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Buğra Kabil, Christian Rohde

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Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik.uni-stuttgart.de/preprints

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PERSISTENCE OF UNDERCOMPRESSIVE PHASE BOUNDARIES FOR ISOTHERMAL EULER EQUATIONS INCLUDING CONFIGURATIONAL FORCES AND SURFACE TENSION

B. KABIL[*] & C. ROHDE[*]

Abstract. The persistence of subsonic phase boundaries in a multidimensional Van der Waals fluid is analyzed. The phase boundary is considered as a sharp free boundary which connects liquid and vapor bulk phase dynamics given by the isothermal Euler equations. The evolution of the boundary is driven by effects of configurational forces as well as surface tension.

To analyze this problem the equations and trace conditions are linearized such that one obtains a general hyperbolic initial boundary value problem with higher-order boundary conditions. A global existence theorem for the linearized system with constant coefficients is shown. The proof relies on the normal mode analysis in [13] and a linear form in suitable spaces which is defined using an associated adjoint problem. Especially, the associated adjoint problem satisfies the uniform backward in time Kreiss-Lopatinskiĭ condition. A new energy-like estimate which also includes surface energy terms leads finally to the uniqueness and regularity for the found solutions of the problem in weighted spaces.

Keywords: Liquid-vapor interface, uniform Kreiss-Lopatinskiĭ condition, uniform stability, Kreiss symmetrizer, energy estimate, linearized well-posedness, adjoint problem.

Mathematics Subject Classification: 35A01, 35D30, 35E15, 35F05, 35L02, 35M12.

1. Introduction

We consider the multidimensional dynamics of an ideal compressible fluid with Van der Waals pressure relation. The nonmonotone shape of the pressure function allows to define a liquid and a vapor phase. It is expected that phase transitions are realized as moving sharp interfaces separating the two phases. Our main interest is in understanding which conditions at an interface lead to an overall well-posed evolution. Figure 1 gives a sketch of the situation.

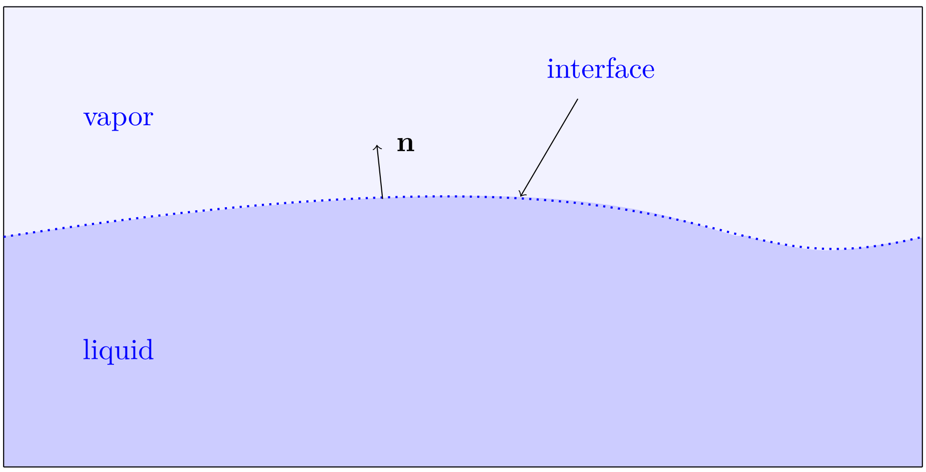

Figure 1. Sketch of a liquid-vapor interface

[*]Institute of Applied Analysis and Numerical Simulation, University of Stuttgart, Germany

Let us assume that the fluid in the bulk phase is governed by the isothermal Euler equations (see (2.1), (2.2) in section 2 with the complete model). At the interface the mass conservation law must hold, and we suppose that a dynamical version of the Young-Laplace law for the momentum balance is valid (see (2.6), (2.7)). Since we consider a first-order system for the bulk evolution we can classify the interface in terms of characteristic analysis. In this treatise we restrict ourselves to interfaces which are subsonic phase transitions. Then it is well-known that the mentioned kinematic conditions do not determine the phase dynamics completely, see [1, 22]. Together with curvature and surface tension, configurational forces drive the dynamics of the interface. This results in an additional Gibbs-Thomson-like condition (2.8) at the interface which is often called kinetic relation in the mathematical framework.

The planar case where surface tension can be neglected is by now quite well understood. The stability of single phase transition fronts is a consequence of the more general work in [10]. There are many results on the existence and stability of weak solutions, we refer to e.g. [7, 19, 9] for Riemann problems, and [16] for a general initial value problem.
For the multidimensional case there are much less results. The first results on the persistence of shock waves in the multidimensional case have been established in [17] for classical (Laxian) shocks where the interface conditions are only given by the Rankine-Hugoniot conditions, see also [5] for an overview. Afterwards undercompressive shock waves were studied in a general framework in [20, 8]. There are results on the existence of multidimensional subsonic phase transition including entropy dissipation, see [24]. For the Euler equations with Van der Waals pressure it was shown that there is a weak solution which contains one shock front and one subsonic phase boundary, see [23, 25]. However surface tension has been neglected. The problem including entropy dissipation and surface tension has been studied for the first time in [13] in the sense of energy estimates for the linearized system. Note that the case of zero entropy dissipation has been the subject of [4].
Up to our knowledge the multidimensional well-posedness for the evolution that takes into account surface tension and kinetic relations which allow nonzero entropy dissipation has not been analyzed. To tackle this issue in this paper we linearize the bulk equations and the trace conditions around a reference solution which is simply a planar phase boundary. For the resulting linearized problem we show the unique existence of weak solutions. Such type of results are classical for interfaces which correspond to compressive shock waves in one-phase flows, see [5, 8, 14, 17, 20]. We will follow the analysis in these works but the different type of the interface and the more complex trace conditions does not allow a simple transfer of the used techniques. Note in particular that the curvature in the Young-Laplace law leads to second order derivatives for the front solution. The starting point is a normal mode analysis for the linear constant coefficient system which has been derived in [13]. The proof of the existence is based on the construction of an associated adjoint problem which satisfies backward in time the uniform Kreiss-Lopatinskiĭ condition. This enables us to construct as for the original problem a symmetrizer (see also [13]) to show an energy estimate for the associated problem. This inequality will be used for a suitable defined linear form on a subspace to the solution space. Duality arguments gurantee then the existence of weak solutions. Finally we will use the energy estimates to improve the regularity of the weak solution and to ensure uniqueness. The local well-posedness of the fully nonlinear problem will be done as a future continuation of this work.

The paper is organized as follows. First we introduce in the second section the system of equations for the model of the motion of a fluid in $\mathbb{R}^d$, for $d > 1$. The equations will be formulated as a general hyperbolic initial boundary value problem which results from linearizing the original equations about a constant reference state. The main result on the existence of weak solutions is given by theorem 1 in section 2. In the third section an associated adjoint problem is constructed which is needed for proving the existence. Section 4 completes the proof of theorem 1. Smoothing this weak solution by using mollifiers improves the regularity such that the energy estimate is fullfilled which ensures uniqueness. The paper is closed with section 5 which contains the final existence result for the linearized initial boundary value problem in corollary 7.

Notation. In this work, we use the Sobolev spaces $H^s(\partial\mathcal{U})$ for $s \geq 0$ defined by

$$H^s(\partial\mathcal{U}) := \left\{ u \in L^2(\mathbb{R}^d) \,\middle|\, p \mapsto |p|^s \widehat{u}(p) \text{ is in } L^2(\mathbb{R}^d) \right\},$$

where $\widehat{\cdot}$ denotes the Fourier transformation and $\mathcal{U} := \left\{(t,y,z) \in \mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}\,\middle|\, z>0\right\}$. The scalar product for $H^s(\partial\mathcal{U})$ is given by

$$\langle u,\, v\rangle_{H^s} := \int\limits_{\mathbb{R}^d} \overline{\widehat{u}(p)}\,\widehat{v}(p)(1+p^2)^s \mathrm{d}p.$$

The dual space is denoted by $H^{-s}(\partial\mathcal{U})$. The dual space can be identified with $H^s(\partial\mathcal{U})$ by an isomorphism. We note that $\partial\mathcal{U} = \mathbb{R}\times\mathbb{R}^{d-1}$, where we also used the notation $(t,y) \in \mathbb{R}_t \times \mathbb{R}^{d-1}$ to indicate the time and the space variable.
We also use $L^2\left(\mathbb{R}, H^1(\mathbb{R}^{d-1})\right)$ as the Sobolev space $\left\{u \in L^2(\mathbb{R}^d)\,\middle|\, \check{\nabla} u \in L^2(\mathbb{R}^d)\right\}$ with $\check{\nabla} := (\partial_2, ..., \partial_d)$. In our context the function has to be understood in its space variables $y \in \mathbb{R}^{d-1}$ as H^1-function, where it is in its time variable a L^2-function. The dual space of $L^2\left(\mathbb{R}, H^1(\mathbb{R}^{d-1})\right)$ is denoted by $L^2\left(\mathbb{R}, H^{-1}(\mathbb{R}^{d-1})\right)$.

2. System of Equations

In this section we formulate the mathematical model and specify the conditions for the motion of a phase boundary. The starting point are the bulk equations for the dynamics of an ideal compressible fluid. Smooth solutions in the bulk will be considered such that we will study the propagating phase boundary as a shock wave which has to be constrained by certain trace conditions. We present a linearization of the problem and an equivalent technically more appropiate reformulation.

2.1. **Bulk Equations.** Let us consider the motion of an ideal fluid in $\mathbb{R}^d$ with constant temperature, for $d>1$. The system of equations is given for space variable $x=(x_1,...,x_d)\in\mathbb{R}^d$, time variable $t>0$, unknown density $\rho=\rho(x,t)>0$ and velocity $\mathbf{u}(x,t)=(u_1(x,t),...,u_d(x,t))\in\mathbb{R}^d$ by the Euler equations

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2.1}$$

$$(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = 0. \tag{2.2}$$

In this system p denotes the given pressure of the fluid. We choose a nonmonotone –Van der Waals– pressure law $p=p(\rho)$ such that there are constants $l^* > v^* > 0$ with (see figure 2)

$$\begin{cases} p'(\rho)>0, & \text{if } \quad 0<\rho<v^* \qquad \text{(vapor states)},\\ p'(\rho)<0, & \text{if } \quad v^*<\rho<l^* \qquad \text{(spinodal states)},\\ p'(\rho)>0, & \text{if } \quad l^*<\rho \qquad \text{(liquid states)}.\end{cases} \tag{2.3}$$

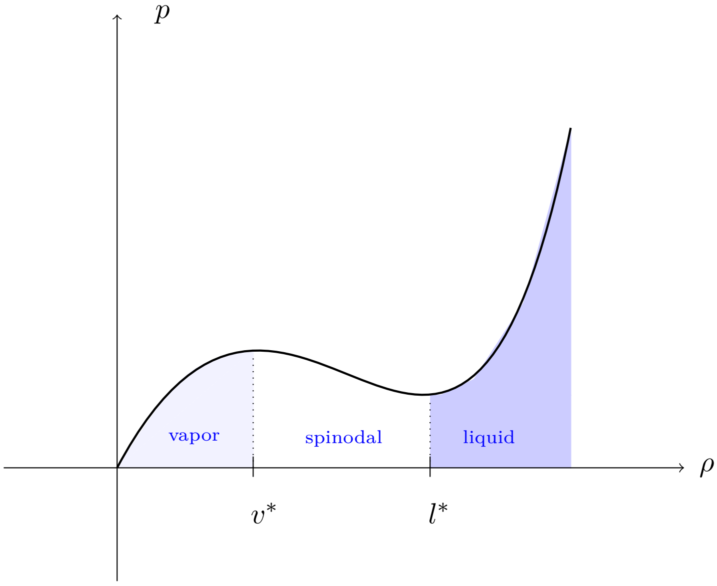

Figure 2. Nonmonotone –Van der Waals– pressure

We consider the system (2.1)-(2.2) outside the spinodal region such that the system is hyperbolic there, see for more details [13]. Such an assumption allows the coexistence of liquid and vapor phases where one observes phase boundaries which are discontinuous solutions to the system (2.1)-(2.2), see also [2, 3]. Precisely we seek for a smooth hypersurface $\Sigma(t)$ and two smooth functions $(\rho^+, \mathbf{u}^+)$ and $(\rho^-, \mathbf{u}^-)$ with either ρ^+ (ρ^-) in the liquid (vapor) region or ρ^- (ρ^+) in the vapor (liquid) region defined on respective domains $V_+(t)$ and $V_-(t)$ on either side of the hypersurface $\Sigma(t)$ such that

$$\rho_t^\pm + \nabla \cdot (\rho^\pm \mathbf{u}^\pm) = 0, \quad \text{in} \quad V_\pm(t), \tag{2.4}$$

$$(\rho^\pm \mathbf{u}^\pm)_t + \nabla \cdot (\rho^\pm \mathbf{u}^\pm \otimes \mathbf{u}^\pm) + \nabla p^\pm = 0, \quad \text{in} \quad V_\pm(t), \tag{2.5}$$

see figure 3.

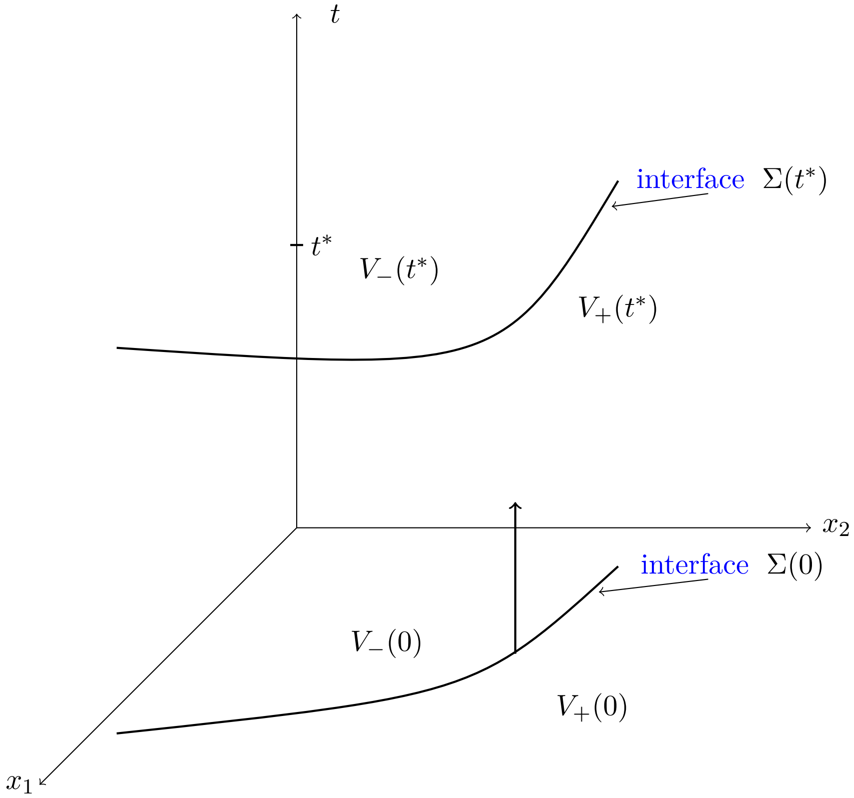

FIGURE 3. Moving interface for $d = 2$ at time $t = 0$ and at time $t = t^* > 0$

We consider with respect to (2.4)-(2.5) the following conditions at the interface

$$[\rho(\mathbf{u} \cdot \mathbf{n} - \sigma)] = 0, \tag{2.6}$$

$$[\rho(\mathbf{u} \cdot \mathbf{n} - \sigma)\mathbf{u} + p\mathbf{n}] = (d-1)\kappa s \mathbf{n}, \tag{2.7}$$

$$\left[g + \frac{j^2}{2\rho^2}\right] = -\mathrm{B} j, \tag{2.8}$$

where the brackets denote the jump of some quantity f across the interface, i.e.

$$[f] = \lim_{\varepsilon \searrow 0} f(x + \varepsilon \mathbf{n}) - f(x - \varepsilon \mathbf{n})$$

for any $x \in \Sigma(t)$, $s > 0$ the surface tension, $\mathrm{B} > 0$ the interfacial mobility, κ the mean curvature, $\mathbf{n} \in \mathbb{R}^d$ the unit normal vector to the moving interface in x and $\sigma \in \mathbb{R}$ the normal speed of propagation of the interface in x. The normal vector $\mathbf{n}$ is oriented such that it points into the vapor bulk domain. In this context we have introduced the mass transfer flux as

$$\begin{aligned} j &:= \lim_{\varepsilon \searrow 0} (\rho(x - \varepsilon \mathbf{n})(\mathbf{u}(x - \varepsilon \mathbf{n}) \cdot \mathbf{n} - \sigma)) \\ &= \lim_{\varepsilon \searrow 0} (\rho(x + \varepsilon \mathbf{n})(\mathbf{u}(x + \varepsilon \mathbf{n}) \cdot \mathbf{n} - \sigma)) \end{aligned}$$

and the chemical potential g which can be determined by $g'(\rho) = \rho^{-1}p'(\rho)$. For more details on the model we refer to [4, 2, 13, 3].

Now we reformulate as in [4, 13] the jump conditions (2.6)-(2.8) and assume that the interface $\Sigma(t)$ can be represented by $X \in C^2\left(\mathbb{R}^{d-1} \times [0,\infty)\right)$ through

$$\Sigma(t) = \left\{\mathbf{x} = (x_1,\dots,x_d) \,\middle|\, x_d = X(x_1,\dots,x_{d-1},t)\right\}.$$

Then we write the geometrical quantities in (2.6)-(2.8) in terms of X as

$$\mathbf{n} = \frac{1}{\sqrt{1+\|\check{\nabla}X\|^2}}\left(-\check{\nabla}X, 1\right)^{\mathsf{T}}, \quad \sigma = \frac{\partial_t X}{\sqrt{1+\|\check{\nabla}X\|^2}}, \quad \kappa = \frac{1}{d-1}\,\check{\nabla}\cdot\left(\frac{\check{\nabla}X}{\sqrt{1+\|\check{\nabla}X\|^2}}\right), \tag{2.9}$$

where we used

$$\check{\nabla} = (\partial_{x_1},\dots,\partial_{x_{d-1}})^{\mathsf{T}}.$$

We collect the first components in $y = (x_1,...,x_{d-1})^{\mathsf{T}}$.
Let us consider now the hyperplane $\left\{\mathbf{x} \in \mathbb{R}^d \,\middle|\, x_d = 0\right\}$ and decompose the velocity $\mathbf{u}$ as $\mathbf{u} = (\mathbf{v}, u)$, where $\mathbf{v}$ denotes the tangential part and u its normal part with respect to the hyperplane $\Sigma(t)$.
With these notations all jump conditions (2.6)-(2.8) can be expressed in terms of X as

$$\left[\rho\,(u - \partial_t X - \mathbf{v}\cdot\check{\nabla}X)\right] = 0, \tag{2.10}$$

$$\left[\rho\,(u - \partial_t X - \mathbf{v}\cdot\check{\nabla}X)\mathbf{v} - p\check{\nabla}X\right] = 0, \tag{2.11}$$

$$\left[\rho\,(u - \partial_t X - \mathbf{v}\cdot\check{\nabla}X)u + p\right] = s\check{\nabla}\cdot\check{\nabla}X, \tag{2.12}$$

$$\left[\left(1+\|\check{\nabla}X\|^2\right)g + \tfrac{1}{2}(u - \partial_t X - \mathbf{v}\cdot\check{\nabla}X)^2\right] = -\mathrm{B}j\left(1+\|\check{\nabla}X\|^2\right). \tag{2.13}$$

We choose now a planar dynamical interface as a reference interface for the linearized stability analysis and for obtaining the linearized system. That means, we consider a planar shock wave for (2.4)-(2.5) and (2.6)-(2.8) of the form

$$(\rho^{\pm}, \mathbf{u}^{\pm}) = \begin{pmatrix} \rho_{r,l} \\ \mathbf{v}_{r,l} \\ u_{r,l} \end{pmatrix}, \tag{2.14}$$

where $(\rho_{r,l}, \mathbf{v}_{r,l}, u_{r,l})$ is a constant vector such that (2.6)-(2.8) are satisfied and ρ_r, ρ_l are in different phases. The last condition (2.8) (respectively (2.13)) seems to be superfluous at first glance. In fact we restrict our study to the undercompressive case where only $d+1$ characteristics impinge into the shock (instead of $d+2$ for a Laxian shock) such that it is justified to put one more jump condition. We also refer to [10] for a general setting.
Physically speaking undercompressivity occuse if the shock wave is subsonic. Subsonic phase transitions are those which satisfy

$$0 < M_{r,l} < 1, \quad \text{for} \quad M_{r,l} := \frac{|\mathbf{u}_{r,l}\cdot\mathbf{n} - \sigma|}{c_{r,l}} \quad \text{and} \quad c_{r,l}^2 = p'(\rho_{r,l}). \tag{2.15}$$

Further, we assume

$$(u_r - u_l)(c_r - c_l) < 0 \tag{2.16}$$

which ensures the entropy admissibility of the wave. Note that (2.15) implies that the mass transfer flux $j = \rho_l(\mathbf{u}_l\cdot\mathbf{n} - \sigma) = \rho_r(\mathbf{u}_r\cdot\mathbf{n} - \sigma)$ is non-zero. By Galilean invariance we can assume that $\mathbf{v}_r = \mathbf{v}_l = 0$ and $\sigma = 0$, w.l.o.g..

For technical reasons we transform the system (2.4)-(2.5) in the last variable to get a boundary value problem in one domain. Therefore we define

$$(\tilde{\rho}^{\pm}, \tilde{\mathbf{u}}^{\pm})(t,y,z) := (\rho^{\pm}(t,y,\pm z + X(t,y)), \mathbf{u}^{\pm}(t,y,\pm z + X(t,y))) \tag{2.17}$$

where the transformed coordinate is $z = \pm(x_d - X(y,t))$. Plugging this function (2.17) into the system (2.4)-(2.5), we obtain

$$\tilde{\rho}^{\pm}_t \mp \partial_z \tilde{\rho}^{\pm} \cdot X_t + \check{\nabla} \cdot (\tilde{\rho}^{\pm} \tilde{\mathbf{v}}^{\pm}) \mp \partial_z (\tilde{\rho}^{\pm} \tilde{\mathbf{v}}^{\pm}) \cdot \check{\nabla} X \pm \partial_z (\tilde{\rho}^{\pm} \tilde{u}^{\pm}) = 0, \tag{2.18}$$

$$\begin{aligned}(\tilde{\rho}^{\pm} \tilde{\mathbf{v}}^{\pm})_t \mp \partial_z (\tilde{\rho}^{\pm} \tilde{\mathbf{v}}^{\pm}) \cdot X_t + \check{\nabla} \cdot (\tilde{\rho}^{\pm} \tilde{\mathbf{v}}^{\pm} \otimes \tilde{\mathbf{v}}^{\pm}) \mp \partial_z (\tilde{\rho}^{\pm} \tilde{\mathbf{v}}^{\pm} \otimes \tilde{\mathbf{v}}^{\pm}) \cdot \check{\nabla} X & \\ \pm \partial_z (\tilde{\rho}^{\pm} \tilde{u}^{\pm} \tilde{\mathbf{v}}^{\pm}) + \check{\nabla} p(\tilde{\rho}^{\pm}) \mp \partial_z (\tilde{\rho}^{\pm}) \cdot \check{\nabla} X &= 0,\end{aligned} \tag{2.19}$$

$$\begin{aligned}(\tilde{\rho}^{\pm} \tilde{u}^{\pm})_t \mp \partial_z (\tilde{\rho}^{\pm} \tilde{u}^{\pm}) \cdot X_t + \check{\nabla} \cdot (\tilde{\rho}^{\pm} \tilde{u}^{\pm} \tilde{\mathbf{v}}^{\pm}) \mp \partial_z (\tilde{\rho}^{\pm} \tilde{u}^{\pm} \tilde{\mathbf{v}}^{\pm}) \cdot \check{\nabla} X & \\ \pm \partial_z (\tilde{\rho}^{\pm} (\tilde{u}^{\pm})^2) + \partial_z p(\tilde{\rho}^{\pm}) &= 0\end{aligned} \tag{2.20}$$

for $(t, y, z) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$. In what follows we will drop the tildas in (2.18)-(2.20) and consider this boundary value problem in $z > 0$.

2.2. **Linearized Equations.** We linearize the system (2.18)-(2.20) about the reference solution (2.14), i.e. we plug

$$(\rho^{\pm}(x,t), \mathbf{v}^{\pm}(x,t), u^{\pm}(x,t)) = (\rho_{r,l}\,,\, \mathbf{v}_{r,l},\, u_{r,l}\,) + \delta \cdot (\rho_{\pm}(x,t), \mathbf{v}_{\pm}(x,t), u_{\pm}(x,t))$$

and

$$X(t,y) = \varphi + \delta\, \mathrm{X}(t,y)$$

for $\varphi = 0$, $\delta > 0$ and some perturbation functions $(\rho_{\pm}(x,t), \mathbf{v}_{\pm}(x,t), u_{\pm}(x,t))$ into the equations (2.18)-(2.20), differentiate with respect to δ and evaluate in $\delta = 0$. That means we obtain a system for the perturbation functions $(\rho_{\pm}(x,t), \mathbf{v}_{\pm}(x,t), u_{\pm}(x,t))$, see figure 4.

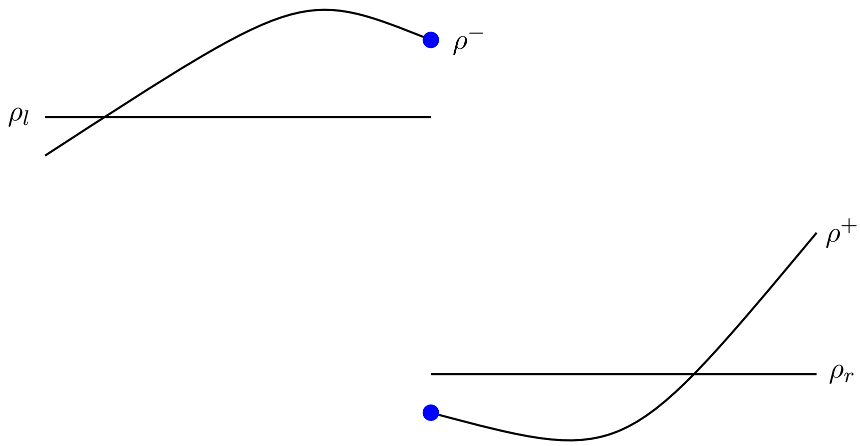

FIGURE 4. Linearization of density about a constant state

Then we obtain for the perturbation functions with $c^2_{r,l} = p'(\rho_{r,l}) > 0$ the linearized system

$$\partial_t \rho_{\pm} + \rho_{r,l}\, \check{\nabla} \cdot \mathbf{v}_{\pm} \pm \rho_{r,l}\, \partial_z u_{\pm} \pm u_{r,l}\, \partial_z \rho_{\pm} = 0, \quad \text{in} \quad z > 0, \tag{2.21}$$

$$\partial_t \mathbf{v}_{\pm} \pm u_{r,l}\, \partial_z \mathbf{v}_{\pm} + \frac{c^2_{r,l}}{\rho_{r,l}} \check{\nabla} \rho_{\pm} = 0, \quad \text{in} \quad z > 0, \tag{2.22}$$

$$\begin{aligned} u_{r,l}\, \partial_t \rho_{\pm} + \rho_{r,l} \partial_t u_{\pm} + \rho_{r,l}\, u_{r,l}\, \check{\nabla} \cdot \mathbf{v}_{\pm} \pm 2 \rho_{r,l}\, u_{r,l}\, \partial_z u_{\pm} & \\ \pm u^2_{r,l} \partial_z \rho_{\pm} \pm c^2_{r.l} \partial_z \rho_{\pm} &= 0, \quad \text{in} \quad z > 0.\end{aligned} \tag{2.23}$$

Using the first equation (2.21) we get the non-conservative form

$$\partial_t \rho_{\pm} + \rho_{r,l}\, \check{\nabla} \cdot \mathbf{v}_{\pm} \pm \rho_{r,l}\, \partial_z u_{\pm} \pm u_{r,l}\, \partial_z \rho_{\pm} = 0, \quad \text{in} \quad z > 0, \tag{2.24}$$

$$\partial_t \mathbf{v}_{\pm} \pm u_{r,l}\, \partial_z \mathbf{v}_{\pm} + \frac{c^2_{r,l}}{\rho_{r,l}} \check{\nabla} \rho_{\pm} = 0, \quad \text{in} \quad z > 0, \tag{2.25}$$

$$\partial_t u_{\pm} \pm u_{r,l}\, \partial_z u_{\pm} \pm \frac{c^2_{r,l}}{\rho_{r,l}} \partial_z \rho_{\pm} = 0, \quad \text{in} \quad z > 0. \tag{2.26}$$

Further, linearizing the boundary conditions (2.10)-(2.13) about $(\rho_{r,l}, \mathbf{v}_{r,l}, u_{r,l})$ and $\varphi = 0$ yields $d+2$ boundary conditions at $z = 0$. The linearized conditions are

$$u_r\,\rho_+ + \rho_r\,u_+ - u_l\,\rho_- - \rho_l\,u_- - [\rho]\,\partial_t X \;=\; 0, \tag{2.27}$$

$$\rho_r\,u_r\,\mathbf{v}_+ - \rho_l\,u_l\,\mathbf{v}_- - [p]\,\check{\nabla} X \;=\; 0, \tag{2.28}$$

$$(c_r^2 + u_r^2)\,\rho_+ + 2\rho_r\,u_r\,u_+ - (c_l^2 + u_l^2)\,\rho_- - 2\rho_l\,u_l\,u_- \;=\; s\check{\nabla}\cdot\check{\nabla} X, \tag{2.29}$$

$$\begin{aligned} &\frac{c_r^2}{\rho_r}\,\rho_+ + u_r\,u_+ - \frac{c_l^2}{\rho_l}\,\rho_- - u_l\,u_- - [u]\,\partial_t X \\ &+\mathrm{B}\,\rho_l\,u_- + \mathrm{B}\,u_l\,\rho_- - \mathrm{B}\,\rho_l\,\partial_t X \;=\; 0. \end{aligned} \tag{2.30}$$

Now we are going to formulate the linearized problem (2.24)-(2.26) and the boundary conditions (2.27)-(2.30) in a more compact form, following [5, 8, 14, 20].

Let $\mathbf{U} := (\rho_-, \mathbf{v}_-, u_-, \rho_+, \mathbf{v}_+, u_+)$; we can write the system (2.24)-(2.26) and (2.27)-(2.30) in the form

$$\begin{cases} \mathbf{L}[\mathbf{U}] = 0 & \text{for} \quad z > 0, \\ \mathbf{b}[\mathrm{X}] + \mathbf{M}\mathbf{U} = 0 & \text{for} \quad z = 0, \end{cases} \tag{2.31}$$

where

$$\mathbf{L} = \mathbf{L}(\partial_t, \check{\nabla}, \partial_z) = \begin{pmatrix} \mathbf{L}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^+ \end{pmatrix} \tag{2.32}$$

is the operator with

$$\mathbf{L}^- = \begin{pmatrix} \partial_t - u_l\partial_z & \rho_l\check{\nabla}^{\mathsf{T}} & -\rho_l\partial_z \\ \frac{c_l^2}{\rho_l}\check{\nabla} & \mathrm{diag}(\partial_t - u_l\partial_z) & \mathbf{0}_{d-1} \\ -\frac{c_l^2}{\rho_l}\partial_z & \mathbf{0}_{d-1}^{\mathsf{T}} & \partial_t - u_l\partial_z \end{pmatrix} \tag{2.33}$$

and

$$\mathbf{L}^+ = \begin{pmatrix} \partial_t + u_r\partial_z & \rho_r\check{\nabla}^{\mathsf{T}} & -\rho_r\partial_z \\ \frac{c_r^2}{\rho_r}\check{\nabla} & \mathrm{diag}(\partial_t + u_r\partial_z) & \mathbf{0}_{d-1} \\ -\frac{c_r^2}{\rho_r}\partial_z & \mathbf{0}_{d-1}^{\mathsf{T}} & \partial_t + u_r\partial_z \end{pmatrix}. \tag{2.34}$$

Here we used the notations $\mathbf{0}_{d-1} = (0, ..., 0)^{\mathsf{T}} \in \mathbb{R}^{d-1}$ and $\mathrm{diag}(a) = a \cdot \mathbf{I}_{d-1}$, where $\mathbf{I}_{d-1}$ is the identity matrix in $\mathbb{R}^{(d-1)\times(d-1)}$. The boundary operator $\mathbf{b}(\partial_t, \check{\nabla})$ is given by

$$\mathbf{b}(\partial_t, \check{\nabla}) = \begin{pmatrix} -[\rho]\partial_t \\ -[p]\check{\nabla} \\ -s\,\check{\nabla}\cdot\check{\nabla} \\ -\left([u] + \mathrm{B}\rho_l\right)\partial_t \end{pmatrix} \tag{2.35}$$

and $\mathbf{M} \in \mathbb{R}^{(d+2)\times 2(d+1)}$ by

$$\mathbf{M} = \begin{pmatrix} -u_l & \mathbf{0}_{d-1}^{\mathsf{T}} & -\rho_l & u_r & \mathbf{0}_{d-1}^{\mathsf{T}} & \rho_r \\ \mathbf{0}_{d-1} & -\mathrm{diag}(\rho_l u_l) & 0 & 0 & \mathrm{diag}(\rho_r u_r) & 0 \\ -c_l^2 - u_l^2 & \mathbf{0}_{d-1}^{\mathsf{T}} & -2\,\rho_l\,u_l & c_r^2 + u_r^2 & \mathbf{0}_{d-1}^{\mathsf{T}} & 2\,\rho_r\,u_r \\ -\frac{c_l^2}{\rho_l} + \mathrm{B}\,u_l & \mathbf{0}_{d-1}^{\mathsf{T}} & -u_l + \mathrm{B}\,\rho_l & \frac{c_r^2}{\rho_r} & \mathbf{0}_{d-1}^{\mathsf{T}} & u_r \end{pmatrix}. \tag{2.36}$$

To be congruent with the general theory (see [5, 8, 20]) we write the system (2.31) additionally in the form

$$\begin{cases} \mathbf{L}[\mathbf{U}] =: \sum\limits_{j=0}^{d-1} \mathbf{A}_j\partial_j\mathbf{U} + \mathbf{A}_z\partial_z\mathbf{U} = 0, & \text{for} \quad z > 0, \\ \mathbf{b}[\mathrm{X}] + \mathbf{M}\mathbf{U} =: \sum\limits_{j=0}^{d-1} \mathbf{b}_j\partial_j\mathrm{X} + \mathtt{b}\check{\Delta}\mathrm{X} + \mathbf{M}\mathbf{U} = 0, & \text{for} \quad z = 0, \end{cases} \tag{2.37}$$

where we used the notations

$$\partial_0 = \partial_t, \qquad \check{\Delta} = \partial_1^2 + \cdots + \partial_{d-1}^2\,,$$

$$\mathbf{b}_0 = \begin{pmatrix} -[\rho] \\ \mathbf{0}_{d-1} \\ 0 \\ -([u] + \mathrm{B}\rho_l) \end{pmatrix}, \qquad \mathtt{b} = \begin{pmatrix} 0 \\ \mathbf{0}_{d-1} \\ -s \\ 0 \end{pmatrix} \tag{2.38}$$

and for $1 \leq j \leq d-1$

$$\mathbf{b}_j = -[p(\rho)] \begin{pmatrix} 0 \\ \mathbf{e}_j \\ 0 \\ 0 \end{pmatrix}, \tag{2.39}$$

where $\mathbf{e}_j$ is the unit vector in $\mathbb{R}^{d-1}$. The matrices $\mathbf{A}_0, \mathbf{A}_1, ..., \mathbf{A}_{d-1}, \mathbf{A}_z \in \mathbb{R}^{2(d+1)\times 2(d+1)}$ are given by

$$\mathbf{A}_0 = \mathbf{I}_{2(d+1)} \tag{2.40}$$

and for $1 \leq j \leq d-1$ through

$$\mathbf{A}_j = \begin{pmatrix} \mathbf{A}_j^- & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_j^+ \end{pmatrix} \quad \text{and} \quad \mathbf{A}_z = \begin{pmatrix} \mathbf{A}_z^- & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_z^+ \end{pmatrix}, \tag{2.41}$$

where

$$\mathbf{A}_j^- = \begin{pmatrix} 0 & \rho_l\, \mathbf{e}_j^\intercal & 0 \\ \frac{c_l^2}{\rho_l}\mathbf{e}_j & \mathbf{0}_{(d-1)\times(d-1)} & 0 \\ 0 & \mathbf{0}_{d-1}^\intercal & 0 \end{pmatrix}, \qquad \mathbf{A}_j^+ = \begin{pmatrix} 0 & \rho_r\, \mathbf{e}_j^\intercal & 0 \\ \frac{c_r^2}{\rho_r}\mathbf{e}_j & \mathbf{0}_{(d-1)\times(d-1)} & 0 \\ 0 & \mathbf{0}_{d-1}^\intercal & 0 \end{pmatrix} \tag{2.42}$$

and

$$\mathbf{A}_z^- = \begin{pmatrix} -u_l & \mathbf{0}_{d-1}^\intercal & -\rho_l \\ 0 & -\mathrm{diag}(u_l) & \mathbf{0}_{d-1} \\ -\frac{c_l^2}{\rho_l} & \mathbf{0}_{d-1}^\intercal & -u_l \end{pmatrix}, \qquad \mathbf{A}_z^+ = \begin{pmatrix} -u_r & \mathbf{0}_{d-1}^\intercal & -\rho_r \\ 0 & -\mathrm{diag}(u_r) & \mathbf{0}_{d-1} \\ -\frac{c_r^2}{\rho_r} & \mathbf{0}_{d-1}^\intercal & -u_r \end{pmatrix}. \tag{2.43}$$

The main result of this section will be given for the nonhomogeneous weighted system

$$\begin{cases} \mathbf{L}_\gamma[\mathbf{U}] = e^{-\gamma t}\mathbf{f}, & \text{for} \quad z > 0, \\ \mathbf{b}[\mathrm{X}] + \mathbf{M}\mathbf{U} + \gamma\mathbf{b}_0\mathrm{X} = e^{-\gamma t}\mathbf{g}, & \text{for} \quad z = 0, \end{cases} \tag{2.44}$$

where $\mathbf{L}_\gamma$ is given by replacing ∂_t by $\gamma + \partial_t$ in the definition of $\mathbf{L}$ in (2.32).

Theorem 1. *Assume that* (2.15) *and* (2.16) *hold for the reference solution* (2.14). *Then there are constants* $s_0 > 0$, $\mathrm{B}_0 > 0$ *and* $\gamma_0 \geq 1$ *such that for all* $\mathrm{B} \in (0, \mathrm{B}_0)$, $s \in (0, s_0)$, $\gamma \geq \gamma_0$ *and for all*

$$(e^{-\gamma t}\boldsymbol{f}, e^{-\gamma t}\boldsymbol{g}) \in L^2(\mathcal{U}) \times L^2(\partial\mathcal{U})$$

there exists a weak solution

$$(e^{-\gamma t}\mathbf{U}, e^{-\gamma t}\mathrm{X}) \in L^2(\mathcal{U}) \times \left(H^{1/2}(\partial\mathcal{U}) \cap L^2\left(\mathbb{R}_t, H^1(\mathbb{R}^{d-1})\right)\right)$$

of the system (2.44), *i.e. for all* $\mathbf{W} \in C_0^\infty(\overline{\mathcal{U}})$

$$\langle \mathbf{L}[\mathbf{U}],\, \mathbf{W}\rangle_{L^2(\mathcal{U})} = \left\langle \mathbf{U},\, -\sum_{j=0}^{d-1} \mathbf{A}_j^\intercal \partial_j \mathbf{W} - \mathbf{A}_z^\intercal \partial_z \mathbf{W} \right\rangle_{L^2(\mathcal{U})} - \langle \mathbf{A}_z\mathbf{U}(z=0),\, \mathbf{W}(z=0)\rangle_{L^2(\partial\mathcal{U})}. \tag{2.45}$$

The proof will be presented in section 4. Theorem 1 is concerned with $t \in \mathbb{R}$. The initial boundary value problem for $t \geq 0$ is then considered in section 5.

We conclude this section with further preliminaries for the proof of theorem 1. We collect the vectors $(\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_{d-1}, \mathtt{b})$ into the boundary matrix

$$\mathcal{B} := (\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_{d-1}, \mathtt{b}) \in \mathbb{R}^{(d+2)\times(d+1)}. \tag{2.46}$$

For this matrix we have the following property.

Lemma 2. *There is a (pseudo inverse) matrix* $\mathcal{B}^{-1} \in \mathbb{R}^{(d+1)\times(d+2)}$ *such that*

$$\mathcal{B}^{-1}\mathcal{B}\,\mathcal{B}^{-1} = \mathcal{B}^{-1} \qquad \text{and} \qquad \mathcal{B}\mathcal{B}^{-1}\,\mathcal{B} = \mathcal{B},$$

Furthermore, one has

$$\mathcal{B}^{-1}\mathcal{B} = \mathbf{I}_{d+1}.$$

Proof. It is easy to see that

$$\mathcal{B}^{-1} := \begin{pmatrix} \frac{-[\rho]}{[\rho]^2+([u]+\mathrm{B}\,\rho_l)^2} & \mathbf{0}_{d-1}^{\mathsf{T}} & 0 & \frac{-([u]+\mathrm{B}\,\rho_l)}{[\rho]^2+([u]+\mathrm{B}\,\rho_l)^2} \\ \mathbf{0}_{d-1} & \frac{-1}{[\rho]}\mathbf{I}_{d-1} & \mathbf{0}_{d-1} & \mathbf{0}_{d-1}^{\mathsf{T}} \\ 0 & \mathbf{0}_{d-1}^{\mathsf{T}} & \frac{1}{s} & 0 \end{pmatrix} \tag{2.47}$$

satisfies the statement of the lemma. □

One checks readily that $\mathcal{B}$ has full rank. Using the pseudo inverse matrix we write the system (2.37) in the form

$$\begin{cases} \sum_{j=0}^{d-1} \mathbf{A}_j\partial_j\mathbf{U} + \mathbf{A}_z\partial_z\mathbf{U} = 0, & \text{for} \quad z>0, \\ \begin{pmatrix} \check{\nabla}^0 \\ \check{\Delta} \end{pmatrix} \mathrm{X} + \mathcal{B}^{-1}\mathbf{M}\mathbf{U} = 0, & \text{for} \quad z=0, \end{cases} \tag{2.48}$$

where $\check{\nabla}^0 := (\partial_0, \partial_1, ..., \partial_{d-1}) := (\partial_t, \partial_1, ..., \partial_{d-1})$.

3. Associated Adjoint Problem

Now we are going to construct a boundary value problem associated with (2.37) which satisfies the uniform Kreiss-Lopatinskiĭ condition such that it is possible to get an energy estimate as in [13] for this adjoint system. The existence of a solution for the original problem (2.37) will be given then by using the Riesz theorem for a linear form on special spaces. In the construction of this adjoint problem we are following the method introduced in [5, 20, 8].

We consider the original problem (2.37) and choose a for now arbitrary matrix

$$N \in \mathbb{R}^{(d+1)\times 2(d+1)}$$

such that

$$\begin{pmatrix} \mathcal{B}^{-1}\mathbf{M} \\ N \end{pmatrix} \in \mathbb{R}^{2(d+1)\times 2(d+1)} \tag{3.1}$$

is invertible, where we note that $\mathcal{B}^{-1}\mathbf{M}$ has full rank for small surface tension s and small interfacial mobility constant B, see also [13, Theorem 7]. Let the inverse of (3.1) be represented through

$$\begin{pmatrix} Y \,|\, D \end{pmatrix}, \tag{3.2}$$

where

$$Y, D \in \mathbb{R}^{2(d+1)\times(d+1)}.$$

From

$$\begin{pmatrix} Y \,|\, D \end{pmatrix} \begin{pmatrix} \mathcal{B}^{-1}\mathbf{M} \\ N \end{pmatrix} = \mathbf{I}_{2(d+1)}, \tag{3.3}$$

we conclude

$$Y\left(\mathcal{B}^{-1}\mathbf{M}\right) + DN = \mathbf{I}_{2(d+1)}. \tag{3.4}$$

We define for $\mathbf{A}_z$

$$\mathbf{C}^{\mathrm{adj}} := (\mathbf{A}_z D)^{\mathsf{T}} \qquad \text{and} \qquad \mathbf{N}^{\mathrm{adj}} := (\mathbf{A}_z Y)^{\mathsf{T}} \tag{3.5}$$

and obtain with (3.4)

$$\mathbf{A}_z = (\mathbf{C}^{\mathrm{adj}})^{\mathsf{T}} N + (\mathbf{N}^{\mathrm{adj}})^{\mathsf{T}} \mathcal{B}^{-1}\mathbf{M}. \tag{3.6}$$

Especially, we have the following property.

Lemma 3. *The full space $\mathbb{R}^{2(d+1)}$ can be written as a direct sum of the kernel of the matrices $\mathcal{B}^{-1}\boldsymbol{M}$ and N, i.e.*

$$\mathbb{R}^{2(d+1)} = \ker(\mathcal{B}^{-1}\mathbf{M}) \oplus \ker(N). \tag{3.7}$$

Proof. By having chosen the matrix N such that $\begin{pmatrix} \mathcal{B}^{-1}\mathbf{M} \\ N \end{pmatrix}$ is invertible we have

$$\ker(\mathcal{B}^{-1}\mathbf{M}) \cap \ker(N) = \{0\}.$$

It is easy to see that the direct sum of these kernels generates $\mathbb{R}^{2(d+1)}$ since the matrix $\mathcal{B}^{-1}\mathbf{M}$ has full rank. ☐

The original system (2.37) is symmetrizable by a symmetrizer $\mathbf{K}$ given by [13, Theorem 7], i.e. all the matrices $\mathbf{KA}_j$ for $j = 0, ..., d-1$ and $\mathbf{KA}_z$ are symmetric and $\mathbf{KA}_0$ is positive definite. Defining

$$\mathbf{C}^{\mathrm{adj}}_{\mathbf{K}} := \mathbf{KC}^{\mathrm{adj}} \qquad \text{and} \qquad \mathbf{N}^{\mathrm{adj}}_{\mathbf{K}} := \mathbf{KN}^{\mathrm{adj}}, \tag{3.8}$$

yields then

$$\mathbf{KA}_z = (\mathbf{C}^{\mathrm{adj}}_{\mathbf{K}})^{\intercal} N + (\mathbf{N}^{\mathrm{adj}}_{\mathbf{K}})^{\intercal} \mathcal{B}^{-1}\mathbf{M}. \tag{3.9}$$

As in lemma 3 one can show that

$$\mathbb{R}^{2(d+1)} = \ker(\mathbf{C}^{\mathrm{adj}}_{\mathbf{K}}) \oplus \ker(\mathbf{N}^{\mathrm{adj}}_{\mathbf{K}}). \tag{3.10}$$

Especially, we note that we have

$$\ker(\mathbf{C}^{\mathrm{adj}}) = (\mathbf{A}_z \mathrm{ran}(D))^{\perp} = \left(\mathbf{A}_z \ker(\mathcal{B}^{-1}\mathbf{M})\right)^{\perp},$$

which shows that $\ker(\mathbf{C}^{\mathrm{adj}}_{\mathbf{K}})$ does not depend on the choice of the matrix N, see for instance [5].

The adjoint operator to $\mathbf{L}$ is given by

$$\mathbf{L}^* := -\sum_{j=0}^{d-1} \mathbf{A}_j^{\intercal} \partial_j - \mathbf{A}_z^{\intercal} \partial_z. \tag{3.11}$$

As usual for hyperbolic problems we also introduce the weighted-in-time adjoint operator

$$\mathbf{L}^*_{\gamma} =: \gamma - \sum_{j=0}^{d-1} \mathbf{A}_j^{\intercal} \partial_j - \mathbf{A}_z^{\intercal} \partial_z, \quad \gamma \in \mathbb{R}^+ . \tag{3.12}$$

Now we collect some properties for the adjoint operator. First we have

$$\langle \mathbf{KL}[\mathbf{U}],\, \mathbf{V} \rangle_{L^2(\mathcal{U})} = \langle \mathbf{U},\, \mathbf{KL}^*[\mathbf{V}] \rangle_{L^2(\mathcal{U})} \tag{3.13}$$

for all $\mathbf{U}, \mathbf{V} \in C_0^{\infty}(\mathcal{U})$. The scalar product $\langle \cdot, \cdot \rangle_{L^2(\mathcal{U})}$ in (3.13) has to be understood for $\mathbf{U}, \mathbf{V} \in C_0^{\infty}(\mathcal{U}, \mathbb{R}^{2(d+1)})$ as

$$\langle \mathbf{U},\, \mathbf{V} \rangle_{L^2(\mathcal{U})} := \int_{\mathcal{U}} \langle \mathbf{U}(t,y,z),\, \mathbf{V}(t,y,z) \rangle_{\mathbb{C}^{2(d+1)}} \,\mathrm{d}t\mathrm{d}y\mathrm{d}z. \tag{3.14}$$

In weighted norms we obtain in the same way

$$\langle \mathbf{KL}_{\gamma}[\mathbf{U}],\, \mathbf{V} \rangle_{L^2(\mathcal{U})} = \left\langle \mathbf{U},\, \mathbf{KL}^*_{\gamma}[\mathbf{V}] \right\rangle_{L^2(\mathcal{U})}. \tag{3.15}$$

For $\mathbf{U}, \mathbf{V} \in H^1(\mathcal{U}, \mathbb{R}^{2(d+1)})$ the scalar product identity is

$$\langle \mathbf{KL}[\mathbf{U}],\, \mathbf{V} \rangle_{L^2(\mathcal{U})} = \langle \mathbf{U},\, \mathbf{KL}^*[\mathbf{V}] \rangle_{L^2(\mathcal{U})} - \langle \mathbf{KA}_z \mathbf{U}(z=0),\, \mathbf{V}(z=0) \rangle_{L^2(\partial\mathcal{U})}, \tag{3.16}$$

where we note that the functions are well-defined on the boundary by [12, 6]. Especially for $\mathbf{L}[\mathbf{U}], \mathbf{U} \in L^2(\mathcal{U})$ and $\mathbf{V} \in H^1(\mathcal{U})$ one can identify by the classical result [12]

$$\langle \mathbf{KA}_z \mathbf{U}(z=0),\, \mathbf{V}(z=0) \rangle_{L^2(\partial\mathcal{U})} = \langle \mathbf{KA}_z \mathbf{U}(z=0),\, \mathbf{V}(z=0) \rangle_{H^{-1/2}(\partial\mathcal{U}) \times H^{1/2}(\partial\mathcal{U})}. \tag{3.17}$$

Now consider again the nonhomogeneous weighted system (2.44) for $e^{-\gamma t}\mathbf{f} \in L^2(\mathcal{U})$ and $e^{-\gamma t}\mathbf{g} \in L^2(\partial\mathcal{U})$. We assume that $(\mathbf{U}, X) \in L^2(\mathcal{U}) \times \left(H^{1/2}(\partial\mathcal{U}) \cap L^2\left(\mathbb{R}_t, H^1(\mathbb{R}^{d-1})\right)\right)$ is a weak solution of the weighted system (2.44). At this point we refer to (2.45) for the definition of a weak solution. Looking for a simplified formulation of the scalar product (3.17) by using the special form of (3.9), we obtain for $\mathbf{W} \in C_0^\infty(\overline{\mathcal{U}})$

$$
\begin{aligned}
\langle \mathbf{K}\mathbf{A}_z\mathbf{U},\, \mathbf{W}\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} \;=\;& \left\langle N\mathbf{U},\, \mathbf{C}_{\mathbf{K}}^{\mathrm{adj}}\mathbf{W}\right\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} \\
&+ \left\langle \mathcal{B}^{-1}e^{-\gamma t}\mathbf{g},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}}\mathbf{W}\right\rangle_{L^2(\partial\mathcal{U})} \\
&+ \left\langle \mathrm{X},\, \check{\nabla}_\gamma^0\cdot\mathbf{N}_{\mathbf{1}}^{\mathrm{adj}}\mathbf{W}\right\rangle_{L^2(\partial\mathcal{U})} - \left\langle \mathrm{X},\, \check{\Delta}\mathbf{N}_{\mathbf{2}}^{\mathrm{adj}}\mathbf{W}\right\rangle_{L^2(\partial\mathcal{U})},
\end{aligned}
\tag{3.18}
$$

where we note that we have used $\check{\nabla}_\gamma^0 := \check{\nabla}^0 + \gamma\mathbf{e}_0$ and $\mathcal{B}^{-1}\mathbf{b}_0 = \mathbf{e}_0$. The matrices $\mathbf{N}_{\mathbf{1}}^{\mathrm{adj}}$ and $\mathbf{N}_{\mathbf{2}}^{\mathrm{adj}}$ are given by

$$
\mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} = \begin{pmatrix} \mathbf{N}_{\mathbf{1}}^{\mathrm{adj}} \\ \mathbf{N}_{\mathbf{2}}^{\mathrm{adj}} \end{pmatrix},
\tag{3.19}
$$

where $\mathbf{N}_{\mathbf{2}}^{\mathrm{adj}}$ is the last row of the matrix $\mathbf{N}_{\mathbf{K}}^{\mathrm{adj}}$.

We can now define the adjoint problem. The associated semi-homogeneous adjoint problem to (2.37) is for $\mathbf{F} \in L^2(\mathcal{U})$

$$
\begin{cases}
\mathbf{L}^*[\mathbf{V}] = \mathbf{F}, & \text{for } \quad z > 0, \\
\mathbf{C}_{\mathbf{K}}^{\mathrm{adj}}\mathbf{V} = 0,\ \check{\nabla}^0\cdot\mathbf{N}_{\mathbf{1}}^{\mathrm{adj}}\mathbf{V} = 0,\ \check{\Delta}\,\mathbf{N}_{\mathbf{2}}^{\mathrm{adj}}\mathbf{V} = 0, & \text{for } \quad z = 0.
\end{cases}
\tag{3.20}
$$

The following proposition for zero surface tension can also be found in [20, 5, 8].

Proposition 4. *Let* $(\mathbf{U}, \mathrm{X}) \in L^2(\mathcal{U}) \times L^2(\partial\mathcal{U})$ *be a weak solution of the system* (2.44) *with* $\gamma \geq 1$, $e^{-\gamma t}\mathbf{f} \in L^2(\mathcal{U})$ *and* $e^{-\gamma t}\mathbf{g} \in L^2(\partial\mathcal{U})$. *Then for all* $\mathbf{W} \in C_0^\infty(\overline{\mathcal{U}})$ *such that on the boundary* $\partial\mathcal{U}$

$$
\mathbf{C}_{\mathbf{K}}^{\mathrm{adj}}\mathbf{W} = 0,\ \check{\nabla}^0\cdot\mathbf{N}_{\mathbf{1}}^{\mathrm{adj}}\mathbf{W} = 0,\ \check{\Delta}\,\mathbf{N}_{\mathbf{2}}^{\mathrm{adj}}\mathbf{W} = 0
\tag{3.21}
$$

holds, we have

$$
\langle \mathbf{K}\mathbf{L}_\gamma[\mathbf{U}],\, \mathbf{W}\rangle_{L^2(\mathcal{U})} = \langle \mathbf{U},\, \mathbf{K}\mathbf{L}_\gamma^*[\mathbf{W}]\rangle_{L^2(\mathcal{U})} - \left\langle \mathcal{B}^{-1}e^{-\gamma t}\mathbf{g},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}}\,\boldsymbol{W}\right\rangle_{L^2(\partial\mathcal{U})}.
\tag{3.22}
$$

Proof. The proof follows immediately from (3.16) and (3.18). ☐

The next step is to check the backward uniform Kreiss-Lopatinskiĭ condition for the adjoint problem (3.20) such that we will get an energy estimate for this associated problem. This energy estimate in weighted norms will be applied to a linear form such that the existence of a solution to the weighted system will be found. To check the uniform Kreiss-Lopatinskiĭ condition, one uses as a practical tool the so-called Lopatinskiĭ determinant. The roots of this polynomial in several variables have to be studied to ensure the condition. This method called normal mode analysis can be found in a more general context in [11]. We cite here an energy estimate in weighted norms which has been derived in [13].

Proposition 5 ([13])**.** *Assume that* (2.15), (2.16) *hold. Then there are constants* $s_0 \geq 0$ *and* $\mathrm{B}_0 > 0$ *such that for all* $\mathrm{B} \in (0, \mathrm{B}_0)$, $s \in [0, s_0)$ *and* $\gamma_0 > 0$, *there exists a constant* $C > 0$ *such that for all* $\gamma \geq \gamma_0$ *and all solutions*

$$
(e^{-\gamma t}\mathbf{U}, e^{-\gamma t}\mathrm{X}) \in C^1(\mathbb{R}^+, L^2(\mathbb{R}^d)) \times H^1(\mathbb{R}^d)
\tag{3.23}
$$

of (2.44) *with* $e^{-\gamma t}\mathbf{f} \in L^2(\mathcal{U})$, $e^{-\gamma t}\mathbf{g} \in L^2(\partial\mathcal{U})$ *the following inequality holds*

$$
\begin{aligned}
\gamma\|e^{-\gamma t}\mathbf{U}\|^2_{L^2(\mathcal{U})} + \|e^{-\gamma t}\mathbf{U}(0)\|^2_{L^2(\partial\mathcal{U})} + \|e^{-\gamma t}\mathrm{X}\|^2_{H^1_\gamma(\partial\mathcal{U})} \\
\leq C\left(\|e^{-\gamma t}\mathbf{g}\,\|^2_{L^2(\partial\mathcal{U})} + \frac{1}{\gamma}\|e^{-\gamma t}\mathbf{f}\,\|^2_{L^2(\mathcal{U})}\right).
\end{aligned}
$$

Especially, the uniform Kreiss-Lopatinskiĭ condition is satisfied.

The result of proposition 5 can be improved. We have in fact for all solutions

$$
(e^{-\gamma t}\mathbf{U}, e^{-\gamma t}\mathrm{X}) \in C^1(\mathbb{R}^+, L^2(\mathbb{R}^d)) \times \left(H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}, H^2(\mathbb{R}^{d-1}))\right)
$$

the estimate

$$\gamma\|e^{-\gamma t}\mathbf{U}\|^2_{L^2(\mathcal{U})} + \|e^{-\gamma t}\mathbf{U}(0)\|^2_{L^2(\partial\mathcal{U})} + \|e^{-\gamma t}\mathrm{X}\|^2_{H^1_\gamma(\partial\mathcal{U})} + \frac{1}{\gamma^2}\|e^{-\gamma t}\mathrm{X}\|^2_{L^2(\mathbb{R},H^2_\gamma(\mathbb{R}^{d-1}))} \leq C\left(\|e^{-\gamma t}\mathbf{g}\|^2_{L^2(\partial\mathcal{U})} + \frac{1}{\gamma}\|e^{-\gamma t}\mathbf{f}\|^2_{L^2(\mathcal{U})}\right). \tag{3.24}$$

Using Plancherel's theorem we have

$$\frac{1}{\gamma^2}\|e^{-\gamma t}\mathrm{X}\|^2_{L^2(\mathbb{R},H^2_\gamma(\mathbb{R}^{d-1}))} = \frac{1}{\gamma^2}\int\limits_{\mathbb{R}^d}(\gamma^2+\|\eta\|^2)^2|\widehat{X(t)}(\eta)|^2\mathbf{d}\eta\mathrm{d}t.$$

We obtain using the second equation of (2.44) and again Plancherel's theorem

$$\begin{aligned}\frac{1}{\gamma^2}\|e^{-\gamma t}\mathrm{X}\|^2_{L^2(\mathbb{R},H^2_\gamma(\mathbb{R}^{d-1}))} &\leq \int\limits_{\mathbb{R}^d}(\gamma^2+\|\eta\|^2+s^2\|\eta\|^4)|\widehat{X(t)}(\eta)|^2\mathbf{d}\eta\mathrm{d}t\\ &\leq c\left(\|e^{-\gamma t}\mathbf{g}\|^2_{L^2(\mathbb{R}^d)} + \|e^{-\gamma t}\mathbf{U}(0)\|^2_{L^2(\mathbb{R}^d)}\right).\end{aligned}$$

The improved estimate contains a second order term of the front solution X. This can be seen as a natural surface energy which should be included in this phase transition process.

Now we are going to prove the proposition which gives the energy estimate for the weighted adjoint problem (3.20).

Proposition 6. *Let be the assumptions as in proposition 5.*
Then the problem (3.20) *satisfies the backward (in time) uniform Kreiss-Lopatinskiĭ condition. Further, there exists a constant* $c>0$ *such that a weak solution* $\mathbf{V}\in C_0^\infty(\overline{\mathcal{U}})$ *of* (3.20) *satisfies the energy estimate*

$$\gamma\|e^{-\gamma t}\mathbf{V}\|^2_{L^2(\mathcal{U})} + \|e^{-\gamma t}\mathbf{V}(0)\|^2_{L^2(\partial\mathcal{U})} \leq \frac{c}{\gamma}\|\boldsymbol{L}^*_\gamma[\mathbf{V}]\|^2_{L^2(\mathcal{U})}. \tag{3.25}$$

Proof. The uniform Kreiss-Lopatinskiĭ condition for the adjoint problem (3.20) reads as follows, especially we give the property of the stable subspace[1] of the associated matrix.
There exists $C>0$ such that for all $(\eta_0,\eta_1,..,\eta_{d-1})\in\mathbb{R}^d$, and all $\gamma\geq 0$ with $\gamma^2+|\eta|^2=1$ the inequality

$$\|W\| \leq C\left(\|\mathbf{C}^{\mathrm{adj}}_{\mathbf{K}}\mathbf{W}\| + \|(\gamma-\mathrm{i}\eta_0,-\mathrm{i}\eta_1,..,-\mathrm{i}\eta_{d-1})\cdot\mathbf{N}^{\mathrm{adj}}_{\mathbf{1}}\mathbf{W}\| + \|\left(|\eta|^2-\eta_0^2\right)\cdot\mathbf{N}^{\mathrm{adj}}_{\mathbf{2}}\mathbf{W}\|\right)$$

holds for all W in the stable subspace of the matrix $\mathcal{A}_*(\gamma,\eta)$. Here $\mathcal{A}_*(\gamma,\eta)$ is given

$$\mathcal{A}_*(\gamma,\eta) = \left(\mathbf{A}_z^{\intercal}\right)^{-1}\left(\gamma - \mathrm{i}\sum_{j=0}^{d-1}\eta_j\mathbf{A}_j^{\intercal}\right).$$

We denote the stable subspace of $\mathcal{A}_*(\gamma,\eta)$ by $\mathcal{E}^-(\mathcal{A}_*(\gamma,\eta))$.
It was shown in [5] that the crucial property for the stable subspace is given by the relation

$$\mathcal{E}^-(\mathcal{A}_*(\gamma,\eta)) = \left(\mathbf{A}_z\mathcal{E}^-(\mathcal{A}(\gamma,\eta))\right)^\perp,$$

where

$$\mathcal{A}(\gamma,\eta) = -\mathbf{A}_z^{-1}\left(\gamma + \mathrm{i}\sum_{j=0}^{d-1}\eta_j\mathbf{A}_j\right).$$

Now we have to check that the space

$$\left\{\mathbf{W}\in\left(\mathbf{A}_z\mathcal{E}^-(\mathcal{A}(\gamma,\eta))\right)^\perp \;\middle|\; \mathbf{C}^{\mathrm{adj}}_{\mathbf{K}}\mathbf{W}=0,\ (\gamma+\mathrm{i}\eta_0,\mathrm{i}\eta_1,..,\mathrm{i}\eta_{d-1})\cdot\mathbf{N}^{\mathrm{adj}}_{\mathbf{1}}\mathbf{W}=0,\ \mathbf{N}^{\mathrm{adj}}_{\mathbf{2}}\mathbf{W}=0\right\}$$

is trivial. Using theorem 8 from [13] we can write any $Z\in\mathbb{C}^{2(d+1)}$ such that

$$Z = \chi\begin{pmatrix}(\gamma+\eta_0,\eta_1,..,\eta_{d-1})^{\intercal}\\ \eta_1^2+\cdots+\eta_{d-1}^2\end{pmatrix} + \mathcal{B}^{-1}\mathbf{M}\mathrm{V}, \tag{3.26}$$

[1] A stable invariant subspace of a matrix A with n rows, n columns and entries in $\mathbb{C}$ is formed of vectors $v\in\mathbb{C}^n$ such that $(\exp(tA))v$ tends to zero as $t\to\infty$ and the decay is exponentially fast.

where $\chi \in \mathbb{C}$ and $\mathrm{V} \in \mathcal{E}^-(\mathcal{A}(\gamma,\eta))$ are given. Now, we obtain for any $Z \in \mathbb{C}^{2(d+1)}$ by using (3.9)

$$\left\langle \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} W,\, \mathrm{Z} \right\rangle = 0,$$

where $W \in \left\{ \mathbf{W} \in \left(\mathbf{A}_z \mathcal{E}^-(\mathcal{A}(\gamma,\eta))\right)^{\perp} \,\middle|\, \mathbf{C}_{\mathbf{K}}^{\mathrm{adj}}\mathbf{W} = 0,\, (\gamma + \mathrm{i}\eta_0, \mathrm{i}\eta_1, .., \mathrm{i}\eta_{d-1}) \cdot \mathbf{N}_{\mathbf{1}}^{\mathrm{adj}}\mathbf{W} = 0,\, \mathbf{N}_{\mathbf{2}}^{\mathrm{adj}}\mathbf{W} = 0 \right\}$.
Using (3.10) yields that $\mathrm{W} = 0$, which means that the space includes only the null which completes the first part of the proof. The energy estimate is given by the standard theory since for a system which satisfies the uniform Kreiss-Lopatinskiĭ condition one can construct a symmetrizer to get this estimate. ☐

4. Well-Posedness of the Boundary Value Problem

Now we are going to show the existence of a solution of (2.44) in weighted norms. The existence will be given by using the Riesz lemma for a suitably defined linear form on special spaces. Our plan is to define a linear form on some space $\mathcal{F}$ (see (4.2)) such that we will find a weak solution, see figure 5 for the idea of the proof. $\mathcal{F}$ takes in particular into account that the boundary conditions contain second order derivatives.

$$\begin{tikzcd}
(*)\begin{cases}\mathbf{L}[\mathbf{U}] = \mathbf{f} \\ \mathcal{B}(\mathbf{U},\mathrm{X}) = \mathbf{g}\end{cases} \arrow[rr, "\text{Construction of the associated adjoint problem}"] & & \begin{cases}\mathbf{L}^*[\mathbf{V}] = \mathbf{F} \\ \mathcal{C}(\mathbf{V}) = \mathbf{G}\end{cases} \arrow[dl, "\text{UKL, energy estimate}"] \\
& \textbf{Linearform } l : \mathbf{L}_\gamma^* \mathcal{F} \to \mathbb{R} \arrow[ul, "\text{Existence for } (*) \text{ using Riesz}"] &
\end{tikzcd}$$

Figure 5. Sketch of the proof for theorem 1

In this context we note that the standard approach to analyze the existence and stability of compressive (Laxian, see [15]) shocks can be found in e.g. [17, 18].

We consider the nonhomogeneous weighted system (see also (2.44))

$$\begin{cases} \mathbf{L}_\gamma[\mathbf{U}] = e^{-\gamma t}\mathbf{f}, & \text{for} \quad z > 0, \\ \mathbf{b}[\mathrm{X}] + \mathbf{M}\mathbf{U} + \gamma \mathbf{b}_0 \mathrm{X} = e^{-\gamma t}\mathbf{g}, & \text{for} \quad z = 0. \end{cases} \tag{4.1}$$

We will show the existence of a weak solution

$$(e^{-\gamma t}\mathbf{U}, e^{-\gamma t}\mathrm{X}) \in L^2(\mathcal{U}) \times \left(H^{1/2}(\mathcal{U}) \cap L^2\left(\mathbb{R}_t, H^1(\mathbb{R}^{d-1})\right)\right)$$

of the system (4.1) for all

$$(e^{-\gamma t}\mathbf{f}, e^{-\gamma t}\mathbf{g}) \in L^2(\mathcal{U}) \times L^2(\partial\mathcal{U}).$$

Here we denote by $L^2\left(\mathbb{R}_t, H^1(\mathbb{R}^{d-1})\right)$ the Sobolev space (see [21, Chapter 11])

$$\left\{ \mathbf{U} \in L^2(\partial\mathcal{U}) \,\middle|\, \check{\nabla}\mathbf{U} \in L^2(\partial\mathcal{U}) \right\}$$

with the usual norm

$$\|\mathbf{U}\|^2_{L^2(\mathbb{R}_t, H^1(\mathbb{R}^{d-1}))} := \int\limits_{\mathbb{R}_t} \|\mathbf{U}(t,\cdot)\|^2_{H^1(\mathbb{R}^{d-1})} \mathrm{d}t.$$

Now we are going to prove the main result which we stated in section 2.

4.1. **Proof of Theorem 1.** The proof of theorem 1 will be based on the theorems of Hahn&Banach and Riesz. Therefore we are going to define a linear form on the space $\mathbf{L}_\gamma^*\mathcal{F}$, where $\mathcal{F}$ is given by

$$\mathcal{F} := \left\{ w \in C_0^\infty(\overline{\mathcal{U}}) \,\middle|\, \mathbf{C}_{\mathbf{K}}^{\mathrm{adj}} w = 0,\ \check{\nabla}_{-\gamma}^0 \cdot \mathbf{N}_{\mathbf{1}}^{\mathrm{adj}} w = 0,\ \check{\Delta}\,\mathbf{N}_{\mathbf{2}}^{\mathrm{adj}} w = 0 \ \text{ for } \ z = 0 \right\}. \tag{4.2}$$

Here is $\check{\nabla}_\gamma^0 := \check{\nabla}^0 + \gamma \mathbf{e}_0$. $\mathcal{F}$ is a subset of $L^2(\mathcal{U})$ such that we can define

$$l : \mathbf{L}_\gamma^*\mathcal{F} \to \mathbb{R}$$

by

$$\mathbf{L}_\gamma^*[w] \mapsto l\left(\mathbf{L}_\gamma^*[w]\right) := \left\langle e^{-\gamma t}\mathbf{f},\, w \right\rangle_{L^2(\mathcal{U})} + \left\langle \mathcal{B}^{-1} e^{-\gamma t}\mathbf{g},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{L^2(\partial\mathcal{U})}.$$

Using the fact, that the adjoint problem (3.20) satisfies the backward uniform Kreiss-Lopatinskiĭ condition and especially the energy estimate for the adjoint problem, we obtain for a generic constant $c > 0$ the inequality

$$\begin{aligned} \left| l\left(\mathbf{L}_\gamma^*[w]\right) \right| &\le \left| \left\langle e^{-\gamma t}\mathbf{f},\, w \right\rangle_{L^2(\mathcal{U})} \right| + \left| \left\langle \mathcal{B}^{-1} e^{-\gamma t}\mathbf{g},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{L^2(\partial\mathcal{U})} \right| \\ &\le \|e^{-\gamma t}\mathbf{f}\| \cdot \|w\| + \|\mathcal{B}^{-1} e^{-\gamma t}\mathbf{g}\| \cdot \|\mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w\| \\ &\le c \left\{ \|e^{-\gamma t}\mathbf{f}\| \cdot \|w\| + \|e^{-\gamma t}\mathbf{g}\| \cdot \|w\| \right\} \\ &\le c \left\{ \frac{1}{\gamma}\|e^{-\gamma t}\mathbf{f}\| + \frac{1}{\gamma^{1/2}}\|e^{-\gamma t}\mathbf{g}\| \right\} \cdot \|\mathbf{L}_\gamma^*[w]\|, \end{aligned}$$

where we have used (3.25) in the last inequality. We have omitted in the norms the space notation, which is clear from the context. Altogether we have

$$\left| l\left(\mathbf{L}_\gamma^*[w]\right) \right| \le c\, \|\mathbf{L}_\gamma^*[w]\|_{L^2(\mathcal{U})}. \tag{4.3}$$

This yields the continuation of the linear form to the subset $\mathbf{L}_\gamma^*\mathcal{F}$. By using Hahn&Banach, the linear form can be continued on $e^{\gamma t}L^2(\mathcal{U})$ and by using the lemma of Riesz, we find exactly one

$$\mathbf{U} \in e^{\gamma t} L^2(\mathcal{U})$$

such that for all $w \in \mathcal{F}$

$$l\left(\mathbf{L}_\gamma^*[w]\right) = \left\langle e^{-\gamma t}\mathbf{U},\, \mathbf{L}_\gamma^*[w] \right\rangle_{L^2(\mathcal{U})}. \tag{4.4}$$

By definition this is equivalent to $\mathbf{U}$ being a weak solution of

$$\mathbf{L}_\gamma[\mathbf{U}] = e^{-\gamma t}\mathbf{f},$$

since we have for all $w \in C_0^\infty(\mathcal{U})$

$$\left\langle e^{-\gamma t}\mathbf{f},\, w \right\rangle_{L^2(\mathcal{U})} = \left\langle e^{-\gamma t}\mathbf{U},\, \mathbf{L}_\gamma^*[w] \right\rangle_{L^2(\mathcal{U})}. \tag{4.5}$$

We have found a weak solution for the weighted equation (4.1). It remains to show that $\mathbf{U}$ satisfies the boundary condition for some X.

By the classical result of Friedrichs [12], we obtain that $e^{-\gamma t}\mathbf{U}$ is well-defined on the boundary $\partial\mathcal{U}$ and belongs to $H^{-1/2}(\partial\mathcal{U})$, especially $H^{-1}(\partial\mathcal{U})$. We know from the identity (3.17) and using (3.9) the following

$$\langle \mathbf{K}\mathbf{A}_z\mathbf{U},\, w \rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} = \left\langle N\mathbf{U},\, \mathbf{C}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} + \left\langle \mathcal{B}^{-1}\mathbf{M}\mathbf{U},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{L^2(\partial\mathcal{U})}.$$

If we restrict to all $w \in \mathcal{F}$ we have

$$\langle \mathbf{K}\mathbf{A}_z\mathbf{U},\, w \rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} = \left\langle \mathcal{B}^{-1}\mathbf{M}\mathbf{U},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})}.$$

That yields

$$\left\langle \mathcal{B}^{-1}\mathbf{M}\mathbf{U},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} = \left\langle e^{-\gamma t}\mathbf{U},\, \mathbf{L}_\gamma^*[w] \right\rangle_{L^2(\mathcal{U})} - \left\langle e^{-\gamma t}\mathbf{f},\, w \right\rangle_{L^2(\mathcal{U})}. \tag{4.6}$$

We compare (4.6) with (4.4) and obtain for all $w \in \mathcal{F}$

$$\left\langle \mathcal{B}^{-1}\mathbf{M}\mathbf{U},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} = \left\langle \mathcal{B}^{-1} e^{-\gamma t}\mathbf{g},\, \mathbf{N}_{\mathbf{K}}^{\mathrm{adj}} w \right\rangle_{L^2(\partial\mathcal{U})}. \tag{4.7}$$

We note that (4.7) makes sense by the result of [12, 6]. We write (4.7) in the following form

$$\begin{aligned}(4.8)\qquad & \Big\langle (\mathcal{B}^{-1}\mathbf{MU})_1,\, \mathbf{N}_1^{\mathrm{adj}}w \Big\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} + \int_{\mathbb{R}_t} \Big\langle (\mathcal{B}^{-1}\mathbf{MU}(t))_2,\, \mathbf{N}_2^{\mathrm{adj}}w(t) \Big\rangle_{H^{-1}\times H^1} \mathrm{d}t \\ = \ & \Big\langle (\mathcal{B}^{-1}e^{-\gamma t}\mathbf{g})_1,\, \mathbf{N}_1^{\mathrm{adj}}w \Big\rangle_{L^2(\partial\mathcal{U})} + \int_{\mathbb{R}_t} \Big\langle (\mathcal{B}^{-1}e^{-\gamma t}\mathbf{g}(t))_2,\, \mathbf{N}_2^{\mathrm{adj}}w(t) \Big\rangle_{L^2(\partial\mathcal{U})} \mathrm{d}t,\end{aligned}$$

where we used the notations

$$\mathcal{B}^{-1}\mathbf{MU} =: \begin{pmatrix} (\mathcal{B}^{-1}\mathbf{MU})_1 \\ (\mathcal{B}^{-1}\mathbf{MU})_2 \end{pmatrix},$$

i.e. $(\mathcal{B}^{-1}\mathbf{MU})_2$ is the last row of $\mathcal{B}^{-1}\mathbf{MU}$. By a continuity argument (4.8) holds for all $w \in H^{3/2}(\mathcal{U})$ such that

$$\mathbf{C}_{\mathbf{K}}^{\mathrm{adj}}w = 0, \quad \check{\nabla}^0_{-\gamma}\cdot \mathbf{N}_{\mathbf{1}}^{\mathrm{adj}}w = 0, \quad \check{\Delta}\,\mathbf{N}_{\mathbf{2}}^{\mathrm{adj}}w = 0$$

on the boundary $\partial\mathcal{U}$. Therefore, the trace of this function from $H^{3/2}(\mathcal{U})$ is well-defined by [6] and belongs to $H^1(\partial\mathcal{U})$, especially

$$\mathbf{N}_2^{\mathrm{adj}}w(t) \in H^1(\mathbb{R}^{d-1}).$$

In this context we recall that $(t, y) = (t, x_1, ..., x_{d-1}) \in \mathbb{R}_t \times \mathbb{R}^{d-1}$. To verify the existence of the front, we consider for $\gamma \geq 1$ the mapping

$$(4.9)\qquad \begin{pmatrix} \check{\nabla}^0_{\gamma} \\ \check{\Delta} \end{pmatrix} : H^{1/2}(\partial\mathcal{U}) \cap L^2(\mathbb{R}_t, H^1(\mathbb{R}^{d-1})) \to H^{-1/2}(\partial\mathcal{U}) \times L^2(\mathbb{R}_t, H^{-1}(\mathbb{R}^{d-1}))$$

defined by

$$(4.10)\qquad \psi \mapsto \begin{pmatrix} \check{\nabla}^0_{\gamma}\psi \\ \check{\Delta}\psi \end{pmatrix}.$$

This mapping is well defined and self-adjoint in the sense, that the range of this mapping is given by

$$\left(\ker\begin{pmatrix} \check{\nabla}^0_{\gamma} \\ \check{\Delta} \end{pmatrix}\right)^{\perp},$$

see also the appendix. Here the orthogonal complement has to be understood in the product space $H^{-1/2}(\partial\mathcal{U}) \times L^2(\mathbb{R}_t, H^{-1}(\mathbb{R}^{d-1}))$. Especially, the range can be written as

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{pmatrix} H^{-1/2}(\partial\mathcal{U}) \\ L^2(\mathbb{R}_t, H^{-1}(\mathbb{R}^{d-1})) \end{pmatrix} \;\middle|\; \begin{array}{c} \langle u,\, \theta_1\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} + \int_{\mathbb{R}_t} \langle v(t),\, \theta_2(t)\rangle_{H^{-1}\times H^1}\,\mathrm{d}t = 0 \\ \text{for all } \theta_1 \in H^{1/2}(\partial\mathcal{U}) \text{ and } \theta_2 \in L^2(\mathbb{R}_t, H^{-1}(\mathbb{R}^{d-1})) \\ \text{such that } \check{\nabla}^0_{-\gamma}\cdot\theta_1 = 0 \text{ and } \check{\Delta}\theta_2 = 0 \end{array} \right\}.$$

For the proof of this relation we use the result from [5, Lemma 12.3] and the fact of the selfadjointness of the mapping

$$-\check{\Delta} : H^2(\mathbb{R}^{d-1}) \subset L^2(\mathbb{R}^{d-1}) \to L^2(\mathbb{R}^{d-1}),$$

such that we can write the range as the orthogonal complement of the kernel. The mapping (the proof can be found in [5, Lemma 12.3])

$$\check{\nabla}^0_{\gamma} : H^{1/2}(\partial\mathcal{U}) \to H^{-1/2}(\partial\mathcal{U})$$

has the range

$$\left(\ker(\check{\nabla}^0_{\gamma})\right)^{\perp} = \left\{ u \in H^{-1/2}(\partial\mathcal{U}) \;\middle|\; \begin{array}{c} \langle u,\, \theta\rangle_{H^{-1/2}(\partial\mathcal{U})\times H^{1/2}(\partial\mathcal{U})} = 0 \\ \text{for all}\, \theta \in H^{1/2}(\partial\mathcal{U})\, \text{such that}\, \check{\nabla}^0_{\gamma}\cdot\theta = 0 \end{array} \right\}.$$

In comparison to (4.8) we see that

$$\mathcal{B}^{-1}\mathbf{MU} - \mathcal{B}^{-1}e^{-\gamma t}\mathbf{g}$$

is in the range of the mapping (4.9), i.e. there exists $\mathrm{X} \in H^{1/2}(\partial\mathcal{U}) \cap L^2(\mathbb{R}_t, H^1(\mathbb{R}^{d-1}))$ such that

$$\begin{pmatrix} \check{\nabla}^0_{\gamma} \\ \check{\Delta} \end{pmatrix} \mathrm{X} + \mathcal{B}^{-1}\mathbf{MU} = \mathcal{B}^{-1}e^{-\gamma t}\mathbf{g}$$

on the boundary. That means we have found with $\mathbf{U}$ and X, respectively $e^{-\gamma t}\mathbf{U}$ and $e^{-\gamma t}\mathrm{X}$ a solution of the weighted boundary value problem (4.1). This completes the proof of theorem 1. ☐

5. Regularity and Initial Boundary Value Problem

At this point, we can not say anything about the uniqueness for the found weak solutions, since we can not apply the energy estimate (3.24). Therefore we have to show that the found weak solutions are strong in the sense of finding a weak solution with the right regularity such that we can apply the energy estimate, for instance see also [5, 8, 20]. We smooth the solutions from theorem 1 by using mollifiers in the (y,t)-variables. As in the standard theory, we denote by

$$(\mathbf{U}^\varepsilon, X^\varepsilon)$$

the regularized solutions, such that for $\varepsilon > 0$

$$e^{-\gamma t}\mathbf{U}^\varepsilon \in H^1(\mathcal{U}) \qquad \text{and} \qquad e^{-\gamma t}X^\varepsilon \in H^1(\partial\mathcal{U}) \cap L^2(\mathbb{R}_t, H^2(\mathbb{R}^{d-1})),$$

and also the energy estimate (3.24) is satisfied. Now we can study the convergence of the sequence $(e^{-\gamma t}\mathbf{U}^\varepsilon, e^{-\gamma t}X^\varepsilon)$ for $\varepsilon \to 0$. We follow the results given by [5, 20]. We see that $e^{-\gamma t}\mathbf{U}^\varepsilon$ and $e^{-\gamma t}X^\varepsilon$ are Cauchy sequences by using the energy estimate (3.24). The fact of uniqueness of limits implies that

$$e^{-\gamma t}\mathbf{U}^\varepsilon \to e^{-\gamma t}\mathbf{U} \quad \text{in} \quad L^2(\mathcal{U}) \qquad \Rightarrow \qquad e^{-\gamma t}\mathbf{U} \in H^1(\mathcal{U})$$

and

$$e^{-\gamma t}X^\varepsilon \to e^{-\gamma t}X \quad \text{in} \quad H^{1/2}(\partial\mathcal{U}) \cap L^2(\mathbb{R}_t, H^1(\mathbb{R}^{d-1})) \quad \Rightarrow \quad e^{-\gamma t}X \in H^1(\partial\mathcal{U}) \cap L^2(\mathbb{R}_t, H^2(\mathbb{R}^{d-1})).$$

Furthermore we have

$$\mathbf{L}_\gamma[e^{-\gamma t}\mathbf{U}^\varepsilon] \to e^{-\gamma t}\mathbf{f} \quad \text{in} \quad L^2(\mathcal{U}), \quad e^{-\gamma t}\mathbf{U}^\varepsilon(z=0) \to e^{-\gamma t}\mathbf{U}(z=0) \quad \text{in} \quad H^{-1/2}(\partial\mathcal{U})$$

and

$$\mathbf{b}[e^{-\gamma t}X^\varepsilon] + e^{-\gamma t}\mathbf{M}\mathbf{U}^\varepsilon + \gamma e^{-\gamma t}\mathbf{b}_0 X^\varepsilon \to e^{-\gamma t}\mathbf{g} \quad \text{in} \quad L^2(\partial\mathcal{U}).$$

Altogether we have found a strong solution of the weighted system (4.1) with the regularity

$$(e^{-\gamma t}\mathbf{U}, e^{-\gamma t}\mathrm{X}) \in H^1(\mathcal{U}) \times \left(H^1(\partial\mathcal{U}) \cap L^2\left(\mathbb{R}_t, H^2(\mathbb{R}^{d-1})\right)\right).$$

Now we are going to study the initial boundary value problem, since up to now we have ignored possible initial conditions. The main result for the initial boundary value problem is given by the following corollary.

Corollary 7 (Existence for the initial boundary value problem)**.** *Let the assumptions as in theorem 1 be satisfied. Consider for fixed $T > 0$*

$$\mathbf{f} \in L^2\left((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}^+\right) \qquad \textit{and} \qquad \mathbf{g} \in L^2\left((0,T) \times \mathbb{R}^{d-1}\right)$$

and

$$\mathbf{U}_0 \in L^2\left(\mathbb{R}^{d-1} \times \mathbb{R}^+\right) \qquad \textit{and} \qquad X_0 \in H^{3/2}\left(\mathbb{R}^{d-1}\right).$$

The system

$$\begin{cases} \mathbf{L}[\mathbf{U}] = \boldsymbol{f}, & \textit{for} \quad z > 0,\ t \in (0,T), \\ \mathbf{b}[\mathrm{X}] + \mathbf{M}\mathbf{U} = \mathbf{g}, & \textit{for} \quad z = 0,\ t \in (0,T), \end{cases} \tag{5.1}$$

with inital conditions

$$\mathbf{U}(t=0) = \mathbf{U}_0 \qquad \textit{and} \qquad X(t=0) = X_0$$

has a unique solution $(\mathbf{U}, X) \in L^2((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}^+) \times \left(H^1((0,T) \times \mathbb{R}^{d-1}) \cap L^2\left((0,T), H^2(\mathbb{R}^{d-1})\right)\right)$ which satisfies also $\mathbf{U} \in C^0([0,T], L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+))$.

Further, we have for all $t \in [0,T]$ and all $\gamma \geq \gamma_0$ that the solution satisfies for a constant $C > 0$

$$\begin{aligned} & e^{-2\gamma t}\|\mathbf{U}(t)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} + \gamma\|\mathbf{U}\|^2_{L^2_\gamma((0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^+)} + \|\mathbf{U}(z=0)\|^2_{L^2_\gamma((0,T)\times\mathbb{R}^{d-1})} \\ & + \|X\|^2_{H^1_\gamma((0,T)\times\mathbb{R}^{d-1})} + \frac{1}{\gamma^2}\|X\|^2_{L^2((0,T),H^2_\gamma(\mathbb{R}^{d-1}))} \\ \leq \ & C\left(\frac{1}{\gamma}\|\mathbf{f}\|^2_{L^2_\gamma((0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^+)} + \|\mathbf{g}\|^2_{L^2_\gamma((0,T)\times\mathbb{R}^{d-1})} + \|\mathbf{U}_0\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} + \|X_0\|^2_{H^{3/2}(\mathbb{R}^{d-1})}\right). \end{aligned} \tag{5.2}$$

We note that we used the abbreviation L^2_γ for the weighted space $e^{\gamma t}L^2$.

Proof. The proof of this theorem can be handled as in [20, 5, 8]. First one shows that the solutions for the boundary value problem without initial data vanish for vanishing data in $t < 0$, see also [5]. After that one considers the initial boundary value problem and uses the fact of symmetrizability of the system to get the energy estimate as in [20]. (5.2) relies on the improved estimate (3.24). $\square$

The estimate (5.2) in corollary 7 can be seen as the natural energy estimate for the linearized problem (5.1) (see (2.24)-(2.30) for the original hydromechanical formulation). In particular the interfacial quantities

$$\|\mathbf{U}(z=0)\|^2_{L^2_\gamma((0,T)\times\mathbb{R}^{d-1})} \quad \text{and} \quad \frac{1}{\gamma^2}\|X\|^2_{L^2((0,T),H^2_\gamma(\mathbb{R}^{d-1}))}$$

are remarkable. They represent the entropy dissipation rate and the energy associated to surface tension. This expresses the stability effect of both mechanisms.

Appendix. Range as the orthogonal complement of the kernel

In section 4 we use some Hilbert space analysis for product spaces which is summarized here for the sake of completeness. We consider a more general densely defined mapping

$$A : X \cap Y \to X' \times Y',$$

where X, Y are real Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_X, \langle\cdot,\cdot\rangle_Y$ and subspaces of $L^2(\mathcal{U})$. We assume that $C_0^\infty(\mathcal{U}) \subset X \cap Y$ is not trivial and that the ranges $R(A_1)$ and $R(A_2)$ of the mappings

$$A_1 : X \to X' \qquad \text{and} \qquad A_2 : Y \to Y'$$

are known. It is easy to see that $R(A) = R(A_1) \times R(A_2)$, where we note that real Hilbert spaces are isomorphic to their dual spaces, such that the ranges $R(A_1)$ and $R(A_2)$ can be expressed by using their scalar product as for the mapping (4.9), respectively operator. The selfadjointness for A is clear if A_1 and A_2 are selfadjoint, we have to give the hint that the scalar product for $X \times Y$ has to be understood as

$$\langle\cdot,\cdot\rangle_X + \langle\cdot,\cdot\rangle_Y$$

for real Hilbert spaces X, Y.

Here also note that the selfadjointness of $-\check{\Delta}$ implies the selfadjointness of the operator (4.9) by the isomorphism of real Hilbert spaces to their dual spaces. The fact that the range can be written as the orthogonal complement is clear for selfadjoint operators, i.e.

$$R(A) = (\ker A)^\perp.$$

References

[1] R. Abeyratne and J. K. Knowles. *Kinetic relations and the propagation of phase boundaries in solids. Arch. Rat. Mech. Anal.*, 114:119 – 154, 1991.

[2] S. Benzoni-Gavage. *Stability of multi-dimensional phase transitions in a Van der Waals fluid. Nonl. Anal.: T.M.A.*, 31(1/2):243 – 263, 1998.

[3] S. Benzoni-Gavage. *Stability of subsonic planar phase phase boundaries in a Van der Waals fluid . Arch. Rat. Mech. Anal.*, 150(1):23 – 55, 1999.

[4] S. Benzoni-Gavage and H. Freistühler. *Effects of surface tension on the stability of dynamical liquid-vapor interfaces. Arch. Rat. Mech. Anal.*, 174(1):111 – 150, 2004.

[5] S. Benzoni-Gavage and D. Serre. *Multi-dimensional hyperbolic partial differential equations: First-order systems and applications. Oxford Mathematical Monographs*, 2001. Clarendon Press.

[6] J. Chazarain and A. Piriou. *Introduction to the theory of linear partial differential equations. Studies in Mathematics and its Applications*, 14. Elsevier.

[7] R. M. Colombo and A. Corli. *Continuous dependence in conservation laws with phase transitions. SIAM J. Math. Anal.*, 31(1):34–62, 1999.

[8] J.-F. Coulombel. *Stability of multidimensional undercompressive shock waves. Interfaces and Free Boundaries*, 5(4):367 – 390, 2003.

[9] W. Dreyer, M. Hantke, and G. Warnecke. *Exact solutions to the Riemann problem for compressible isothermal Euler equations for two phase flows with and without phase transition. J. Hyp. Diff. Eq.*, 9:403 – 449, 2012.

[10] H. Freistühler. *A short note on the persistence of ideal shock waves. Arch. Math.*, 64:344 – 352, 1995.

[11] H. Freistühler and R. Plaza. *Normal modes and nonlinear stability behavior of dynamical phase boundaries in elastic materials. Arch. Rat. Mech. Anal.*, 186:1 – 24, 2007.

[12] K. O. Friedrichs. *Symmetric positive linear differential equations. Comm. on Pure and Appl. Math.*, 11(3):333 – 418, 1958.

[13] B. Kabil and C. Rohde. *The influence of surface tension and configurational forces on the stability of liquid-vapor interfaces. Nonl. Anal.: T.M.A.*, 107(0):63 – 75, 2013.

[14] H. O. Kreiss. *Initial boundary value problems for hyperbolic systems. Comm. on Pure and Appl. Math.*, 23:277 – 298, 1970.

[15] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves. CBMS-NSF*, 1973. SIAM.

[16] P. LeFloch. *Propagating phase boundaries: formulation of the problem and existence via the Glimm method. Arch. Rational Mech. Anal.*, 123(2):153–197, 1993.

[17] A. Majda. *The existence of multidimensional shock fronts. Mem. Amer. Math. Soc.*, 43(281), 1983.

[18] A. Majda. *The stability of multidimensional shock fronts. Mem. Amer. Math. Soc.*, 41(275), 1984.

[19] C. Merkle and C. Rohde. *The sharp-interface approach for fluids with phase change: Riemann problems and ghost fluid techniques. M2AN Math. Model. Numer. Anal.*, 41(6):1089–1123, 2007.

[20] G. Métivier. *Stability of multidimensional shocks. Advances in the Theory of Shock Waves*, 47:25 – 103, 2001.

[21] R. Renardy. *An introduction to partial differential equations. Texts in applied mathematics*, 13, 2004. Springer.

[22] L. Truskinovsky. *Shock induced transitions and phase structures in general media. Springer*, pages 185 – 229, 1993.

[23] Y.-G. Wang and Z. Xin. *Stability and existence of multidimensional subsonic phase transitions. Acta Math. Appl. Sin. Engl. Ser.*, 19(4):529–558, 2003.

[24] S.-Y. Zhang. *Existence of multidimensional phase transitions in a steady Van Der Waals flow. Dyn. Partial Differ. Equ.*, 10(1):79–97, 2013.

[25] S.-Y. Zhang and Y.-G. Wang. *Existence of multidimensional shock waves and phase boundaries. J. Differential Equations*, 244(7):1571–1602, 2008.

Buğra Kabil
Pfaffenwaldring 57
70569 Stuttgart, Germany
E-Mail: Bugra.Kabil@mathematik.uni-stuttgart.de
WWW: www.mathematik.uni-stuttgart.de/fak8/ians/lehrstuhl/LstAngMath/mitarbeiter/kabil/

Christian Rohde
Pfaffenwaldring 57
70569 Stuttgart, Germany
E-Mail: Christian.Rohde@mathematik.uni-stuttgart.de
WWW: www.mathematik.uni-stuttgart.de/fak8/ians/lehrstuhl/LstAngMath/mitarbeiter/rohde/

Buğra Kabil
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: Bugra.Kabil@mathematik.uni-stuttgart.de
WWW: http:www.mathematik.uni-stuttgart.de/fak8/ians/lehrstuhl/LstAngMath/mitarbeiter/kabil/

Christian Rohde
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: crohde@mathematik.uni-stuttgart.de
WWW: http://www.mathematik.uni-stuttgart.de/fak8/ians/lehrstuhl/LstAngMath/mitarbeiter/rohde

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