Universität Stuttgart

Fachbereich Mathematik

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Abstract

We extend the concepts of de Casteljau and de Boor algorithms as well as splines to geodesic spaces and establish a link to minimization of certain convex functionals. Moreover, we investigate the relation to Karcher equation and geometric mean in Riemannian manifolds and present some applications in geometric modeling.

1 Introduction

A geodesic space is a metric space where any two points can be joined by a shortest geodesic realizing the distance between those points. If there is a unique shortest geodesic between any two points (unique geodesic spaces), affine combination of points is well defined and in turn, the natural and general setting for de Casteljau and de Boor algorithms is provided. This construction shares many interesting properties with the special case of manifold-valued data considered in [10] and [15] (as well as [19] and [16] for subdivision schemes). Moreover, geodesic spaces give rise to a wider range of applications. A survey on particular geodesic spaces including Alexandrov and Busemann spaces can be found in [11] and [18]. A recent related work, considering subdivision schemes in metric spaces and link to barycenters in Hadamard spaces is [3]. Furthermore, an extrinsic approach to spline curves in embedded manifolds by minimizing energy, and several significant examples can be found in [6] and [7]. Data refinement in nonlinear geometries is a substantial problem in geometric processing and has a wide range of applications. Furthermore, many properties and applications of Bernstein polynomials immediately extend to analogous constructions in geodesic spaces. For example, an $\mathbb{R}^m$-valued function f can be approximated by the polynomial $\sum_{i=0}^n f(i/n)B_i^n$ where B_i^n denotes the i-th Bernstein polynomial of degree n. Similar results for manifold-valued functions are desirable. For Bernstein polynomials on spheres and sphere-lie surfaces we refer to [2]. The intrinsic approach in the present work can be used to construct the minimizer (e.g. geometric mean) of certain convex functionals, Bézier curves and more generally splines. Usually, to ensure well definedness of our approach, restriction of control points to a small enough neighbourhood is sufficient. In Riemannian manifolds, the size of this neighbourhood (for the algorithmic construction of splines to be well defined) is determined by the injectivity radius and for the minimization task concerning geometric mean and median, by the convexity radius. There are interesting recent progresses and extensions in this direction to general $CAT(k)$ spaces, for which we refer to [12], [9] and [17]. In the present work we focus on geometric modeling.

This work is organized as follows. In the next section, we introduce the construction of Bézier curves via de Casteljau algorithm in unique geodesic sapces and their relation to certain convex functionals. Section 3 is devoted to the link between geometric mean, Karcher equation and preceding results in Riemannian manifolds. The last two sections present rational Bézier curves, de Boor algorithm, splines and some examples.

2 De Casteljau algorithm and minimizing convex functionals

We recall a few definitions. Let (M, d) be a metric space. The length L of a curve $c \in C^0([0, 1], M)$ is

$$L(c) := \sup\{\sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) \,:\, 0 = t_0, \cdots, t_{n-1} = 1,\, n \in N\}.$$

c is called geodesic iff there exists $\epsilon > 0$ with

$$L(c|_{[s,t]}) = d(c(s), c(t)) \quad \text{whenever } |s - t| < \epsilon.$$

Definition 1. *We call M a unique geodesic space[1] iff there exists a continuous map $\Phi : M \times M \to C^0([0,1], M)$ such that for any two points $x, y \in M$*

$$L(\Phi_{[0,t]}(x,y)) = td(x,y) \quad \text{for all } 0 \leq t \leq 1.$$

Any reparametrization of Φ proportional to arclength, provides another shortest geodesic. Now, suppose that there is a second map $\tilde{\Phi}$ with the same properties as Φ. Then

$$d(\Phi_t(x,y), \tilde{\Phi}_t(x,y)) \leq d(x, \tilde{\Phi}_t(x,y)) + d(x, \Phi_t(x,y)) = td(x,y) - td(x,y) = 0.$$

Therefore Φ is unique. We call Φ the affine map of M and use the interval notation for the image of Φ, i.e., we set

$$[x,y] = \Phi_{[0,1]}(x,y), \;]x,y[= \Phi_{]0,1[}(x,y).$$

Note that $\Phi(x,y)$ is a homeomorphism from $[0,1]$ onto the geodesic segment joining x and y. Moreover, in this setting, the notation of betweenness also makes sense:

$$y \in [x,z] \iff d(x,y) + d(y,z) = d(x,z).$$

For example, any neighbourhood of a complete Riemannian manifold within injectivity radius (in particular, any Hadamard-Cartan manifold) is a unique geodesic space. Euclidean trees, and more generally, Bruhat-Titz buildings provide examples of nonmanifold geodesic unique spaces. For any neighbourhood in an open hemisphere of S^2 we have

$$\Phi_t(x,y) = \frac{\sin((1-t)\varphi)}{\sin\varphi}x + \frac{\sin(t\varphi)}{\sin\varphi}y, \; \text{ with } \varphi := \arccos(\langle x,y \rangle).$$

Note that in general for geodesic spaces, restriction to a small neighbourhood, does not result in a unique geodesic space. For instance, every neighbourhood in $\mathbb{R}^2$ endowed with the Manhattan (Taxicab) metric defined by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|,$$

is geodesic but not unique. Nevertheless, fixing a representative of identified geodesics, an affine map can be simply defined. For example, let k be any function of the slope of the line through x and y, $c = (x+y)/2$, g the line passing through the point c with slope k and x^* resp. y^* the nearest point of g to x resp. y. Consider the geodesic $[x, x*] \cup [x*, y*] \cup [y*, y]$ with $[\,,\,]$ being Euclidean. Obviously, the corresponding affine map reads

$$\Phi_t^k(x,y) = \begin{cases} (1 - \frac{L_3 t}{L_1})x + \frac{L_3 t}{L_1}x^* & \text{for } 0 \leq t < L_1/L_3, \\ \frac{L_2 - L_3 t}{L_2 - L_1}x^* + \frac{L_3 t - L_1}{L_2 - L_1}y^* & \text{for } L_1/L_3 \leq t \leq L_2/L_3, \\ \frac{L_3 - L_3 t}{L_3 - L_2}y^* + \frac{L_3 t - L_2}{L_3 - L_2}y & \text{for } L_2/L_3 < t \leq 1, \end{cases}$$

where $L_1 = |x - x^*|$, $L_2 = L_1 + |x^* - y^*|$ and $L_3 = L_2 + |y^* - y|$.

Throughout this work, unless otherwise stated explicitly, (M,d) will denote a unique geodesic space and Φ its affine map. Now, we define the de Casteljau algorithm in unique geodesic spaces. Let k be the smoothness of M and $I := [0,1]$.

Definition 2. *Let $i = 0, \cdots, n$, $p_i \in M$ and $\psi_0, \cdots, \psi_n \in C^0(I,I)$ bijections with $\psi_j(s) = s$ for $s = 0, 1$. Fix $t \in I$.*

$$r = 1, \cdots, n,$$
$$p_i^0 := p_i, \; i = 0, \ldots, n - r,$$
$$t_i^r = \psi_i^r(t) := \psi_{i+n-r}(t), \; p_i^r := \Phi_{t_i^r}(p_i^{r-1}, p_{i+1}^{r-1}).$$

---

[1]Some authors use the terminology, geodesic length space.

We call the functions ψ_i^r parameter maps and $I \ni t \mapsto p_0^n(t)$ corresponding Bézier curve. Moreover, we call the maps $I \ni t \mapsto B_{\psi_0^r}, \cdots, B_{\psi_{n-r}^r}$, defined by the identity

$$\sum_{i=0}^{r} B_{\psi_i^r} p_i := p_0^r \ \text{ with } M = \mathbb{R},$$

Bernstein characteristics of degree r. Usually parameter maps are piecewise linear or broken linear (for rational Bézier curves). If parameter maps are fixed throughout a discussion, we drop the subscript ψ and write B_i^r. Note that for each r, the functions B_i^r constitute a partition of the unity. Moreover, for any homeomorphism (resp. C^k diffeomorphism) $\tau : [a, b] \rightarrow I$, replacing ψ_i^r by $\psi_i^r \circ \tau$ just reparametrizes the Bézier curve. Obviously this curve lies in the convex hull

$$\{\Phi_I([p_i, p_{i+1}], [p_j, p_{j+1}]) : i, j = 0, \cdots, n-1\}$$

of the control points and has the end point property $p_0^n(0) = p_0$, $p_0^n(1) = p_n$.

Example 3. For $\psi_i^r = Id_I$ we get the natural generalization of the classic de Casteljau algorithm to unique geodesic spaces. In this case, if M is Euclidean, then Bernstein characteristics are just the ordinary Bernstein polynomials of degree n.

Applications of de Casteljau algorithm to some Riemannian manifolds including Lie groups and more generally, symmetric spaces can be found in [15]. Next we consider a metric real tree.

Example 4. Denote $[x, y, z] = 0$ iff x, y and z lie in the same geodesic. Let $c \in M$. The Paris metric d_p with respect to c can be defined as

$$d_p(x, y) = \begin{cases} d(x, y) & \text{if } [x, y, c] = 0, \\ d(x, c) + d(y, c) & \text{else.} \end{cases}$$

If (M, d) is the Euclidean plane, then obviously

$$\Phi_t^p(x, y) = \begin{cases} (1 - t)x + ty & \text{if } [x, y, c] = 0, \\ R_\alpha((1 - t)x + tR_\alpha(c - y)) & \text{else} \end{cases}$$

where R_α denotes the rotation by the angle $s\alpha$ with $s = sign(det(c - x, y - c))$ and $\alpha = \angle(x - c, y - c)$.

Bézier curves produced by the algorithm 2 enjoy the following subdivision property.

Theorem 5. *For any partition $0 < s_1 \leq \cdots \leq s_k < 1$ the Bézier curve p_0^n can be split into $k + 1$ Bézier curves $p_0^{0,n}$ on $[0, s_1], \cdots, p_0^{k,n}$ on $[s_k, 1]$.*

Proof. For completeness we sketch the proof which is similar to the Euclidean case. Let $k = 1$ and $s \in]0, 1[$. The map $t \mapsto st$ resp. $t \mapsto s + t - st$ is a bijection from $[0, 1]$ onto $[0, s]$ resp. $[s, 1]$. Hence applying 2 with parameter maps $\psi_i^{0,r}(t) := s\psi_i^r(t)$ resp. $\psi_i^{1,r}(t) := s + \psi_i^r(t) - s\psi_i^r(t)$ with $t \in [0, 1]$ yields the desired subdivision $p_0^n([0, 1]) = p_0^{0,n}([0, s]) \cup p_0^{1,n}([s, 1])$, $p_0^{0,n}([0, s]) \cap p_0^{1,n}([s, 1]) = \{p_0^n(s)\}$. Iterating k completes the proof. $\qquad \square$

The following theorem gives further properties of the de Casteljau algorithm. Proofs are slight modifications of the Riemannian case presented in [15] and we outline them for the reader's convenience.

Theorem 6. *Consider control points $P := (p_0, \ldots, p_n)$ and $Q := (q_0, \ldots, q_n)$ in $U \subset M$. Then the following holds.*
a) Transformation invariance: Suppose a Lie group H acts on M by

$$H \times M \ni (h, x) \mapsto hx \in M$$

leaving U invariant, i.e., $HU \subset U$. If the action is segment-equivariant, i.e., for every $h \in G$

$$h[x,y] = [hx, hy] \text{ for all } x, y \in U.$$

Then

$$hB(p_0, \ldots, p_n) = B(hp_0, \ldots, hp_n) \text{ for all } h \in H.$$

b) Local control: Suppose that M as embedded in an Euclidean space with any norm $\|.\|$. Then

$$\|p - q\|_\infty \leq C\|P - Q\|_\infty$$

where C denotes a positive constant depending only on n, U.

Proof. a) The segments $[hp_i, hp_{i+1}]$ and $h[p_i, p_{i+1}]$ have the same endpoints:

$$\Phi_0(hp_i, hp_{i+1}) = hp_i = h\Phi_0(p_i, p_{i+1}),$$
$$\Phi_1(hp_i, hp_{i+1}) = hp_{i+1} = h\Phi_1(p_i, p_{i+1}).$$

b) Fix $t \in I$. Due to finiteness of n there is a positive constant K determined by U and Lipschitz constants of Φ on U such that

$$\|p_i^r(t) - q_i^r(t)\| = \|\Phi_{t_i^r}(p_i^{r-1}(t), p_{i+1}^{r-1}(t)) - \Phi_{t_i^r}(q_i^{r-1}(t), q_{i+1}^{r-1}(t))\|$$
$$\leq K\|(p_i^{r-1}(t), p_{i+1}^{r-1}(t)) - (q_i^{r-1}(t), q_{i+1}^{r-1}(t))\|.$$

Iteration yields

$$\|p(t) - q(t)\| = \|p_0^n(t) - q_0^n(t)\| \leq K^n\|P - Q\|_\infty$$

which immediately implies the desired inequality. $\qquad\square$

For our approach to minimization of convex functionals using de Casteljau algorithm, we recall some definitions. Let M be a geodesic space. A function $r : M \to \mathbb{R}$ is called (strictly) convex iff $r \circ c$ is (strictly) convex for any nonconstant geodesic c. In the context of geodesic geometry and tasks to consider here, we introduce the following notations.

Definition 7. *We call a functional $e : M \times M \to \mathbb{R}$ (uniquely) Casteljau-like iff it is convex in the second argument and for any $t \in I$ and $p_0, p_1 \in M$ the point p_0^1 (uniquely) minimizes*

$$M \ni x \mapsto E(x) := (1-t) + e(x, p_0) + te(x, p_1).$$

For example, if e is (uniquely) Casteljau-like, $f : \mathbb{R} \to \mathbb{R}$ and $u : M \to \mathbb{R}$ are convex and f attains its minimum, then for $\tilde{e}(x, y) := f(e(x, y)) + u(y)$ we have

$$(1-t)\tilde{e}(x, p_0) + t\tilde{e}(x, p_1) \geq f(E(p_0^1)) + u(p_0^1).$$

Therefore, $\tilde{e}$ is also (uniquely) Casteljau-like. In some important applications e has strong convexity properties, hence the main issue is just the minimizing property.

Example 8. Suppose that e is uniformly convex with modulus ϕ and there is an $\epsilon \in \mathbb{R}$ such that $e(x, y) \geq \epsilon$ with $e(x, y) = \epsilon$ if and only if $x = y$. Then

$$E(x) \geq e(x, p_0^1) + t(1-t)\phi((d \times d)((x, p_0), (x, p_1)))$$
$$= e(x, p_0^1) + t(1-t)\phi(d(p_0, p_1)) \geq t(1-t)\phi(d(p_0, p_1)) + \epsilon.$$

Last inequality becomes an equality if and only if $x = p_0^1$. Moreover, with e, also E attains its minimum. Therefore, e is uniquely Casteljau-like.

Now, let b_i^n be some nonnegative real weights satisfying (w.l.g.) $\sum_{i=0}^n b_i^n = 1$ and consider the following optimization task

$$E_n(.) := \sum_{i=0}^n b_i^n e(., p_i) \to Min.$$

on M. If there is a unique minimizer, then it is called (weighted) geometric median for $e = d$ center of mass or geometric mean[2] for $e = d^2$. Next, we reduce the above minimization task to a 1-dimensional (along shortest geodesics) one and use the de Casteljau algorithm as an approach to the solution for the case that $b_i^n = B_i^n(t)$ for some $t \in I$. Let us first consider just two control points and their geometric median, i.e., $n = 1$ and $e = d$. We show that e is not Casteljau-like, nevertheless any minimizer is in the geodesic segment from p_0 to p_1. Of course, this example also shows that convexity of e does not imply its Casteljau-likeness. Suppose that d is convex. Fix $x \in M$ and $t \in I$. Let B_i denote the geodesic ball with radius $r_i = d(x, p_i)$ around p_i and $l := d(p_0, p_1)$. If $r_i > l$ for $i = 1$ or 0, then we have $E_1(x) > l \geq E_1(q)$ for any $q \in [p_0, p_1]$. Hence, x cannot be a minimizer. Therefore, we may and do assume that $r_i \leq l$. In view of the triangle inequality, there is a point $q \in [p_0, p_1]$ in the intersection of B_0 and B_1, i.e., $d(x, p_i) \geq d(q, p_i)$. Thus, $e(x, p_i) \geq e(q, p_i)$ implying $E_1(x) \geq E_1(q) = E_1(\Phi_s(p_0, p_1))$ for some $s \in I$ with $q = \Phi_s(p_0, p_1)$, and we arrive at $E_1(x) \geq ((1-t)s + t(1-s))l$. Therefore, E_1 has a minimizer in $[p_0, p_1]$ (Clearly, this result remains valid, if e is any proper convex function of d.). Indeed, $d(x, .)$ is linearly increasing on $[p_0, p_0^1(1/2)]$ and linearly decreasing on $[p_0, p_0^1(1/2)]$ for any x and we have

$$argMin(E_1) = \begin{cases} p_0 & \text{for } 0 \leq t < 1/2, \\ p_0^1(1/2) & \text{for } t = 1/2, \\ p_1 & \text{for } 1/2 < t \leq 1. \end{cases}$$

For geometric mean the situation changes, and we can construct the minimizer via de Casteljau algorithm.

Theorem 9. *Let $p_0, \cdots, p_n \in M$ and*

$$E_n(.) = \sum_{i=0}^n b_i^n e(., p_i).$$

Then the following holds.
a) The functional E_n has a unique minimizer, provided e is proper and strictly convex.
b) Suppose that $b_i^n = B_i^n(t)$ for some $t \in [0, 1]$ and e is (uniquely) Casteljau-like. Then $p_0^n(t)$ (uniquely) minimizes E_n.

Proof. a) E_n is as positive linear combination of proper strictly convex functionals $e(x, p_i)$ proper and strictly convex. Therefore it attains its unique minimum.
b) Fix $x \in M$. Set $e_i := e(x, p_i)$ with $i = 0, \ldots, n+1$. Then we have

$$E_{n+1}(x) = \sum_{i=0}^{n+1} B_i^{n+1}(t)e_i = B_0^{n+1}(t)e_0 + \sum_{i=1}^n B_i^{n+1}(t)e_i + B_{n+1}^{n+1}(t)e_{n+1}$$

$$= B_0^{n+1}(t)e_0 + \sum_{i=1}^n ((1 - t_i^{n+1})B_i^n(t) + t_i^{n+1}B_{i-1}^n(t))e_i + B_{n+1}^{n+1}(t)e_{n+1}$$

$$= \sum_{i=0}^n B_i^n(t)((1 - t_i^{n+1})e_i + t_i^{n+1}e_{i+1})$$

Due to convexity of e in the second argument, we may write

$$E_{n+1}(x) \geq \sum_{i=0}^n B_i^n(t)e(x, p_i^1(t)).$$

If our claim is true for some $n \geq 2$, then we arrive at

$$E_{n+1}(x) \geq E_{n+1}(p_0^{n+1}(t)).$$

With $e_0, \cdots, e_{n+1}$, also E_{n+1} attains its (unique) minimum. This completes the proof. $\qquad\square$

Corrolary 10. *Suppose that d^2 is convex. Then the de Casteljau point p_0^n is the unique minimizer of $M \ni x \mapsto \sum_{i=0}^n B_i^n d^2(x, p_i)$.*

Proof. For arbitrary $p_0, p_1 \in M$ and $0 \leq t \leq 1$ we have

$$(1-t)d^2(x, p_0) + td^2(x, p_1) \geq t(1-t)(d(x, p_0) + d(x, p_1))^2$$
$$\geq t(1-t)d^2(p_0, p_1)$$

with equality if and only if $x = p_0^1$. The preceding theorem completes the proof. $\qquad\square$

We borrow the following definition from [14]. Let M be a C_k domain, i.e., an open set in a geodesic space satisfying

$$E_1(x) \geq d^2(x, p_0^1) + \frac{k}{2}t(1-t)d^2(p_0, x, p_1)$$

with some $k \in]0, 2]$. Obviously, $e = d^2$ is uniquely Casteljau-like on M. Due to the preceding theorem p_0^n is the unique minimizer of E_n. Note that for $CAT(0)$ spaces the above inequality (strong convexity) holds with $k = 2$ and in particular, for Euclidean spaces it becomes equality.

3 Karcher equation and Bernstein representation for manifold-valued data

In this section, we assume that (M, d) is a smooth complete Riemannian manifold without conjugate points and give a characterization of Bézier curves as solution of Karcher equation. Note that within the injectivity radius of the Riemannian exponential map exp in terms of local representatives, the affine map of (M, d) is given by

$$\Phi_t(x, y) = \exp_x(t \log_x(y))$$

where $\log_x$ denotes the local inverse of the exponential map at x. Next, we consider the case $e = d^2$ more closely. We recall two important well known special cases when for any control points the geometric mean is well defined. First, if the sectional curvatures of M are bounded above by $k > 0$ and $diam(M) < \pi/(2\sqrt{k})$. Second, if the sectional curvatures of M are semi-negative. In this case, e.g., for surfaces with semi-negative Gaussian curvature (particularly ruled surfaces) as well as Cartan-Hadamard manifolds like the space of positive definite symmetric matrices. Many progresses concern the latter (for which there are also recent extensions to the infinite-dimensional setting presented in [12]). Next, we look at the Karcher equation in the tangent space $T_x M$ determining the centroid. For details as well as the proof (based on Jacobi field estimates) of Karcher's theorem below we refer to [1] and the classic reference [13]. For some similar results on geometric median and extension of Weiszfeld algorithm for its computation, we refer to [4]. We shall use the following version of Karcher's theorem.

Theorem 11. *Suppose that M has no conjugate points and sectional curvatures of M are bounded above by k. Set $\frac{1}{\sqrt{k}} := \infty$ if $k \leq 0$. If $diam(M) < \pi/(2\sqrt{k})$, then the squared distance function d^2 is strictly convex and E has a unique minimizer x determined by the Karcher equation*

$$\sum_i b_i^n \log_x p_i = 0. \tag{1}$$

Note that for $k \leq 0$ there is no restriction on $diam(M)$. For $k > 0$ (thick geodesic triangles), denoting the injectivity radius of M by r_{inj}, we may replace M by a neighbourhood with diameter less than $min(r_{inj}, \pi/(2\sqrt{k}))$. Now, in analogy to the Bernstein representation of Bézier curves in the Euclidean case and under the assumptions of the preceding theorem, it seems appropriate to denote the solution of Karcher equation by $\bigoplus_i b_i^n p_i$. Although the following result is an immediate consequence of 10 and Karcher's theorem, we present an alternate direct proof using the smooth structure.

Theorem 12. *Suppose that $b_i^n = B_i^n(t)$ for some $t \in I$. Then, under the assumptions of Karcher's theorem, we have $p_0^n(t) = \bigoplus_i b_i p_i$.*

Proof. Fix $t \in I$. Suppose that $n = 1$. Then $p := p_0^1$ parametrizes the geodesic from p_0 to p_1 with initial velocity $v := \log_{p_0}(p_1)$. Hence $\log_p(p_0) = -tw$ and $\log_p(p_1) = (1-t)w$ where P_t^p denotes the parallel transport along p at t and $w := P_t^p v$. Therefore

$$b_0^1(t) \log_{p(t)}(p_0) + b_1^1(t) \log_{p(t)}(p_1) = -(1-t)tw + t(1-t)w = 0 \text{ in } T_{p(t)}M.$$

Hence, the point $p(t)$ is critical. With d^2, the weighted sum E_1 is also strictly convex, implying that the point $p(t)$ is its unique minimizer. Theorems 9 and 11 complete the proof. $\square$

4 Rational Bézier curves

In a geodesic space (M, d) the rational Bézier curve with control points $p_0, \cdots, p_n$ and positive weights $w_0, \cdots, w_n$ can be produced by applying the weighted version of de Casteljau algorithm

$$
\begin{aligned}
&r = 1, \ldots, n, \\
&(p_i^0, w_i^0) := (p_i, w_i), \ i = 0, \ldots, n-r, \\
&w_i^r = (1-t)w_i^{r-1} + tw_{i+1}^{r-1}, \\
&t_i^r = t\frac{w_{i+1}^{r-1}}{w_i^r}, \ p_i^r := \Phi_{t_i^r}(p_i^{r-1}, p_{i+1}^{r-1}).
\end{aligned}
$$

Due to 10, minimizing

$$M \ni x \mapsto \sum_{i=0}^n \frac{w_i B_i^n}{\sum w_j B_j^n} d^2(x, p_i)$$

provides the same result. The following figures show the effect of weights. We remark that in general, for a surface of revolution the geodesic differential equation reduces to a first order one and can efficiently be solved using e.g. ode45 of MATLAB.

Example 13. In order to reflect constraints caused by the presence of some objects (without changing the degree of the Bézier curve), we may choose weights as functions of distances from control points to the objects. In the following we treat the small disc B in M as an attracting object. Here each weight is simply chosen as inverse of the distance between corresponding control point and B, i.e., $w_i = 1/d(p_i, B)$. Similarly, avoiding objects can be treated. Of course, due to the convex hull property of Bézier curves, the gained flexibility is restricted. For a treatment of obstacles via a variational approach we refer to [7] and [6].

5 Splines and de Boor algorithm

For a knot vector, i.e., a finite or bi-infinite nondecreasing sequence

$$\xi : \cdots \leq \xi_0 \leq \xi_1 \leq \xi_2 \leq \cdots$$
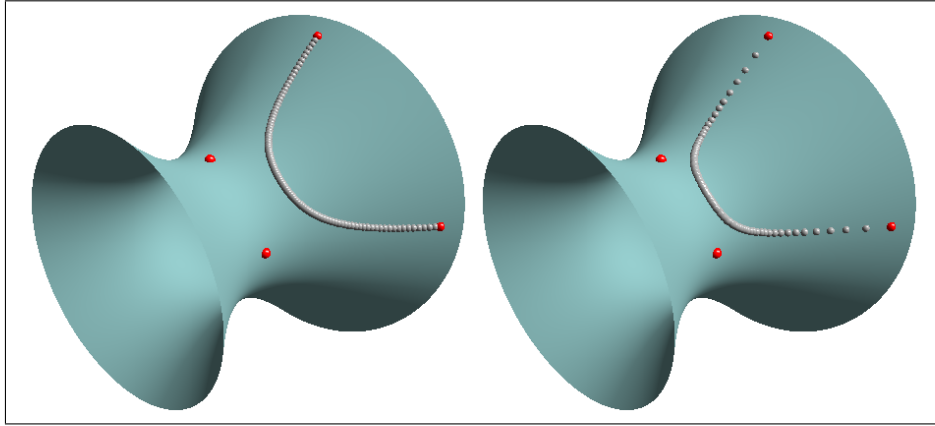
Figure 1: Bézier curve in a one-sheeted hyperboloid: left) cubic , right) rational with weights $1, 5, 5, 1$.
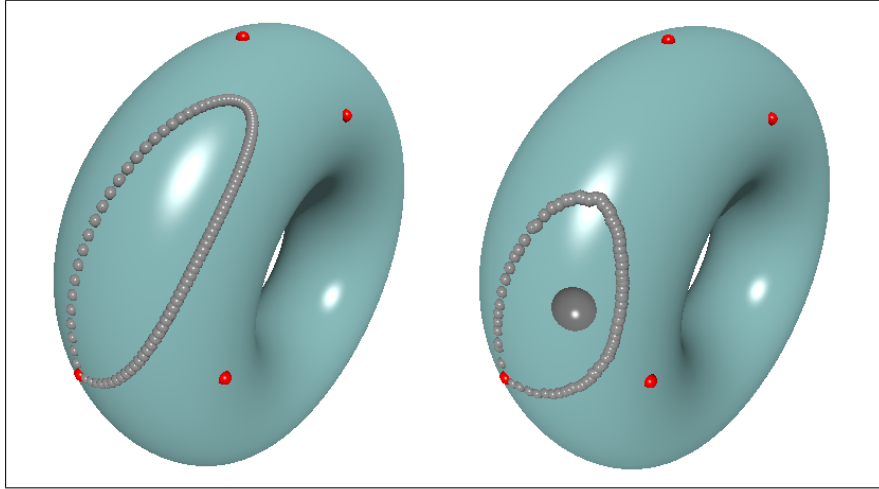

Figure 2: Bézier curve in a torus: left) cubic , right) rational with weights equal to inverse of distances between the small disc and control points.

of real numbers without accumulation points, control points $p_0, \cdots, p_n$ in a geodesic space M with metric d, we define the de Boor algorithm by

$$
\begin{aligned}
& r = 1, \ldots, m, \\
& p_i^0 := p_i, \ i = l - m, \ldots, l, \\
& t_{i,\xi}^r = \frac{t - \xi_i}{\xi_{i+n-r} - \xi_i}, \ p_i^r := \Phi_{t_{i,\xi}^r}(p_i^{r-1}, p_{i+1}^{r-1}).
\end{aligned}
$$

For a knot interval $I_l := [\xi_l, \xi_{l+1}[$ the spline curve p of degree $\leq m$ evaluated at $t \in I_l$ is obtained as the final value $p_{l-\mu}^{m-\mu}$ where μ denotes the multiplicity of t. Moreover, for $t \in I_l$ the above algorithm coincides with the de Casteljau algorithm applied to control points $p_{l-m}, \cdots, p_{l-\mu}$ and due to 10 after reindexing

$$
p(t) = argmin_{x \in M} \sum_{i=0}^{m-\mu} B_i^{m-\mu}(t) d^2(x, p_{i+l-m}).
$$

In particular, if M is Riemannian, then $p(t)$ is the unique solution of

$$\sum_{i=0}^{m-\mu} B_i^{m-\mu}(t) \log_x(p_{i+l-m}) = 0.$$
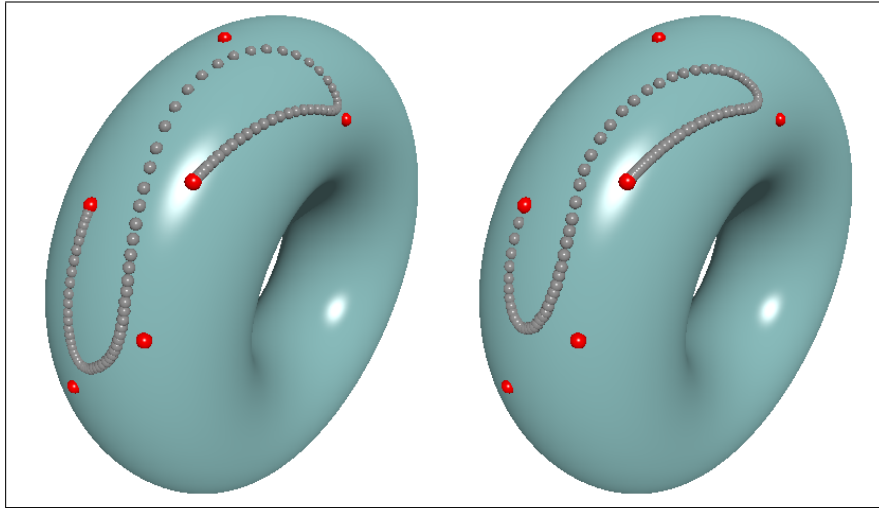
Figure 3: Cubic spline curve in a torus: left) uniform: $\xi = [0, 1, \cdots, 9]/9$, right) with double knots: $\xi = [0, 0, 1, 1, 2, 2, 3, 3, 3, 3]/3$.

Example 14. Poses of a rigid body can be visualized as a curve in the Euclidean motion group

$$E_3 = \{ \begin{pmatrix} 1 & 0 \\ b & R \end{pmatrix} : \ R \in SO(3), \ b \in \mathbb{R}^3 \}$$

for which convexity radius is $\pi/2$ and

$$\Phi_t(x, y) = \exp(tx \log((x^{-1}y)).$$
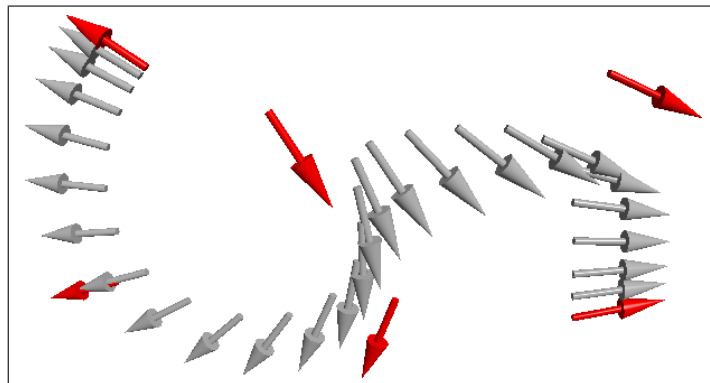
Figure 4: Cubic uniform spline curve in Euclidean motion group.

6 Conclusion

In this paper we extended the setting for de Casteljau algorithm as well as de Boor algorithm to unique geodesic spaces and presented some examples. In turn, Bézier and more generally spline curves in these spaces can be constructed via iteration. We proved that for certain choice of weights, geometric mean, i.e., minimizer of weighted sum of distance squared, can also be constructed using those algorithms. Of course, the main issue remains the fact that one needs to compute geodesics. We expect besides Riemannian manifolds, applications of our results in other geodesic spaces including trees and Bruhat-Titz buildings. Furthermore, we expect similar results concerning subdivision schemes as well as the bivariate case to produce nets and spline surfaces in geodesic spaces. Moreover, weighted extended B-splines (WEB-splines, [5] and [8]) can be combined with our approach to provide efficient approximations with high accuracy for solutions of boundary value problems on manifolds.

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