

**Universität
Stuttgart**

**Fachbereich
Mathematik**

Some remarks on polar actions

Claudio Gorodski, Andreas Kollross

Preprint 2015/010

Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: <http://www.mathematik.uni-stuttgart.de/preprints>

ISSN **1613-8309**

© Alle Rechte vorbehalten. Nachdruck nur mit Genehmigung des Autors.
L^AT_EX-Style: Winfried Geis, Thomas Merkle

SOME REMARKS ON POLAR ACTIONS

CLAUDIO GORODSKI AND ANDREAS KOLLROSS

ABSTRACT. We classify infinitesimally polar actions on compact Riemannian symmetric spaces of rank one. We also prove that every polar action on one of those spaces has the same orbits as an asystatic action.

1. INTRODUCTION AND RESULTS

A Riemannian orbifold is a metric space which is locally modeled on quotients of Riemannian manifolds by finite groups of isometries. It has been shown by Lytchak and Thorbergsson [LT10] that the orbit space of a proper and isometric action of a Lie group on a Riemannian manifold, with the quotient metric space structure, is a Riemannian orbifold if and only if the action is *infinitesimally polar*, which means that all of its slice representations are polar. Recall that a proper and isometric action of a Lie group on a complete Riemannian manifold is called *polar* if there exists a connected complete isometrically immersed submanifold, called a *section*, meeting all orbits and always orthogonally. A section is automatically totally geodesic, and if it is flat in the induced metric then the action is called *hyperpolar* [PT88, BCO03].

Infinitesimally polar actions of connected compact Lie groups on Euclidean spheres have been classified in [GL14]. It is obvious that infinitesimally polar actions on real projective spaces are exactly those actions induced from infinitesimally polar actions on spheres. In the present paper, we classify infinitesimally polar actions on the remaining compact rank one symmetric spaces. Note that general polar actions are infinitesimally polar ([PT88, Thm 4.6] or [BCO03, Prop. 3.2.2]), and polar actions on compact rank one symmetric spaces have been classified by Podestà and Thorbergsson in [PT99] (although they have overlooked one case, cf. subsection 3.4). Our result is as follows.

Theorem 1.1. *Assume a compact connected Lie group acts isometrically and effectively on a compact rank one symmetric space M .*

- (a) *If M is a complex projective space $\mathbb{C}P^m$, then the action is infinitesimally polar if and only if it is polar or it is orbit equivalent to the action induced from one of the following representations ρ of G :*

G	ρ	<i>Conditions</i>	m	<i>Orbit space</i>
$\mathrm{SO}(2) \times \mathrm{Spin}(9)$	$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^{16}$	—	15	$S_{+++}^2(\frac{1}{2})$
$\mathrm{U}(2) \times \mathrm{Sp}(n)$	$\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^{2n}$	$n \geq 2$	$4n - 1$	$S_{+++}^2(\frac{1}{2})$
$\mathrm{U}(n)$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$n \geq 2$	$2n - 1$	$S_+^3(\frac{1}{2})$
$\mathrm{T}^2 \times \mathrm{SU}(n)$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$n \geq 2$	$2n - 1$	$S_{++}^2(\frac{1}{2})$
$\mathrm{U}(1) \times \mathrm{Sp}(n)$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$n \geq 2$	$4n - 1$	$S_{++}^4(\frac{1}{2})$
$\mathrm{T}^2 \times \mathrm{Sp}(n)$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$n \geq 2$	$4n - 1$	$S_{+++}^3(\frac{1}{2})$

TABLE 1

Date: May 12, 2015.

2010 Mathematics Subject Classification. 53C35, 57S15.

Key words and phrases. Cayley plane, polar action, asystatic action, compact rank one symmetric space.

The first author was partially supported by the CNPq grant 303038/2013-6 and FAPESP project 2011/21362-2.

- (b) If M is a quaternionic projective space $\mathbb{H}P^m$ with $m > 1$, then the action is infinitesimally polar if and only if it is polar or it is orbit equivalent to the action induced from the following representation ρ of G :

G	ρ	Conditions	m	Orbit space
$\mathrm{Sp}(n) \times \mathrm{Sp}(1)$	$\mathbb{R}^{4n} \oplus \mathbb{R}^{4n}$	$n \geq 2$	$2n - 1$	$S_{++}^3(\frac{1}{2})$

TABLE 2

- (c) If M is the Cayley projective plane $\mathbb{O}P^2$, then the action is infinitesimally polar if and only if it is polar. In this case the action is conjugate to one given by the following subgroups G of F_4 :

G	Cohom	Type	Multiplicities
$\mathrm{Spin}(9)$	1	\tilde{A}_1	$\begin{array}{c} 15 \\ \circ \text{---} \infty \text{---} \circ \\ 7 \end{array}$
$\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ $\mathrm{Sp}(3) \cdot \mathrm{U}(1)$ $\mathrm{Sp}(3)$	1	\tilde{A}_1	$\begin{array}{c} 7 \\ \circ \text{---} \infty \text{---} \circ \\ 4 \end{array}$
$\mathrm{Spin}(8)$	2	$A_1 \times A_1 \times A_1$	$\begin{array}{ccc} 7 & 7 & 7 \\ \circ & \circ & \circ \end{array}$
$\mathrm{Spin}(7) \cdot \mathrm{SO}(2)$	2	$A_1 \times C_2$	$\begin{array}{ccc} 7 & 6 & 1 \\ \circ & \circ \text{---} \text{---} \circ \end{array}$
$\mathrm{SU}(4) \cdot \mathrm{SU}(2)$	2	$A_1 \times C_2$	$\begin{array}{ccc} 7 & 5 & 2 \\ \circ & \circ \text{---} \text{---} \circ \end{array}$
$\mathrm{SU}(3) \cdot \mathrm{SU}(3)$	2	C_3	$\begin{array}{ccc} 2 & 2 & 3 \\ \circ \text{---} \text{---} \circ \text{---} \text{---} \circ \end{array}$
$\mathrm{SO}(3) \cdot G_2$	2	C_3	$\begin{array}{ccc} 1 & 1 & 5 \\ \circ \text{---} \text{---} \circ \text{---} \text{---} \circ \end{array}$

TABLE 3

The notation $S_+^n(r)$, $S_{++}^n(r)$ and $S_{+++}^n(r)$ in Tables 1 and 2 stands for the quotient of the n -sphere of radius r by a group generated by 1, 2, resp. 3, commuting reflections. In Table 3 we indicate the type of the Coxeter group acting on the universal covering of the section and its multiplicities (these are the dimensions of the unit spheres in the normal space, at a generic point, to a sub-principal stratum of M — that is, a stratum in M projecting to a codimension one stratum in the orbit space).

Following Thurston [Thu80], we call a Riemannian orbifold *good* if it is globally isometric to the quotient of a Riemannian manifold by a discrete group of isometries.

Corollary 1.2. *An infinitesimally polar action on a compact rank one symmetric space has a good Riemannian orbifold as a quotient.*

It is relevant to mention the fundamental work of Dadok, who classified polar representations [Dad85]. It follows from his result that a polar representation of a connected compact Lie group is orbit equivalent to the isotropy representation of a Riemannian symmetric space. Herein we say that two isometric Lie group actions on Riemannian manifolds are *orbit equivalent* if they have the same orbits after a suitable isometric identification of the manifolds.

Recall that a homogeneous manifold is called *asystatic* if sufficiently close points in the manifold have different isotropy groups; equivalent definitions are: that the manifold has no nonzero invariant vector field; or the isotropy representation has no nonzero fixed vectors; or that the

normalizer of the isotropy group is a discrete extension thereof (this concept can be traced back to S. Lie [LE88, p.501]; see also [PT02]). Finally, a proper action of a Lie group on a smooth manifold is called *asystatic* if one (and hence all) principal orbits are asystatic homogeneous manifolds [AA93a]. The relevance to us is that asystatic actions are automatically polar with respect to any invariant metric: a section is given by a connected component of the set of fixed points which contains regular points of the action (therefore, in [PT87] they are called *G-manifolds with canonical sections*). Note also that the asystatic property actually depends on the group and not only on its orbits. It follows from our discussion below that:

Scholium 1.3. *Every polar action of a compact connected Lie group on a compact rank one symmetric space is orbit equivalent to an asystatic action.*

More precisely, each group with a polar action on such a space admits an extension, by a finite group, which acts with the same orbits and asystatically. The action of the original group will thus be polar with respect to any Riemannian metric invariant under the enlarged group.

Note that the sections of asystatic actions are automatically properly embedded submanifolds, but this is not true for general polar actions, see e.g. [GT02, p. 47]. It is an interesting open question, communicated to us by W. Ziller, to decide whether sections of polar actions can admit self-intersections (in the case of cohomogeneity one actions it is known that they cannot [AA93b, Thm. 6.1]). Since sections of polar actions on compact irreducible symmetric spaces of higher rank are known to be properly embedded [HPTT95, KL12], one can ask:

Questions 1.4. Can the result in Scholium 1.3 be extended to the case of compact irreducible symmetric spaces of higher rank? What is the most general class of polar actions on complete Riemannian manifolds for which it is true?

The authors would like to express their gratitude to Alexander Lytchak for very informative discussions and his kind hospitality during their stay at the University of Cologne.

2. ACTIONS ON CLASSICAL PROJECTIVE SPACES

2.1. Infinitesimally polar actions. We view the complex and quaternionic projective spaces as quotients of unit spheres under the corresponding Hopf actions so that their quotient Riemannian metrics have sectional curvatures lying between 1 and 4. Moreover, a 2-plane with sectional curvature equal to 1 must be totally real.

Assume an isometric Lie group action on $\mathbb{C}P^m$ is given by specifying a closed subgroup G of $SU(m+1)$. This action lifts, via the Hopf fibration, to an isometric action of the group $\tilde{G} := G \times U(1)$ on S^{2m+1} with the same orbit space, where we can view S^{2m+1} as the unit sphere in \mathbb{C}^{m+1} and $U(1)$ acts by multiplication by unit complex numbers. It follows that the G -action on $\mathbb{C}P^m$ is infinitesimally polar if and only if the \tilde{G} -action on S^{2m+1} is infinitesimally polar. Moreover, it follows from O’Neill’s formula for Riemannian submersions that an isometric action on the unit sphere is polar if and only if it has cohomogeneity one or the principal stratum of the orbit space has constant sectional curvature 1, cf. [GL15, Introd.]. Therefore another application of O’Neill’s formula gives that polarity of the \tilde{G} -action on S^{2m+1} implies polarity of the G -action on $\mathbb{C}P^m$ with totally real sections. Conversely, a polar action on $\mathbb{C}P^m$ must have totally real sections (see [PT02, Thm. 1.1] or [Lyt14, Prop. 9.1]) and thus the lifted action on S^{2m+1} is polar.

Similarly, let an isometric Lie group action $\mathbb{H}P^m$ be given by specifying a closed subgroup G of $Sp(m+1)$. This action lifts, via the Hopf fibration, to an isometric action of $\tilde{G} := G \times Sp(1)$ on S^{4m+3} with the same orbit space, where we can view S^{4m+3} as the unit sphere in \mathbb{H}^{m+1} and $Sp(1)$ acts by (right) multiplication by unit quaternion numbers. It follows as above that the G -action on $\mathbb{H}P^m$ ($m > 1$, cf. [PT99, p. 161]) is non-polar and infinitesimally polar if and only if the \tilde{G} -action on S^{4m+3} is non-polar and infinitesimally polar.

Therefore Tables 1 and 2 follow from [GL14, Th. 1.3] and [Str94, Table II].

2.2. Asystatic actions. Straume proved that a polar representation is orbit equivalent to an asystatic representation, using the following argument; see [Str94, Completion of proof of Thm. 1.3, pp. 11-12], although he does not use the word “asystatic” explicitly. By [Dad85], one may replace

the representation by one with the same orbits which is the isotropy representation of a symmetric space. In the irreducible case, one checks case-by-case that this representation is already asystatic unless it is of Hermitian type, in which case it becomes asystatic after adjunction of an extra element to the group, namely, complex conjugation, without changing its orbits. In the reducible case, the representation splits as the direct product of irreducible isotropy representations of symmetric spaces, and it also becomes asystatic by adjoining one single extra element that acts as complex conjugation.

As a consequence of Straume's result, also the polar actions on classical projective spaces are asystatic. In fact, a polar G -action, where $G \subset \mathrm{SU}(m+1)$, on a complex projective space $M = \mathbb{C}P^m$ lifts to a polar \tilde{G} -action on the corresponding sphere $\tilde{M} = S^{2m+1}$, which can be replaced by a group acting with the same orbits via the isotropy representation of an Hermitian symmetric space, see [PT99, Theorem 3.1]. In its turn, the latter group can by the above be enlarged to a group \tilde{K} acting asystatically on \tilde{M} with the same orbits, by adjunction of complex conjugation, and which induces an action of a group K on M orbit equivalent to the original G -action. If $\tilde{p} \in \tilde{M}$, $p \in M$ are \tilde{K} -regular, resp. K -regular points, where \tilde{p} lies above p and $\tilde{H} = \tilde{K}_{\tilde{p}}$, $H = K_p$ are the associated principal isotropy groups, then the projection $\tilde{K} \rightarrow K$ induces an isomorphism $\tilde{H} \cong H$ with respect to which the isomorphism $T_p(\tilde{K} \cdot \tilde{p}) \cap \mathcal{H}_{\tilde{p}} \cong T_p(K \cdot p)$ (induced by the projection $\tilde{M} \rightarrow M$, where \mathcal{H} denotes the horizontal distribution) is equivariant. Hence there exist no H -fixed directions in $T_p(K \cdot p)$, i.e. K acts asystatically on M .

For actions on quaternionic projective space, we may use an analogous method, however, the argument has to be somewhat refined. Assume now $G \subset \mathrm{Sp}(m+1)$ acts polarly on the quaternionic projective space $M = \mathbb{H}P^m$. Then the action lifts to a polar \tilde{G} -action on the corresponding sphere $\tilde{M} = S^{4m+3}$. This action can be replaced by a group acting on $\mathbb{H}^{m+1} = \mathbb{H}^{m_1} \oplus \dots \oplus \mathbb{H}^{m_r}$ with the same orbits via the isotropy representation of the product of r quaternion Kähler symmetric spaces Q_1, \dots, Q_r , where Q_1, \dots, Q_{r-1} are of rank one and Q_r can be of arbitrary rank, cf. [PT99, Theorem 4.1]. The isotropy representations of quaternion Kähler symmetric spaces are known to be asystatic in all but one case: there is exactly one case of quaternion Kähler symmetric space which is also a Hermitian symmetric space, namely, the Grassmann manifold $\mathrm{SU}(n+2)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(2))$ of complex 2-planes in \mathbb{C}^{n+2} , in which we need to pass to a \mathbb{Z}_2 -extension to make it asystatic. However, as remarked in [PT99] after Proposition 2A.2, this action does not descend to M if $r \geq 2$. To remedy this, one takes the subgroup $K := \mathrm{Sp}(m_1) \times \dots \times \mathrm{Sp}(m_{r-1}) \times H_r \times \mathrm{Sp}(1)$ which acts on \mathbb{H}^{m+1} with the same orbits, where the $\mathrm{Sp}(1)$ -factor acts by right quaternionic multiplication and $(K_r = H_r \times \mathrm{Sp}(1), \mathbb{H}^{m_r} \otimes_{\mathbb{H}} \mathbb{H})$ corresponds to the isotropy representation of Q_r . The K -action on \tilde{M} ,

$$(K = \mathrm{Sp}(m_1) \times \dots \times \mathrm{Sp}(m_{r-1}) \times H_r \times \mathrm{Sp}(1), \mathbb{H}^{m+1} = \mathbb{H}^{m_1} \oplus \dots \oplus \mathbb{H}^{m_r}),$$

descends to an action on M which is orbit equivalent to the original action; let us show that this K -action is asystatic. Up to conjugation of H_r in $\mathrm{Sp}(m_r)$, we may assume that there exists a regular point $p \in \mathbb{H}^{m+1}$ of the form (p_1, \dots, p_r) , where each p_ℓ is the first element of the canonical basis of \mathbb{H}^{m_ℓ} . Then the corresponding principal isotropy group K_p is isomorphic to

$$\mathrm{Sp}(m_1 - 1) \times \dots \times \mathrm{Sp}(m_{r-1}) \times (K_r)_{p_r},$$

where $(K_r)_{p_r}$ is the principal isotropy group of $(K_r, \mathbb{H}^{m_r} \otimes_{\mathbb{H}} \mathbb{H})$. The group K_p acts on $\mathbb{H}^{m+1} = \mathbb{H}^{m_1} \oplus \dots \oplus \mathbb{H}^{m_r}$ by

$$\begin{aligned} & (A_1, \dots, A_{r-1}, (A_r, q)) \cdot (x_1, \dots, x_r) = \\ & = ((qx_1^0 q^{-1}, A_1 x_1' q^{-1}), \dots, (qx_{r-1}^0 q^{-1}, A_{r-1} x_{r-1}' q^{-1}), A_r x_r q^{-1}), \end{aligned}$$

where $A_\ell \in \mathrm{Sp}(m_\ell - 1)$ for $\ell < r$, $(A_r, q) \in (K_r)_{p_r} \subset H_r \times \mathrm{Sp}(1)$, $x_\ell = (x_\ell^0, x_\ell') \in \mathbb{H} \oplus \mathbb{H}^{m_\ell - 1}$ for $\ell < r$. By polarity, the tangent space to the K -principal orbit through p is the direct sum of the tangent spaces of the K -orbits through the p_ℓ . First assume Q_r is not a Hermitian symmetric space, namely, it is not the Grassmann manifold of complex 2-planes in \mathbb{C}^{m_r+2} . Then, by [Str94], the K_r -action on \mathbb{H}^{m_r} is asystatic. Therefore the $(K_r)_{p_r}$ -action on $T_{p_r}(K_r \cdot p_r)$ has no fixed directions; note that the $(K_r)_{p_r}$ -action on \mathbb{H}^{m_r} is the effectivized K_p -action on that space. In order to see that K_p has no fixed tangent directions in the other summands $\mathbb{H}^{m_1}, \dots, \mathbb{H}^{m_{r-1}}$, we argue as

follows. By [Teb07], the section of (K_r, \mathbb{H}^{m_r}) through p_r is totally real, namely, it is orthogonal to its image under right multiplication by an imaginary unit quaternion. By asystaticity, the only $(K_r)_{p_r}$ -fixed direction in the quaternionic span of p_r is $\mathbb{R}p_r$. Since the effective group acting on \mathbb{H}^{m_ℓ} for $\ell < r$ is the full $\mathrm{Sp}(m_\ell) \cdot \mathrm{Sp}(1)$, this behavior is reproduced in \mathbb{H}^{m_ℓ} , namely, the only K_p -fixed direction in the quaternionic span of p_ℓ is $\mathbb{R}p_\ell$. Since the K_p -action on \mathbb{H}^{m_ℓ} contains the subgroup $\{1\} \times \mathrm{Sp}(m_\ell - 1) \subset \mathrm{Sp}(m_\ell)$ which does not fix non-zero vectors in $\{0\} \oplus \mathbb{H}^{m_\ell - 1}$, this already shows that there are no further fixed directions in \mathbb{H}^{m_ℓ} .

In case Q_r is the complex Grassmannian of complex 2-planes in \mathbb{C}^{m_r+2} , we argue analogously by replacing the group K_r with the group \hat{K}_r generated by K_r and the element σ_r that acts by complex conjugation on $\mathbb{H}^{m_r} = \mathbb{C}^{m_r} \oplus \mathbb{C}^{m_r}j$, and trivially on \mathbb{H}^{m_ℓ} for $\ell < r$. Note that σ_r is given on \mathbb{H}^{m_r} by $L_j \circ R_j^{-1}$ (left and right multiplication), which shows that $\sigma_r \in \mathrm{Sp}(m_r) \times \mathrm{Sp}(1)$. The resulting group \hat{K} has the same orbits as K and acts asystatically on \mathbb{H}^{m+1} . It is generated by K and an element σ that acts as $L_j \circ R_j^{-1}$ on each \mathbb{H}^{m_ℓ} (again due to the fact that the effective group on \mathbb{H}^{m_ℓ} for $\ell < r$ is $\mathrm{Sp}(m_\ell) \cdot \mathrm{Sp}(1)$), which fixes p and preserves quaternionic lines in \mathbb{H}^{m+1} , so \hat{K} induces an asystatic action on $\mathbb{H}P^m$.

3. ACTIONS ON THE CAYLEY PROJECTIVE PLANE

We first prove a simple but useful criterion for asystaticity.

Lemma 3.1. *Let G act properly and isometrically on a complete Riemannian manifold M . Assume there exists a point $q \in M$ and a principal isotropy group H which fixes q and acts on the tangent space T_qM with fixed point set of dimension equal to the cohomogeneity of the G -action on M . Then the G -action on M is asystatic.*

Proof. We may assume the action is non-transitive and the point q is not regular. There is a non-zero normal vector $v \in \nu_q(Gq)$ which is regular for the slice representation of G_q and fixed under H . There is a sequence of regular points $(p_n := \exp_q(t_n v))$, where $t_n \rightarrow 0$ as $n \rightarrow \infty$, which converges to q . The isotropy representations $(H = G_{p_n}, T_{p_n}(G \cdot p_n))$ of the principal orbits $G \cdot p_n$ are all equivalent one to the other, for all n . By continuity, the dimension of the fixed point set of H in $T_{p_n}M$ is not larger than the dimension of the fixed point set of H in T_qM , hence it is equal to the cohomogeneity of the G -action on M . \square

We will also use the following lemma [KP03, Th. 6].

Lemma 3.2. *Let $\rho : G \rightarrow \mathrm{O}(V)$ be a faithful irreducible representation of a compact connected Lie group G of cohomogeneity at least 2. Assume the restriction of ρ to a non-trivial closed connected subgroup H is polar. Then the G - and H -actions on V are orbit equivalent.*

We will throughout use Dynkin's tables of maximal connected closed subgroups of compact Lie groups [Dyn00]. The identity component of the isometry group of $M = \mathbb{O}P^2$ is the exceptional Lie group F_4 , and its isotropy group at a fixed basepoint is $\mathrm{Spin}(9)$. The maximal connected closed subgroups of F_4 are, up to conjugacy [Dyn00, Table 12 and Thm. 14.1]:

$$\mathrm{Spin}(9), \quad \mathrm{Sp}(3) \cdot \mathrm{Sp}(1), \quad \mathrm{SU}(3) \cdot \mathrm{SU}(3), \quad \mathrm{G}_2 \cdot \mathrm{A}_1^8, \quad \mathrm{A}_1^{156}.$$

(By the way, the group $\widetilde{\mathrm{SU}(2)} \cdot \mathrm{SU}(4)$ is not maximal and occurs in Dynkin's tables because of a mistake, see e.g. [GR71].) Here A_1 denotes a simple group of rank 1 and the upper index refers to its Dynkin index as a subgroup of F_4 , as defined in [Dyn00].

3.1. $\mathrm{Spin}(9)$ and its subgroups. The action of $\mathrm{Spin}(9)$ is its isotropy action on the homogeneous space $M = F_4/\mathrm{Spin}(9)$. Since M is a symmetric space of rank 1, this action has cohomogeneity 1 and it is thus hyperpolar. The principal orbits are distance spheres $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$ centered at the basepoint, and their isotropy representation decomposes as $\mathbb{R}^7 \oplus \mathbb{R}^8$, where \mathbb{R}^7 denotes the vector representation of $\mathrm{Spin}(7)$ and \mathbb{R}^8 denotes its spin representation. It follows that $\mathrm{Spin}(9)$ acts asystatically on M .

We proceed to consider the maximal connected closed subgroups of $\text{Spin}(9)$ of rank greater than one (cf. subsection 3.5), which are

$$\text{Spin}(8), \quad \text{Spin}(7) \cdot \text{SO}(2), \quad \text{Spin}(6) \cdot \text{Spin}(3), \quad \text{Spin}(5) \cdot \text{Spin}(4)$$

and an embedding of $\text{Sp}(1) \cdot \text{Sp}(1)$ coming from the representation of $\text{SO}(4)$ on the space of real traceless symmetric 4×4 matrices.

3.1.1. $\text{Spin}(8)$. The slice representation of $\text{Spin}(8)$ at the basepoint decomposes as the direct sum of the half-spin representations, so the principal isotropy of its action on M is G_2 . Now the isotropy representation of a principal orbit $\text{Spin}(8)/\text{G}_2$ is easily seen to be $\mathbb{R}^7 \oplus \mathbb{R}^7$. It follows that $\text{Spin}(8)$ acts asystatically on M .

It remains to determine whether subgroups of $\text{Spin}(8)$ can act infinitesimally polar. To this end, we consider the maximal connected subgroups in $\text{Spin}(8)$ of rank greater than one. Since outer automorphisms of $\text{Spin}(8)$ are restrictions of inner automorphisms of F_4 , it suffices to consider maximal connected subgroups up to arbitrary automorphisms of $\text{Spin}(8)$, and they are (cf. [Kol02, Prop. 3.3]):

$$\text{Spin}(7), \quad \text{Spin}(6) \cdot \text{SO}(2), \quad \text{Spin}(5) \cdot \text{Spin}(3), \quad \text{Spin}(4) \cdot \text{Spin}(4), \quad \pi^{-1}(\text{Ad } \text{SU}(3)),$$

where $\pi: \text{Spin}(8) \rightarrow \text{SO}(8)$ is a covering map and where $\text{Ad } \text{SU}(3)$ is the group given by the 8-dimensional irreducible representation of $\text{SU}(3)$.

We do not need to discuss $\text{Spin}(7)$, $\text{Spin}(6) \cdot \text{SO}(2)$, $\text{Spin}(5) \cdot \text{Spin}(3)$, or $\text{Spin}(4) \cdot \text{Spin}(4)$ or any of their closed subgroups now, since each one of them is contained in one of the subgroups $\text{Spin}(7) \cdot \text{SO}(2)$, $\text{Spin}(6) \cdot \text{Spin}(3)$, $\text{Spin}(5) \cdot \text{Spin}(4)$ of $\text{Spin}(9)$, and the latter will be treated below. We do not need to discuss the group $\pi^{-1}(\text{Ad } \text{SU}(3))$, either, since it is contained in $\text{SU}(3) \cdot \text{SU}(3)$, see [Dyn00, p. 195].

3.1.2. $\text{Spin}(7) \cdot \text{SO}(2)$ and $\text{Spin}(6) \cdot \text{Spin}(3)$. For the next two group actions, it will be convenient to use the notion of Weyl involution of a compact Lie group with respect to a maximal torus [Oni04, §2]. Recall that a regular subalgebra of a compact semisimple Lie algebra \mathfrak{g} is a subalgebra which is normalized by a maximal torus of $\text{Int}(\mathfrak{g})$. Subalgebras of maximal rank are obviously regular. It follows easily that for any regular subalgebra, there exists a Weyl involution preserving the subalgebra and restricting to its Weyl involution. We apply these ideas to the case of the group F_4 , where any Weyl involution is an inner automorphism.

The slice representation of $\text{Spin}(7) \cdot \text{SO}(2)$ at the basepoint p is $\mathbb{R}^8 \otimes_{\mathbb{R}} \mathbb{R}^2$, so the principal isotropy group of its action on M is $\text{SU}(3) \rtimes \mathbb{Z}_2$, where $\text{SU}(3) \subset \text{G}_2 \subset \text{Spin}(7)$ and \mathbb{Z}_2 is diagonally embedded in $\text{Spin}(7) \cdot \text{SO}(2)$. One computes the isotropy representation of a principal orbit to be $2\mathbb{C}^3 \oplus 2\mathbb{R}$, where the \mathbb{Z}_2 -factor acts trivially on $2\mathbb{R}$ (one summand corresponds to $\mathfrak{so}(2)$ and the other to the center of $\mathfrak{u}(3) \subset \mathfrak{so}(7)$). This shows the $\text{Spin}(7) \cdot \text{SO}(2)$ -principal orbit is not asystatic. To deal with this, we enlarge $\text{Spin}(7) \cdot \text{SO}(2)$ by adjoining $w \in \text{F}_4$, where $\psi = \text{Ad}_w$ is a Weyl involution of F_4 relative to a maximal torus contained in $\text{Spin}(7) \cdot \text{SO}(2)$. Note that ψ preserves $\text{Spin}(9)$, which does not have outer automorphisms, so we may take $w \in \text{Spin}(9)$. Now ψ induces an isometry of M that maps $\text{Spin}(7) \cdot \text{SO}(2)$ -orbits to $\text{Spin}(7) \cdot \text{SO}(2)$ -orbits, and thus it induces an isometry of the corresponding orbit space. Since ψ fixes the basepoint p , it also induces an isometry of the orbit space of the slice representation of $\text{Spin}(7) \cdot \text{SO}(2)$ at p . The latter is orbit equivalent to $(\text{SO}(8) \times \text{SO}(2), \mathbb{R}^8 \otimes_{\mathbb{R}} \mathbb{R}^2)$ [Dad85, EH99] and its orbit space is thus isometric to the cone over the interval of length $\pi/4$. The only non-trivial isometry interchanges the endpoints of the interval, but such an isometry cannot be induced by ψ because the endpoints parametrize singular orbits of different dimensions, namely, 8 and 13 [HPT88, p. 436]. It follows that ψ preserves the orbits in $T_p M$, and hence preserves the orbits in M by using the exponential map at p . Now the enlarged group has the same orbits in M and its principal isotropy group contains w . Since ψ acts as minus identity on the Lie algebra of the maximal torus of $\text{Spin}(7) \cdot \text{SO}(2)$, this shows that the enlarged group acts asystatically.

Consider now $\text{Spin}(6) \cdot \text{Spin}(3) = \text{SU}(4) \cdot \text{SU}(2)$; this case is very similar to the previous one. Its slice representation at p is $\mathbb{C}^4 \otimes_{\mathbb{C}} \mathbb{C}^2$, so the principal isotropy group of its action on M is $H \cong \text{U}(2)$, and one computes that the isotropy representation of a principal orbit has fixed point set of

dimension 2, namely, the Killing orthogonal of $\mathfrak{h} \cong \mathfrak{u}(2)$ in the Lie algebra of the maximal torus of $\mathrm{SU}(4) \cdot \mathrm{SU}(2)$. We enlarge $\mathrm{SU}(4) \cdot \mathrm{SU}(2)$ by adjoining $w \in \mathrm{Spin}(9)$ where $\psi = \mathrm{Ad}_w$ is a Weyl involution of F_4 relative to a maximal torus contained in $\mathrm{SU}(4) \cdot \mathrm{SU}(2)$. In order to see that the enlarged group has the same orbits, it suffices, as above, to note that ψ fixes p and that it cannot induce a non-trivial isometry of the orbit space of the slice representation at p . Indeed the latter is orbit equivalent to $(\mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(2)), \mathbb{C}^4 \otimes_{\mathbb{C}} \mathbb{C}^2)$ [Dad85, EH99], so its orbit space is isometric to the cone over the interval of length $\pi/4$, where the endpoints parametrize orbits of dimensions 9 and 12 [HPT88, p. 436]. Now the enlarged group is orbit equivalent and its principal isotropy group contains w , so it has no non-zero fixed vectors in the isotropy representation. This proves that the enlarged group acts asystatically on M .

It is now easy to see that $\mathrm{Spin}(7) \cdot \mathrm{SO}(2)$ and $\mathrm{SU}(4) \cdot \mathrm{SU}(2)$ do not have proper closed subgroups acting infinitesimally on M . In fact, their slice representations at p are irreducible of cohomogeneity two, so, owing to Lemma 3.2, any subgroup of one of those groups acting infinitesimally polar on M must be such that the slice representations at p of the group and its subgroup are orbit-equivalent, but according to [Dad85, EH99] there cannot exist such a subgroup.

The groups obtained above by adjoining w to $\mathrm{Spin}(7) \cdot \mathrm{SO}(2)$ and $\mathrm{SU}(4) \cdot \mathrm{SU}(2)$, respectively, are well known: they are just the subgroups $\pi^{-1}(\mathrm{S}(\mathrm{O}(7) \times \mathrm{O}(2)))$ and $\pi^{-1}(\mathrm{S}(\mathrm{O}(6) \times \mathrm{O}(3)))$ of $\mathrm{Spin}(9)$, where $\pi: \mathrm{Spin}(9) \rightarrow \mathrm{SO}(9)$ is the universal covering map. Indeed, it is easy to see that the action of $\psi \in \mathrm{Aut}(\mathrm{Spin}(9))$ is given by conjugation with elements of those subgroups. However, we have preferred the above alternate point of view, as the Weyl involution appears to be a useful notion to prove asystaticity in this context, cf. subsection 3.3.

3.1.3. $\mathrm{Spin}(5) \cdot \mathrm{Spin}(4)$. Consider the restriction of the spin representation of $\mathrm{Spin}(9)$ to the subgroup $\mathrm{Spin}(5) \cdot \mathrm{Spin}(4)$. This representation can be regarded as $\mathbb{H}^2 \otimes_{\mathbb{H}} (\mathbb{H} \oplus \mathbb{H})$, where the representation of $\mathrm{Spin}(5) \cdot \mathrm{Spin}(4) \cong \mathrm{Sp}(2) \cdot (\mathrm{Sp}(1) \times \mathrm{Sp}(1))$ is given in such a way that the $\mathrm{Sp}(2)$ -factor acts on both copies of \mathbb{H}^2 by its standard representation and where the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathbb{H} \oplus \mathbb{H}$ is given componentwise by the standard representation. This reducible representation is the restriction of $(\mathrm{Sp}(2) \cdot \mathrm{Sp}(2), \mathbb{H}^2 \otimes_{\mathbb{H}} \mathbb{H}^2)$, which is irreducible and of cohomogeneity 2, hence it follows from Lemma 3.2 that neither $\mathrm{Spin}(5) \cdot \mathrm{Spin}(4)$ nor any non-trivial connected closed subgroup thereof can act polarly on M .

3.1.4. $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$. This maximal connected subgroup of $\mathrm{Spin}(9)$ does not act infinitesimally polar on M since the restriction of the spin representation of $\mathrm{Spin}(9)$ on $T_p M = \mathbb{R}^{16}$ to $\mathrm{SU}(2) \cdot \mathrm{SU}(2)$ yields $S^3(\mathbb{C}^2) \otimes_{\mathbb{H}} \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes_{\mathbb{H}} S^3(\mathbb{C}^2)$, namely, a sum of two polar representations, each with finite principal isotropy groups, which cannot be polar by [Dad85, Th. 4]; in addition, since each irreducible summand has cohomogeneity two, no subgroup of $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$ can act infinitesimally polar on M (Lemma 3.2).

3.2. $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ and its subgroups. This is a symmetric subgroup of F_4 so that we have a so-called *Hermann action* on M [Kol02]. One singular orbit is a totally geodesic $\mathbb{H}P^2$ which is also the fixed point set of the $\mathrm{Sp}(1)$ -factor. We choose $p \in \mathbb{H}P^2$. Then the isotropy subgroup at p is $\mathrm{Sp}(1) \cdot \mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$, namely, the group described in subsection 3.1.3. Its action on $T_p M$ is a sum of two representations of cohomogeneity one, that is, the isotropy representation of the singular orbit $T_p(\mathbb{H}P^2) = \mathbb{H} \otimes_{\mathbb{H}} \mathbb{H}^2$ and the slice representation $\nu_p(\mathbb{H}P^2) = \mathbb{H}^2 \otimes_{\mathbb{H}} \mathbb{H}$. In particular the $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ -action on M has cohomogeneity one and it is thus hyperpolar. The principal isotropy group is obtained from the slice representation: it is the $\mathrm{Sp}(1)^3$ -subgroup given by $q_2 = q_4$ in the diagonal embedding

$$(q_1, q_2, q_3, q_4) \in \mathrm{Sp}(1)^4 \mapsto (q_1, \mathrm{diag}(q_2, q_3), q_4) \in \mathrm{Sp}(1) \cdot \mathrm{Sp}(2) \cdot \mathrm{Sp}(1).$$

The action of the principal isotropy group on $(x, y) \in \mathbb{H}^2 \cong \nu_p(\mathbb{H}P^2)$ is given by

$$\begin{pmatrix} q_2 & 0 \\ 0 & q_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} q_2^{-1}$$

with fixed point set $\begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}$, and its action on $(x, y) \in \mathbb{H}^2 \cong T_p(\mathbb{H}P^2)$ is given by

$$\begin{pmatrix} q_2 & 0 \\ 0 & q_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} q_1^{-1}$$

with trivial fixed point set. In view of Lemma 3.1, this proves asystaticity of the $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ -action on M . It remains to determine whether there are any proper closed subgroups of $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ which act infinitesimally polar.

The maximal connected subgroups of $\mathrm{Sp}(3)$ are

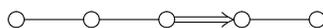
$$\mathrm{Sp}(2) \cdot \mathrm{Sp}(1), \quad \mathrm{U}(3), \quad \mathrm{A}_1^{35}.$$

By [Dyn00, Th. 15.1], it follows that the maximal connected subgroups of $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ are

$$\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1), \quad \mathrm{U}(3) \cdot \mathrm{Sp}(1), \quad \mathrm{A}_1^{35} \cdot \mathrm{Sp}(1), \quad \mathrm{Sp}(3) \cdot \mathrm{U}(1).$$

3.2.1. $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$. This group is exactly the subgroup $\mathrm{Spin}(5) \cdot \mathrm{Spin}(4)$ of $\mathrm{Spin}(9)$, treated in 3.1.3.

3.2.2. $\mathrm{U}(3) \cdot \mathrm{Sp}(1)$. Regular subgroups, in particular, subgroups of maximal rank in a compact semisimple Lie group can be studied using Borel-de-Siebert theory, cf. [Oni94, Ch. 1, §3, Thm. 16]. The maximal connected subgroups of maximal rank are obtained by deleting nodes from the extended Dynkin diagram, the remaining nodes then correspond to a set of simple roots of the subalgebra. The extended Dynkin diagram of F_4 is the following.



The subgroup $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ is obtained by deleting the second node from the left, the subgroup $\mathrm{SU}(3) \cdot \mathrm{SU}(3)$ by deleting the middle node. It follows from [Oni94, §3, Thm. 16] that a system of simple roots of the subgroup $\mathrm{U}(3) \cdot \mathrm{Sp}(1) \subset \mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ is obtained by deleting the second and middle node from the above diagram, which shows that $\mathrm{U}(3) \cdot \mathrm{Sp}(1)$ is also contained in $\mathrm{SU}(3) \cdot \mathrm{SU}(3)$. The latter group will be treated in subsection 3.3.

3.2.3. $\mathrm{A}_1^{35} \cdot \mathrm{Sp}(1)$. Suppose this group or a proper closed subgroup of it acts infinitesimally polar on M . Then this action restricts to an infinitesimally polar action of a rank one group on the fixed point set $\mathbb{H}P^2$ of the $\mathrm{Sp}(1)$ -factor. Due to Theorem 1.1(b), which has already been proved in section 2, such an action is in fact polar. However the section, as a totally geodesic submanifold of $\mathbb{H}P^2$, can have dimension at most 4. On the other hand, the cohomogeneity of the action on $\mathrm{A}_1^{35} \cdot \mathrm{Sp}(1)$ on $\mathbb{H}P^2$ is (since the $\mathrm{Sp}(1)$ -factor acts trivially) at least 5, a contradiction.

3.2.4. $\mathrm{Sp}(3) \cdot \mathrm{U}(1)$. We assert that $\mathrm{Sp}(3) \cdot \mathrm{U}(1)$ and $\mathrm{Sp}(3)$ act on M with the same orbits as $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$; it is enough to prove the second assertion. Indeed, consider the fixed point set $\mathbb{H}P^2$ of the $\mathrm{Sp}(1)$ -factor and fix a basepoint $p \in \mathbb{H}P^2$. Of course $\mathrm{Sp}(3)$ acts transitively on $\mathbb{H}P^2$ with isotropy group $\mathrm{Sp}(1) \cdot \mathrm{Sp}(2)$, and the description of the slice representation given in subsection 3.2 shows that $\mathrm{Sp}(1) \cdot \mathrm{Sp}(2)$ acts with cohomogeneity one on $\nu_p(\mathbb{H}P^2)$, proving the assertion. The other proper closed subgroups of $\mathrm{Sp}(3) \cdot \mathrm{U}(1)$ have already been considered above as subgroups of the other maximal connected subgroups of $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$.

3.3. **$\mathrm{SU}(3) \cdot \mathrm{SU}(3)$ and its subgroups.** One $\mathrm{SU}(3)$ -factor, say the second, is contained in G_2 and its fixed point set in M is a totally geodesic $\mathbb{C}P^2$; the other $\mathrm{SU}(3)$ -factor acts transitively on this $\mathbb{C}P^2$. We fix a basepoint $p \in \mathbb{C}P^2$. Put $G = \mathrm{SU}(3) \cdot \mathrm{SU}(3)$. The isotropy group G_p is $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2)) \cdot \mathrm{SU}(3) \cong \mathrm{U}(2) \times \mathrm{SU}(3)$, and its slice representation is $\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3$. From here we deduce that the principal isotropy group of the G -action on M is the maximal torus T^2 of the diagonal $\mathrm{SU}(3)$ -subgroup of G . Its fixed point set in the isotropy representation of a principal orbit is the Killing orthogonal complement of the Lie algebra of T^2 in the Lie algebra of the maximal torus of G . We enlarge G by adjoining $w \in \mathrm{Spin}(9)$ where $\psi = \mathrm{Ad}_w$ is a Weyl involution of F_4 relative to a maximal torus contained in $\mathrm{SU}(3) \cdot \mathrm{SU}(3) \cap \mathrm{Spin}(9) = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2)) \cdot \mathrm{SU}(3)$. To see that the enlarged group has the same orbits, note first that ψ fixes p so it preserves the singular G -orbit $\mathbb{C}P^2$. Moreover it induces an isometry of the orbit space of the slice representation at p , which must be trivial. Indeed that orbit space is isometric to the cone over the interval of

length $\pi/4$, where the endpoints parametrize orbits of dimensions 7 and 8 [HPT88, p. 436]. Since $\exp_p(\nu_p(\mathbb{C}P^2))$ meets every $SU(3) \cdot SU(3)$ -orbit, this shows that ψ preserves the G -orbits in M . Now the enlarged group has the same orbits and its principal isotropy group contains w . Since ψ acts as minus identity on the maximal torus of G , this proves that the enlarged groups acts asystatically.

Now assume a closed subgroup H of G acts infinitesimally polar on M . We will apply a similar argument as in [Kol07, p. 454]. It follows for all $g \in G$ that also gHg^{-1} acts infinitesimally polar on M . The slice representation of the G -action at p is irreducible of cohomogeneity two and it follows from Lemma 3.2 and [Dad85, EH99] that $G_p \cap gHg^{-1}$ is either finite or contains $SU(2) \times SU(3)$. In the former case, it follows that $\dim H \leq 4$, in the latter case $\dim H \geq 11$. Since these two conditions are mutually exclusive we have that $G_p \cap gHg^{-1}$ either is finite for all $g \in G$ or contains $SU(2) \times SU(3)$ for all $g \in G$. If $G_p \cap gHg^{-1}$ is finite for all $g \in G$, it follows that H is finite, since G_p contains a maximal torus of G . Otherwise, it follows that H contains $g(SU(2) \times SU(3))g^{-1}$ for all $g \in G$; hence H contains the normal subgroup generated by the union of $gSU(2)g^{-1} \times SU(3)$, $g \in G$. Since $SU(3)$ is a simple Lie group, it follows that $H = G$.

3.4. $G_2 \cdot A_1^8$ and its subgroups. We will prove that this group acts asystatically. In order to describe its action on the Cayley projective plane, we will use the models of projective spaces over the normed division algebras \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) given by idempotent Hermitian matrices of trace 1 [CR85]. Let \mathcal{J} be the Jordan algebra of Hermitian 3×3 -matrices with entries in \mathbb{O} , where the multiplication is defined by $x \circ y = \frac{1}{2}(xy + yx)$. Let $\text{Herm}_\varepsilon(3, \mathbb{K})$ be the subspace of \mathcal{J} consisting of Hermitian matrices with entries in \mathbb{K} and trace $\varepsilon \in \mathbb{R}$. Then $M = \mathbb{O}P^2$ is embedded in \mathcal{J} as the smooth real algebraic subvariety $V = \{x \in \text{Herm}_1(3, \mathbb{O}) \mid x^2 = x\}$ of \mathcal{J} . It is well known that the automorphism group of the algebra \mathcal{J} is the compact Lie group F_4 . Furthermore, the action of this group leaves V invariant and acts on it as the isometry group of the Riemannian symmetric space $\mathbb{O}P^2$. We also identify \mathbb{O} with $\mathbb{H} \times \mathbb{H}$ via the Cayley-Dickson process, so that the multiplication in \mathbb{O} is given by

$$(3.1) \quad (a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

and recall that G_2 is the automorphism group of \mathbb{O} . Using (3.1), one sees that the maps

$$\alpha_{p,q}: \mathbb{O} \rightarrow \mathbb{O}, \quad (x, y) \mapsto (pxp^{-1}, qyp^{-1}),$$

where $p, q \in \text{Sp}(1)$, comprise the maximal subgroup $\text{SO}(4)$ of G_2 .

Now the action of $G := \text{SO}(3) \times G_2$ on V is simple to describe:

$$(A, \alpha) \cdot x = A(\alpha(x_{ij}))A^t$$

where $A \in \text{SO}(3)$, $\alpha \in G_2$, $x = (x_{ij}) \in V$. Since both factors are centerless groups, this indeed defines an effective action of $\text{SO}(3) \times G_2$ on V .

Since the fixed point set of G_2 in \mathbb{O} is \mathbb{R} , we see that the fixed point set $V^{G_2} = V \cap \text{Herm}_1(3, \mathbb{R}) = \mathbb{R}P^2$ and that $\text{SO}(3)$ acts transitively on that set. For any $x \in V$, we have

$$(3.2) \quad T_x V = \{y \in \text{Herm}_0(3, \mathbb{O}) \mid xy + yx = y\}.$$

We fix a basepoint

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in V.$$

It is immediate that $G_p = \text{O}(2) \times G_2$, where $\text{O}(2)$ is embedded into $\text{SO}(3)$ via

$$B \mapsto \begin{pmatrix} \det B & 0 \\ 0 & B \end{pmatrix}$$

and a simple calculation using (3.2) shows that the normal and tangent spaces to $V^{G_2} = \mathbb{R}P^2$ in $V = \mathbb{O}P^2$ are:

$$\nu_p(\mathbb{R}P^2) = \left\{ \begin{pmatrix} 0 & a & b \\ \bar{a} & 0 & 0 \\ \bar{b} & 0 & 0 \end{pmatrix} \mid a, b \in \Im \mathbb{O} \right\}$$

and

$$T_p(\mathbb{R}P^2) = \left\{ \left(\begin{array}{ccc} 0 & a & b \\ \bar{a} & 0 & 0 \\ \bar{b} & 0 & 0 \end{array} \right) \mid a, b \in \mathbb{R} \right\}.$$

Therefore the slice representation $(G_p, \nu_p(\mathbb{R}P^2))$ is $(\mathrm{O}(2) \times \mathbf{G}_2, \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^7)$, where $\mathfrak{SO} \cong \mathbb{R}^7$. This representation is orbit equivalent to $(\mathrm{SO}(2) \times \mathrm{SO}(7), \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^7)$, polar, and of cohomogeneity 2 [EH99]. The subspace of $\nu_p(\mathbb{R}P^2)$ spanned by

$$\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & j \\ 0 & 0 & 0 \\ -j & 0 & 0 \end{pmatrix}$$

is a section, from which we find a principal isotropy group

$$H \cong (\mathbb{Z}_2)^2 \times \mathrm{Sp}(1);$$

here $\mathrm{Sp}(1)$ is the normal subgroup of $\mathrm{SO}(4) \subset \mathbf{G}_2$ that acts trivially on $\mathbb{H} \cong \mathbb{H} \times \{0\} \subset \mathbb{O}$, namely, generated by $\alpha_{1,q}$ for $q \in \mathrm{Sp}(1)$, and $(\mathbb{Z}_2)^2$ is embedded diagonally in $\mathrm{SO}(3) \times \mathbf{G}_2$, with generators

$$h_1 = \left(\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \alpha_{i,i} \right)$$

and

$$h_2 = \left(\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \alpha_{j,j} \right).$$

The fixed point set of H in V can be obtained as follows. First note that $V^{H^0} = V \cap \mathrm{Herm}_1(3, \mathbb{H}) = \mathbb{H}P^2$. A direct calculation now shows that $V^H = (V^{H^0})^H$ consists of matrices

$$\begin{pmatrix} a^2 & -abi & -acj \\ abi & b^2 & -bck \\ acj & bck & c^2 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 1$. This is a 2-dimensional submanifold and clearly a copy of $\mathbb{R}P^2$. This proves that the G -action on M is asystatic; in particular, it is polar.

We proceed to compute the generalized Weyl group and the orbit space. For computational ease, we apply the following result, which is also of independent interest. It is a modification of the Luna-Richardson-Straume reduction, cf. [Gor14, Ex. 2.2.1] or [GL14, §2.6], where, instead of a principal isotropy group, only its identity component is used. See [GOT04, Sect. 2] for the definition of a generalized section.

Lemma 3.3. *Assume a Lie group G acts properly and isometrically on a connected complete Riemannian manifold M . Consider the identity component H^0 of a fixed principal isotropy group H . There is a component Σ of the fixed point set M^{H^0} which contains a point with isotropy group H . Then Σ is a generalized section for the G -action on M . In particular, there is an isometry of orbit spaces $M/G = \Sigma/\mathcal{W}$, where the generalized Weyl group is defined by $\mathcal{W} = N_G(\Sigma)/Z_G(\Sigma)$. Moreover, $N_G(\Sigma)$ is an open and closed subgroup of $N_G(H^0)$ and $Z_G(\Sigma)$ is an open and closed subgroup of H .*

Proof. One can always pick a component Σ of M^{H^0} as in the statement. It is clear that Σ is a connected closed totally geodesic submanifold of M such that its tangent space $T_p\Sigma$ contains the normal space $\nu_p(G \cdot p)$ for every G -regular point $p \in \Sigma$, and which thus meets all G -orbits. To see it is a generalized section, it remains only to prove that if $p, q = g \cdot p \in \Sigma$ are G -regular points for some $g \in G$ then $g \cdot \Sigma = \Sigma$.

Assume that $p, q \in \Sigma$ are regular points such that $q = g \cdot p$. Then we have $(G_p)^0 = H^0 = (G_q)^0$ and $G_q = gG_pg^{-1}$. This implies $H^0 = (G_q)^0 = (gG_pg^{-1})^0 = gH^0g^{-1}$, showing that $g \in N_G(H^0)$. Hence $N_G(\Sigma)$ is a closed subgroup of $N_G(H^0)$. Since g normalizes H^0 , we have $g \cdot M^{H^0} = M^{H^0}$ and it follows that g maps the connected component of M^{H^0} containing p onto that component

containing q (the same), that is, $g \cdot \Sigma = \Sigma$. This completes the proof that Σ is a generalized section. The assertion about the orbit space now follows (compare [Mag08, Thm. 2.1.1] and [Gor14, Prop. 2.2.1]). It is also clear that $N_G(\Sigma)$ contains the identity component of $N_G(H^0)$, and hence $N_G(\Sigma)$ is open in that group.

Finally, it is obvious that $H^0 \subset Z_G(\Sigma)$. Let $g \in Z_G(\Sigma)$ and assume $p \in \Sigma$ has $G_p = H$. Then $g \in H$. \square

It is not hard to see that the normalizer $N_{G_2}(H^0) = \mathrm{SO}(4)$, so $N_G(H^0) = \mathrm{SO}(3) \times \mathrm{SO}(4)$. Since this normalizer is connected, it follows from Lemma 3.3 that $N_G(\Sigma) = N_G(H^0)$. It also follows from the lemma that $Z_G(\Sigma) = H^0$, since the elements h_1 and h_2 defined above act nontrivially on M^{H^0} . Now $\mathcal{W} = N_G(H^0)/H^0 = \mathrm{SO}(3) \times \mathrm{SO}(3)$ acts on $V^{H^0} = \mathbb{H}P^2$, and the map

$$(x \in \mathbb{H}^3, \|x\| = 1) \mapsto xx^* \in \mathrm{Herm}_1(3, \mathbb{H})$$

is $\mathrm{Sp}(3)$ -equivariant and induces the standard embedding of $\mathbb{H}P^2$ into $\mathrm{Herm}_1(3, \mathbb{H})$, so we can see the \mathcal{W} -action in homogeneous coordinates as $(A, \pm q) \in \mathrm{SO}(3) \times \mathrm{Sp}(1)/\mathbb{Z}_2 = \mathrm{SO}(3) \times \mathrm{SO}(3)$

acting on $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{H}P^2$ by $A \begin{bmatrix} qx_1q^{-1} \\ qx_2q^{-1} \\ qx_3q^{-1} \end{bmatrix} = A \begin{bmatrix} qx_1 \\ qx_2 \\ qx_3 \end{bmatrix} \in \mathbb{H}P^2$. As in section 2, this action

canonically lifts to a representation of $\mathrm{SO}(3) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on \mathbb{H}^3 of cohomogeneity 3, where the last $\mathrm{Sp}(1)$ -factor acts by right multiplying each coordinate by a unit quaternion. Finally, this representation is, up to a \mathbb{Z}_2 -kernel, equivalent to $(\mathrm{SO}(3) \times \mathrm{SO}(4), \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4)$ and hence polar. The Weyl group for the action of $\mathrm{SO}(3) \times \mathrm{SO}(4)$ on S^{11} is of type C_3 acting on S^2 , so the generalized Weyl group for the action on $\mathbb{H}P^2$ (and for the action on M) is of type C_3/\mathbb{Z}_2 acting on $\mathbb{R}P^2$. Note that this argument also proves polarity of the G -action on M .

An analogous argument as in the last paragraph of 3.3, applied to the slice representation of $G_p = \mathrm{O}(2) \times G_2$ at p , shows that any closed subgroup of G acting infinitesimally polar on M equals G .

Remark 3.4. It follows rather easily from $V \cap \mathrm{Herm}_1(3, \mathbb{K}) \cong \mathbb{K}P^2$ that the fixed point sets of the subgroups in the chain $\mathrm{Spin}(9) \supset G_2 \supset \mathrm{SU}(3) \supset \mathrm{Sp}(1) \supset \{1\}$, where $\mathrm{Sp}(1)$ is an index 1 subgroup of F_4 , yield a chain of totally geodesic submanifolds $\{\mathrm{pt}\} \subset \mathbb{R}P^2 \subset \mathbb{C}P^2 \subset \mathbb{H}P^2 \subset \mathbb{O}P^2$.

3.5. Rank one subgroups. The action of a subgroup G of rank one of F_4 cannot be infinitesimally polar. In fact its maximal torus (a circle subgroup) is conjugate to a subgroup of $\mathrm{Spin}(9)$ so it has a fixed point, say q . The normal space to the G -orbit through q in M has dimension at least 14. The components of the fixed point set in M of any non-trivial subgroup of F_4 are totally geodesic submanifolds of dimension at most 8. It follows that the isotropy group at q acts without fixed directions on a subspace of the normal space of dimension at least 6, but there are no polar representations of $\mathrm{U}(1)$ or $\mathrm{SU}(2)$ without fixed directions in that dimension.

4. ADDENDUM

The following construction yields closed subgroups of F_4 that act on $\mathbb{O}P^2$ with a totally geodesic singular orbit. The list includes all polar actions on $\mathbb{O}P^2$. Let P be a connected closed totally geodesic submanifold of $\mathbb{O}P^2$. Let $N(P)$ and $Z(P)$ denote the identity components of the subgroups of F_4 consisting of elements that preserve P , resp., fix P pointwise. Note that $Z(P)$ is a normal subgroup of $N(P)$. The possibilities for P are well known [Wol63]. In Table 4 we indicate the (almost effectivized) slice representation for the action of $N(P)$ on $\mathbb{O}P^2$ at a point p in P , the cohomogeneity, and whether that action is polar or not. Note $P = S^4$ is the only one case in which the action of $N(P)$ on $\mathbb{O}P^2$ is not polar, see subsection 3.1.3, but its slice representation at p is. In fact, the slice representation is equivalent to the action of $\mathrm{Sp}(1)^4$ on $\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$ given by $(a, b, c, d) \cdot (x, y, z) = (axb^{-1}, axc^{-1}, bxd^{-1})$.

Complements and corrections. The authors take this opportunity to make corrections to some of their previous papers.

P	$Z(P)$	$N(P)$	$Slice\ repr$	$Cohom$	$Polar?$
{pt}	Spin(9)	Spin(9)	(Spin(9), \mathbb{R}^{16})	1	yes
$\mathbb{R}P^2$	G_2	$G_2 \times SO(3)$	($G_2 \times O(2)$, $\mathfrak{S}(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{R}^2$)	2	yes
$\mathbb{C}P^2$	SU(3)	SU(3) · SU(3)	(SU(3) × U(2), $\mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}^2$)	2	yes
$\mathbb{H}P^2$	Sp(1)	Sp(1) · Sp(3)	(Sp(1) × Sp(2), $\mathbb{H}^1 \otimes_{\mathbb{H}} \mathbb{H}^2$)	1	yes
S^1	Spin(7)	Spin(7) · SO(2)	(Spin(7), $\mathbb{R}^7 \oplus \mathbb{R}^8$)	2	yes
S^2	SU(4)	Spin(6) · Spin(3)	(U(4), $\mathbb{R}^6 \oplus \mathbb{C}^4$)	2	yes
S^3	Sp(2)	Spin(5) · Spin(4)	(Sp(1) · Sp(2), $\mathbb{R}^5 \oplus \mathbb{H} \otimes_{\mathbb{H}} \mathbb{H}^2$)	3	no
S^4	Sp(1) · Sp(1)	Spin(4) · Spin(5)	(Sp(1) ⁴ , $\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$)	3	no
S^5	Sp(1)	Spin(3) · Spin(6)	(Sp(1) · Sp(2), $\mathbb{R}^3 \oplus \mathbb{C}^4$)	2	yes
S^6	SO(2)	SO(2) · Spin(7)	(U(4), $\mathbb{R}^2 \oplus \mathbb{C}^4$)	2	yes
S^7	{1}	Spin(8)	(Spin(7), $\mathbb{R} \oplus \mathbb{R}^8$)	2	yes
S^8	{1}	Spin(9)	(Spin(8), \mathbb{R}^8)	1	yes

TABLE 4

In [BG07], the proof that polar actions on the Cayley projective plane are taut was done case-by-case using a reduction argument, but the group $SO(3) \times G_2$ was not considered. It is readily seen that the same method by reduction can be applied to this group. Moreover a short proof of the main result of that paper, that does not rely on classification results, is obtained from [Wie14, Theorem 3.20].

In [Kol11, Sec. 11], a classification of polar actions on the Cayley hyperbolic plane was given under the hypothesis that the group acting is a reductive algebraic subgroup of the isometry group. However, the results of the present article show that, in addition to the actions given there, also the group $SO(2, 1) \cdot G_2$ acts polarly on $\mathbb{O}H^2$ and with an orbit which is a totally geodesic $\mathbb{R}H^2$. Since there are no totally geodesic orbits of $SO(3) \times G_2$ on $\mathbb{O}P^2$ other than the $\mathbb{R}P^2$ on which the $SO(3)$ -factor acts as the isometry group and which is fixed by the G_2 -factor, it follows that the complete list of all connected reductive algebraic subgroups of the noncompact form of F_4 which act polarly on $\mathbb{O}H^2$ is given by the the actions in [Kol11, Thm 11.1] together with the action of $SO(2, 1) \cdot G_2$.

REFERENCES

- [AA93a] A. Alekseevsky and D. Alekseevsky, *Asystatic G-manifolds*, Differential geometry and topology: Alghero (1992), World Sci. Publ., River Edge, NJ, 1993, pp. 1–22.
- [AA93b] ———, *Riemannian G-manifold with one-dimensional orbit space*, Ann. Global Anal. Geom. **11** (1993), 197–211.
- [BCO03] J. Berndt, S. Console, and C. Olmos, *Submanifolds and holonomy*, Research Notes in Mathematics, no. 434, Chapman & Hall/CRC, Boca Raton, 2003.
- [BG07] L. Biliotti and C. Gorodski, *Polar actions on compact rank one symmetric spaces are taut*, Math. Z. **255** (2007), no. 2, 335–342.
- [CR85] T. E. Cecil and P. J. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Mathematics, no. 107, Pitman, 1985.
- [Dad85] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), 125–137.
- [Dyn00] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Selected papers of E. B. Dynkin with commentary (A. L. Onishchik A. A. Yushkevich, G. M. Seitz, ed.), American Mathematical Society and International Press, 2000, pp. 111–312.
- [EH99] J. Eschenburg and E. Heintze, *On the classification of polar representations*, Math. Z. **232** (1999), 391–398.
- [GL14] C. Gorodski and A. Lytchak, *Isometric actions on unit spheres with an orbifold quotient*, E-print arXiv:1407.5863 [math.DG], 2014.
- [GL15] ———, *Representations whose minimal reduction has a toric identity component*, Proc. Amer. Math. Soc. **143** (2015), 379–386.
- [Gor14] C. Gorodski, *A metric approach to representations of compact Lie groups*, Lecture Notes, Ohio State University, Mathematics Research Institute Preprint Series, 2014.
- [GOT04] C. Gorodski, C. Olmos, and R. Tojeiro, *Copolarity of isometric actions*, Trans. Amer. Math. Soc. **356** (2004), 1585–1608.
- [GR71] M. Golubitsky and B. Rothschild, *Primitive subalgebras of exceptional Lie algebras*, Pacific J. Math. **39** (1971), 371–393.

- [GT02] C. Gorodski and G. Thorbergsson, *Variationally complete action on compact symmetric spaces*, J. Differential Geom. **62** (2002), 39–48.
- [HPT88] W.-Y. Hsiang, R. S. Palais, and C.-L. Terng, *The topology of isoparametric submanifolds*, J. Differential Geom. **27** (1988), 423–460.
- [HPTT95] E. Heintze, R. S. Palais, C.-L. Terng, and G. Thorbergsson, *Hyperpolar actions on symmetric spaces*, Geometry, Topology, and Physics for Raoul Bott (S. T. Yau, ed.), Conf. Proc. Lecture Notes Geom. Topology, IV, International Press, Cambridge, MA, 1995, pp. 214–245.
- [KL12] A. Kollross and A. Lytchak, *Polar actions on symmetric spaces of higher rank*, E-print arXiv:1108.3256v3 [math.DG], 2012.
- [Kol02] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), 571–612.
- [Kol07] ———, *Polar actions on symmetric spaces*, J. Differential Geom. **77** (2007), no. 3, 425–482.
- [Kol11] ———, *Duality of symmetric spaces and polar actions*, Journal of Lie Theory **21** (2011), no. 3, 961–986.
- [KP03] A. Kollross and F. Podestà, *Homogeneous spaces with polar isotropy*, manuscripta math. **110** (2003), 487–503.
- [LE88] S. Lie and F. Engel, *Theorie der Transformationsgruppen*, vol. 1, B. G. Teubner, Leipzig, 1888.
- [LT10] A. Lytchak and G. Thorbergsson, *Curvature explosion in quotients and applications*, J. Differential Geom. **85** (2010), 117–140.
- [Lyt14] A. Lytchak, *Polar foliations of symmetric spaces*, Geom. Funct. Anal. **24** (2014), no. 4, 1298–1315.
- [Mag08] F. Magata, *Reductions, resolutions and the copolarity of isometric groups actions*, Ph.D. thesis, University of Münster, 2008, Münstersches Informations- und Archivsystem für multimediale Inhalte.
- [Oni94] A. L. Oniščik, *Topology of transitive transformation groups*, Johann Ambrosius Barth, Leipzig, 1994.
- [Oni04] ———, *Lectures on real semisimple Lie algebras and their representations*, ESI Lectures in Mathematics and Physics, European Mathematical Society, Zürich, 2004.
- [PT87] R. S. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc. **300** (1987), 771–789.
- [PT88] ———, *Critical point theory and submanifold geometry*, Lect. Notes in Math., no. 1353, Springer-Verlag, 1988.
- [PT99] F. Podestà and G. Thorbergsson, *Polar actions on rank one symmetric spaces*, J. Differential Geom. **53** (1999), 131–175.
- [PT02] ———, *Polar and coisotropic actions on Kähler manifolds*, Trans. Amer. Math. Soc. **354** (2002), 1759–1781.
- [Str94] E. Straume, *On the invariant theory and geometry of compact linear groups of cohomogeneity ≤ 3* , Diff. Geom. and its Appl. **4** (1994), 1–23.
- [Teb07] S. Tebege, *Polar actions on Hermitian and quaternion-Kähler symmetric spaces*, Geom. Dedicata **129** (2007), no. 1, 155–171.
- [Thu80] W. P. Thurston, *The geometry and topology of three-manifolds*, Lecture notes available at <http://library.msri.org/books/gt3m>, 1980.
- [Wie14] S. Wiesendorf, *Taut submanifolds and foliations*, J. Differential Geom. **96** (2014), no. 3, 457–505.
- [Wol63] J. A. Wolf, *Elliptic spaces in grassmann manifolds*, Illinois J. Math. **7** (1963), 447–462.

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, RUA DO MATÃO, 1010, SÃO PAULO, SP 05508-090, BRAZIL

E-mail address: gorodski@ime.usp.br

UNIVERSITÄT STUTTGART, INSTITUT FÜR GEOMETRIE UND TOPOLOGIE, PFAFFENWALDRING 57, 70569 STUTTGART, GERMANY

E-mail address: kollross@mathematik.uni-stuttgart.de

Claudio Gorodski
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão, 1010
São Paulo, SP 05508-090, Brazil
E-Mail: gorodski@ime.usp.br
WWW: <http://www.ime.usp.br/~gorodski/>

Andreas Kollross
Fachbereich Mathematik
Universität Stuttgart
D-70569 Stuttgart
Germany
E-Mail: kollross@mathematik.uni-stuttgart.de
WWW: <http://w5.mathematik.uni-stuttgart.de/fachbereich/kollross/>

Erschienenene Preprints ab Nummer 2012-001

Komplette Liste: <http://www.mathematik.uni-stuttgart.de/preprints>

- 2015-010 *Gorodski, C, Kollross, A.:* Some remarks on polar actions
- 2015-009 *Apprich, C.; Höllig, K.; Hörner, J.; Reif, U.:* Collocation with WEB–Splines
- 2015-008 *Kabil, B.; Rodrigues, M.:* Spectral Validation of the Whitham Equations for Periodic Waves of Lattice Dynamical Systems
- 2015-007 *Kollross, A.:* Hyperpolar actions on reducible symmetric spaces
- 2015-006 *Schmid, J.; Griesemer, M.:* Well-posedness of Non-autonomous Linear Evolution Equations in Uniformly Convex Spaces
- 2015-005 *Hinrichs, A.; Markhasin, L.; Oettershagen, J.; Ullrich, T.:* Optimal quasi-Monte Carlo rules on higher order digital nets for the numerical integration of multivariate periodic functions
- 2015-004 *Kutter, M.; Rohde, C.; Sändig, A.-M.:* Well-Posedness of a Two Scale Model for Liquid Phase Epitaxy with Elasticity
- 2015-003 *Rossi, E.; Schleper, V.:* Convergence of a numerical scheme for a mixed hyperbolic-parabolic system in two space dimensions
- 2015-002 *Döring, M.; Györfi, L.; Walk, H.:* Exact rate of convergence of kernel-based classification rule
- 2015-001 *Kohler, M.; Müller, F.; Walk, H.:* Estimation of a regression function corresponding to latent variables
- 2014-021 *Neusser, J.; Rohde, C.; Schleper, V.:* Relaxed Navier-Stokes-Korteweg Equations for Compressible Two-Phase Flow with Phase Transition
- 2014-020 *Kabil, B.; Rohde, C.:* Persistence of undercompressive phase boundaries for isothermal Euler equations including configurational forces and surface tension
- 2014-019 *Bilyk, D.; Markhasin, L.:* BMO and exponential Orlicz space estimates of the discrepancy function in arbitrary dimension
- 2014-018 *Schmid, J.:* Well-posedness of non-autonomous linear evolution equations for generators whose commutators are scalar
- 2014-017 *Margolis, L.:* A Sylow theorem for the integral group ring of $PSL(2, q)$
- 2014-016 *Rybak, I.; Magiera, J.; Helmig, R.; Rohde, C.:* Multirate time integration for coupled saturated/unsaturated porous medium and free flow systems
- 2014-015 *Gaspoz, F.D.; Heine, C.-J.; Siebert, K.G.:* Optimal Grading of the Newest Vertex Bisection and H^1 -Stability of the L_2 -Projection
- 2014-014 *Kohler, M.; Krzyżak, A.; Walk, H.:* Nonparametric recursive quantile estimation
- 2014-013 *Kohler, M.; Krzyżak, A.; Tent, R.; Walk, H.:* Nonparametric quantile estimation using importance sampling
- 2014-012 *Györfi, L.; Ottucsák, G.; Walk, H.:* The growth optimal investment strategy is secure, too.
- 2014-011 *Györfi, L.; Walk, H.:* Strongly consistent detection for nonparametric hypotheses
- 2014-010 *Köster, I.:* Finite Groups with Sylow numbers $\{q^x, a, b\}$
- 2014-009 *Kahnert, D.:* Hausdorff Dimension of Rings
- 2014-008 *Steinwart, I.:* Measuring the Capacity of Sets of Functions in the Analysis of ERM
- 2014-007 *Steinwart, I.:* Convergence Types and Rates in Generic Karhunen-Loève Expansions with Applications to Sample Path Properties

- 2014-006 *Steinwart, I.; Pasin, C.; Williamson, R.; Zhang, S.:* Elicitation and Identification of Properties
- 2014-005 *Schmid, J.; Griesemer, M.:* Integration of Non-Autonomous Linear Evolution Equations
- 2014-004 *Markhasin, L.:* L_2 - and $S_{p,q}^r B$ -discrepancy of (order 2) digital nets
- 2014-003 *Markhasin, L.:* Discrepancy and integration in function spaces with dominating mixed smoothness
- 2014-002 *Eberts, M.; Steinwart, I.:* Optimal Learning Rates for Localized SVMs
- 2014-001 *Giesselmann, J.:* A relative entropy approach to convergence of a low order approximation to a nonlinear elasticity model with viscosity and capillarity
- 2013-016 *Steinwart, I.:* Fully Adaptive Density-Based Clustering
- 2013-015 *Steinwart, I.:* Some Remarks on the Statistical Analysis of SVMs and Related Methods
- 2013-014 *Rohde, C.; Zeiler, C.:* A Relaxation Riemann Solver for Compressible Two-Phase Flow with Phase Transition and Surface Tension
- 2013-013 *Moroianu, A.; Semmelmann, U.:* Generalized Killing spinors on Einstein manifolds
- 2013-012 *Moroianu, A.; Semmelmann, U.:* Generalized Killing Spinors on Spheres
- 2013-011 *Kohls, K.; Rösch, A.; Siebert, K.G.:* Convergence of Adaptive Finite Elements for Control Constrained Optimal Control Problems
- 2013-010 *Corli, A.; Rohde, C.; Schleper, V.:* Parabolic Approximations of Diffusive-Dispersive Equations
- 2013-009 *Nava-Yazdani, E.; Polthier, K.:* De Casteljau's Algorithm on Manifolds
- 2013-008 *Bächle, A.; Margolis, L.:* Rational conjugacy of torsion units in integral group rings of non-solvable groups
- 2013-007 *Knarr, N.; Stroppel, M.J.:* Heisenberg groups over composition algebras
- 2013-006 *Knarr, N.; Stroppel, M.J.:* Heisenberg groups, semifields, and translation planes
- 2013-005 *Eck, C.; Kutter, M.; Sändig, A.-M.; Rohde, C.:* A Two Scale Model for Liquid Phase Epitaxy with Elasticity: An Iterative Procedure
- 2013-004 *Griesemer, M.; Wellig, D.:* The Strong-Coupling Polaron in Electromagnetic Fields
- 2013-003 *Kabil, B.; Rohde, C.:* The Influence of Surface Tension and Configurational Forces on the Stability of Liquid-Vapor Interfaces
- 2013-002 *Devroye, L.; Ferrario, P.G.; Györfi, L.; Walk, H.:* Strong universal consistent estimate of the minimum mean squared error
- 2013-001 *Kohls, K.; Rösch, A.; Siebert, K.G.:* A Posteriori Error Analysis of Optimal Control Problems with Control Constraints
- 2012-013 *Diaz Ramos, J.C.; Dominguez Vázquez, M.; Kollross, A.:* Polar actions on complex hyperbolic spaces
- 2012-012 *Moroianu, A.; Semmelmann, U.:* Weakly complex homogeneous spaces
- 2012-011 *Moroianu, A.; Semmelmann, U.:* Invariant four-forms and symmetric pairs
- 2012-010 *Hamilton, M.J.D.:* The closure of the symplectic cone of elliptic surfaces
- 2012-009 *Hamilton, M.J.D.:* Iterated fibre sums of algebraic Lefschetz fibrations
- 2012-008 *Hamilton, M.J.D.:* The minimal genus problem for elliptic surfaces
- 2012-007 *Ferrario, P.:* Partitioning estimation of local variance based on nearest neighbors under censoring