in $\mathcal{F}_+(\mathfrak{h})$)

$$\|\Phi(f)\psi\| \leq 2^{1/2} \|f\| \|(N+1)^{1/2}\psi\|,$$

the maximal continuity subspace Y° for A contains the dense subspace $D(N^{1/2})$ of X . So, by (3.1) the desired well-posedness statement and the first of the asserted representation formulas for U follow from Theorem 2.1. (Alternatively, one could also

apply Corollary 2.2 with $Y := D(N)$ endowed with the graph norm of N because $A(t)D(N) \subset D(N^{1/2}) \subset \cap_{\tau \in I} D(A(\tau))$ and $[A(s), A(t)]|_{D(N)} \subset -i \operatorname{Im} \langle f_s, f_t \rangle$,

$$N e^{i\Phi(f)} = e^{i\Phi(f)} (N + \Phi(if) + \|f\|^2)$$

for all $f \in \mathfrak{h}$.) In order to see the second representation formula for U , repeatedly apply the identity $W(f)W(g) = W(f+g)e^{-i/2 \operatorname{Im} \langle f, g \rangle}$ to the approximants U_n for U from the proof of Theorem 2.1 and use the strong continuity of $\mathfrak{h} \ni h \mapsto W(h)$. \blacksquare

We point out that alternatively the above result could also be obtained by the strategy sketched in the fourth remark of Section 2.4 (because here $\operatorname{Re} \mu(\tau, \sigma) = 0$ for all $\sigma, \tau \in I$). Also, a slightly weaker result, namely the well-posedness in the wide sense on $Y = D(H_\omega^{1/2})$, could be obtained in yet another way under the slightly stronger condition that both $t \mapsto f_t$ and $t \mapsto f_t/\sqrt{\omega} \in \mathfrak{h} = L^2(\mathbb{R}^3)$ are continuous (where $\omega \in C(\mathbb{R}^3, [0, \infty))$ with $\omega(-k) = \omega(k)$ for $k \in \mathbb{R}^3$ and where H_ω is the second quantization of ω). Indeed, by exploiting the exponential series expansion for Weyl operators on vectors ψ in the finite particle subspace $\mathcal{F}_{0+}(\mathfrak{h})$, one can show that

$$t \mapsto U(t, s)\psi := W\left(\int_s^t f_\tau d\tau\right)\psi e^{-i/2 \int_s^t \int_s^\tau \operatorname{Im} \langle f_\tau, f_\sigma \rangle d\sigma d\tau}$$

is differentiable for $\psi \in \mathcal{F}_{0+}(\mathfrak{h})$ with the desired derivative. Using the fact that $\mathcal{F}_{0+}(\mathfrak{h})$ is a core for $\Phi(f_t)|_{D(H_\omega^{1/2})}$ uniformly in $t \in I$ (see (3.3) below) and the commutation relation (3.1), one then concludes that $t \mapsto U(t, s)\psi$ is continuously differentiable for all $\psi \in D(H_\omega^{1/2})$ with the desired derivative. (Uniqueness of the thus constructed evolution system U for A on Y in the wide sense follows from [11].)

With the help of the above well-posedness result for Segal field operators one can also show the well-posedness of the initial value problems for A on $D(H_\omega)$ with the operators $A(t) := -i(H_\omega + \Phi(f_t))$ which describe a classical particle coupled to a time-dependent quantized field of bosons.

Corollary 3.2. *Suppose $A(t) = -i(H_\omega + \Phi(f_t))$ in $X := \mathcal{F}_+(\mathfrak{h})$, where $\mathfrak{h} := L^2(\mathbb{R}^3)$ and $\omega \in C(\mathbb{R}^3, [0, \infty))$ with $\omega(-k) = \omega(k)$ for $k \in \mathbb{R}^3$, and suppose $t \mapsto f_t, f_t/\sqrt{\omega} \in \mathfrak{h}$ are continuous and $t \mapsto f_t$ is even Lipschitz. Then there exists a unique evolution system U for A on $D(H_\omega)$ and it is given by (3.2) and (3.4).*

Proof. It is well-known that $A(t)$ is skew self-adjoint on $D(H_\omega)$ because $f_t, f_t/\sqrt{\omega} \in \mathfrak{h}$. We define U as the interaction picture evolution,

$$U(t, s) := e^{-iH_\omega t} \tilde{U}(t, s) e^{iH_\omega s}, \tag{3.2}$$

where \tilde{U} denotes the evolution system for \tilde{A} with $\tilde{A}(t) = -ie^{iH_\omega t} \Phi(f_t) e^{-iH_\omega t}$. Since

$$\tilde{A}(t) = -ie^{iH_\omega t} \Phi(f_t) e^{-iH_\omega t} = i\Phi(\tilde{f}_t) \quad \text{with} \quad \tilde{f}_t := -e^{i\omega t} f_t,$$

the maximal continuity subspace \tilde{Y}° for \tilde{A} contains the dense subspace $D(H_\omega^{1/2})$ by the standard estimate

$$\|\Phi(f)\psi\| \leq (\|f\|^2 + \|f/\sqrt{\omega}\|^2)^{1/2} \|(H_\omega + 1)^{1/2}\psi\| \quad (3.3)$$

for $\psi \in D(H_\omega^{1/2})$ and $f, f/\sqrt{\omega} \in \mathfrak{h}$. And therefore, by Corollary 3.1, the evolution system \tilde{U} really exists on \tilde{Y}° and is given by

$$\tilde{U}(t, s) = W\left(\int_s^t \tilde{f}_\tau d\tau\right) e^{-i/2 \int_s^t \int_s^\tau \text{Im}\langle \tilde{f}_\tau, \tilde{f}_\sigma \rangle d\sigma d\tau}. \quad (3.4)$$

We have to show (i) that $\tilde{U}(t, s)D(H_\omega) \subset D(H_\omega)$ for all $(s, t) \in \Delta$ because only then is $t \mapsto U(t, s)\psi$ differentiable for all $\psi \in D(H_\omega)$ with the desired derivative $t \mapsto -i(H_\omega + \Phi(f_t))U(t, s)\psi$ and (ii) that this derivative is continuous. Since, as is well-known,

$$H_\omega W(g) = W(g)(H_\omega + \Phi(i\omega g) + \langle g, \omega g \rangle / 2) \quad (3.5)$$

for every $g \in D(\omega)$, we are led to showing that

$$g_t := \int_s^t \tilde{f}_\tau d\tau \in D(\omega) \quad \text{and} \quad t \mapsto \omega g_t \in \mathfrak{h} \text{ is continuous.} \quad (3.6)$$

In order to do so notice that $t \mapsto f_t$, being a Lipschitz continuous function with values in the reflexive space \mathfrak{h} , belongs to the Sobolev space $W^{1,\infty}(I, \mathfrak{h})$ and therefore we can perform the following integration by parts:

$$\begin{aligned} g_t &= \int_s^t e^{i\omega\tau} (i\omega + 1)^{-1} f_\tau d\tau + \int_s^t e^{i\omega\tau} i\omega (i\omega + 1)^{-1} f_\tau d\tau \\ &= \int_s^t e^{i\omega\tau} (i\omega + 1)^{-1} f_\tau d\tau + e^{i\omega\tau} (i\omega + 1)^{-1} f_\tau \Big|_{\tau=s}^{\tau=t} - \int_s^t e^{i\omega\tau} (i\omega + 1)^{-1} f'_\tau d\tau. \end{aligned}$$

As a consequence, (3.6) ensues and by (3.5), (3.3), (3.1) the desired assertions (i) and (ii) readily follow. \blacksquare

We point out that in order to obtain the above conclusion by the well-posedness theorem from [13], [14] (Section 1), one needs the additional assumption that $t \mapsto f_t/\sqrt{\omega} \in \mathfrak{h}$ is Lipschitz as well.

3.2 Schrödinger operators for external electric fields

In this subsection we apply the well-posedness result of Section 2.2 to Schrödinger operators $-\Delta + b(t) \cdot x$ in $X := L^2(\mathbb{R}^d)$ describing a quantum particle in a time-dependent spatially constant electric field $b(t) \in \mathbb{R}^d$. Setting $A(t) = i\Delta - ib(t) \cdot x$, we obtain by formal computation

$$[A(t_1), A(t_2)] = 2 \sum_{\kappa=1}^d (b_\kappa(t_2) - b_\kappa(t_1)) \partial_\kappa, \quad [[A(t_1), A(t_2)], A(t_3)] = \mu(t_1, t_2, t_3) \quad (3.7)$$

with $\mu(t_1, t_2, t_3) := -2i \sum_{\kappa=1}^d (b_\kappa(t_2) - b_\kappa(t_1)) b_\kappa(t_3) \in \mathbb{C}$ and we therefore expect to be able to apply Theorem 2.3 with $p = 2$. Indeed, we have (see also the remarks below):

Corollary 3.3. *Suppose $A(t) = \overline{A_0 + B(t)}$ in $X := L^2(\mathbb{R}^d)$, where $A_0 := i\Delta$ with $D(A_0) = W^{2,2}(\mathbb{R}^d)$ and where $B(t)$ is multiplication by $-ib(t) \cdot x$, and suppose $t \mapsto b(t) \in \mathbb{R}^d$ is continuous. Then there exists a unique evolution system U for A on the maximal continuity subspace Y° for $A = C^{(0)}$ and $C^{(1)}$ defined after (3.10). Additionally, U is given by (3.11) and (3.12).*

Proof. (i) We first show that $A_0 + B(t_0)$ for every $t_0 \in I$ is essentially skew self-adjoint and that the unitary group generated by $A := \overline{A_0 + B(t_0)}$ is given by

$$e^{At} = e^{A_0 t} e^{Bt} e^{-\partial_1 b_1 t^2} \dots e^{-\partial_d b_d t^2} e^{2ib^2 t^3/3} \quad (t \in \mathbb{R}), \quad (3.8)$$

where $B := B(t_0)$ and $b = (b_1, \dots, b_d) := b(t_0) \in \mathbb{R}^d$. We do so by showing that the right hand side of (3.8), which we abbreviate as $T(t)$, defines a strongly continuous unitary group in X with

$$A_0 + B \subset A_T \quad \text{and} \quad T(t)D(A_0 + B) \subset D(A_0 + B) \quad (t \in \mathbb{R}),$$

where A_T stands for the generator of T . (In order to understand why $e^{A \cdot}$ should decompose as in (3.8), plug the following formal commutators

$$[B, A_0] = -2 \sum_{\kappa=1}^d b_\kappa \partial_\kappa, \quad [[B, A_0], B] = 2ib^2, \quad [[B, A_0], A_0] = 0$$

into the Zassenhaus formula [17], [28], [3] for bounded operators.) With the help of the explicit formulas for the groups $e^{A_0 \cdot}$ (free Schrödinger group), $e^{B \cdot}$ (multiplication group), $e^{\partial_\kappa \cdot}$ (translation group) we find the following commutation relations,

$$\begin{aligned} e^{A_0 t} e^{\partial_\kappa s} &= e^{\partial_\kappa s} e^{A_0 t}, & e^{Bt} e^{\partial_\kappa s} &= e^{\partial_\kappa s} e^{Bt} e^{ib_\kappa t s}, \\ e^{A_0 t} e^{Bs} &= e^{Bs} e^{A_0 t} e^{2\partial_1 b_1 t s} \dots e^{2\partial_d b_d t s} e^{-ib^2 t^2 s} \quad (s, t \in \mathbb{R}). \end{aligned} \quad (3.9)$$

It follows from (3.9) that T is indeed a strongly continuous unitary group and that

$$\begin{aligned} e^{\partial_\kappa s} D(A_0) &\subset D(A_0), & e^{\partial_\kappa s} D(B) &\subset D(B), & e^{Bs} D(A_0) &\subset D(A_0), \\ e^{A_0 t} D(A_0 + B) &\subset D(B) \quad (s, t \in \mathbb{R}), \end{aligned}$$

so that $T(t)D(A_0 + B) \subset D(A_0 + B)$ for all $t \in \mathbb{R}$ and $A_0 + B \subset A_T$. Consequently, $A_0 + B$ is essentially skew self-adjoint and $A = \overline{A_0 + B}$ is equal to A_T . After these preparations we can now verify the assumptions of Theorem 2.3 for $p = 2$. Indeed, using the commutation relations (3.9) we find that

$$e^{C_{12}\sigma} e^{A_3\tau} = e^{A_3\tau} e^{C_{12}\sigma} e^{\mu_{123}\tau\sigma}, \quad e^{A_1\sigma} e^{A_2\tau} = e^{A_2\tau} e^{A_1\sigma} e^{C_{12}\tau\sigma} e^{\mu_{122}\tau^2\sigma/2} e^{\mu_{121}\tau\sigma^2/2} \quad (3.10)$$

for all $\sigma, \tau \in \mathbb{R}$, where $A_j := A(t_j) = C^{(0)}(t_j)$, $b_j := b(t_j)$, $C_{jk} = C^{(1)}(t_j, t_k)$ is the closure of $2 \sum_{\kappa=1}^d (b_{k\kappa} - b_{j\kappa}) \partial_\kappa$ (that is, C_{jk} generates the translation group $t \mapsto e^{2(b_{k1} - b_{j1})\partial_1 t} \dots e^{2(b_{kd} - b_{jd})\partial_d t}$), and $\mu_{jkl} := -2i \sum_{\kappa=1}^d (b_{k\kappa} - b_{j\kappa}) b_{l\kappa}$. And from (3.10),

in turn, the commutation relations imposed in Theorem 2.3 follow by differentiation at $\sigma = 0$. Since, moreover, the maximal continuity subspace for $A = C^{(0)}$ and $C^{(1)}$ contains the dense subspace of Schwartz functions on \mathbb{R}^d , the existence of a unique evolution system U for A on Y° follows by Theorem 2.3.

(ii) We now show the following representation formula for U :

$$U(t, s) = W(t)\tilde{U}(t, s)W(s)^{-1} = e^{\overline{\int_0^t B(\tau) d\tau}^\circ} e^{\overline{\int_s^t \tilde{A}(\tau) d\tau}} e^{-\overline{\int_0^s B(\tau) d\tau}^\circ}, \quad (3.11)$$

where \tilde{U} is the evolution system for \tilde{A} on $D := W^{2,2}(\mathbb{R}^d)$ with $\tilde{A}(t) := -i(-i\nabla - c(t))^2$ and $c(t) := \int_0^t b(\tau) d\tau$ and where the gauge transformation W is the evolution system for B on Z° , the maximal continuity subspace for B . Clearly, since $B(\tau) = -ib(\tau) \cdot x$ and $\tilde{A}(\tau) = -i\mathcal{F}^{-1}(\xi - c(\tau))^2\mathcal{F}$,

$$e^{\overline{\int_0^t B(\tau) d\tau}^\circ} = e^{-i\int_0^t b(\tau) \cdot x d\tau} \quad \text{and} \quad e^{\overline{\int_s^t \tilde{A}(\tau) d\tau}} = \mathcal{F}^{-1} e^{-i\int_s^t (\xi - c(\tau))^2 d\tau} \mathcal{F} \quad (3.12)$$

(which last expression could be cast into a more explicit integral form similar to the explicit integral representation of the free Schrödinger group). It should be noticed that, due to the pairwise commutativity of the operators $\tilde{A}(t)$ and of the operators $B(t)$, the existence of the evolution systems \tilde{U} and W , and the second equality in (3.11) already follow by [9] and [23]. In order to see the first equality in (3.11), one shows by similar arguments as those of part (i) above that the subspace $Y_0^\circ := D \cap Z^\circ$ of Y° is invariant under $W(s)^{-1}$, $\tilde{U}(t, s)$, $W(t)$ and that

$$\begin{aligned} A_0 W(t)f &= W(t)\tilde{A}(t)f \\ B(r)\tilde{U}(t, s)f &= \tilde{U}(t, s) \left(B(r)f - 2 \sum_{\kappa=1}^d b_\kappa(r) (t-s) \partial_\kappa f + 2i \sum_{\kappa=1}^d b_\kappa(r) \int_s^t c_\kappa(\tau) d\tau f \right) \end{aligned}$$

for $f \in Y_0^\circ$. (Show commutation relations for $e^{\tilde{A}(r_1)\sigma}$ and $e^{B(r_2)\tau}$ analogous to (3.9) to obtain commutation relations for $B(r_2)$ with $e^{\tilde{A}(r_1)\sigma}$ and then use the standard product approximants for the evolution systems W and \tilde{U} .) It then follows that U_0 defined by $U_0(t, s) := W(t)\tilde{U}(t, s)W(s)^{-1}$ is an evolution system for A on Y_0° , which by the standard uniqueness argument for evolution systems must coincide with U . \blacksquare

We see from part (ii) of the above proof that the existence of an evolution system U_0 for A on the subspace Y_0° , after a suitable gauge transformation, already follows by [9], [23] – but in order to obtain well-posedness on Y° , the results from [9], [23] do not suffice, because the subspace Y_0° is strictly contained in Y° in general. (Indeed, if for instance $b(t) \equiv 1 \in \mathbb{R}^d$ with $d = 1$, then the function ψ with $\psi(\xi) := e^{i\xi^3/3}/\xi$ for $\xi \in [1, \infty)$ and $\psi(\xi) := 0$ for $\xi \in (-\infty, 1)$ does not belong to the range of $C - i := i\partial_\xi + \xi^2 - i$. Consequently, $-\partial_x^2 + x - i = \mathcal{F}^{-1}(C - i)\mathcal{F}$ is not surjective so that $Y_0^\circ = D(A_0 + B) = D(-\partial_x^2 + x) \subsetneq D(-\partial_x^2 + x) = D(A) = Y^\circ$ by the standard criterion for self-adjointness.) We finally remark that the results of [34] do not apply to the situation of this section.

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