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Families of Separating Hyperplanes

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Representation of Quasi-Monotone Functionals by Families of Separating Hyperplanes

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Abstract

We characterize when the level sets of a continuous quasi-monotone functional defined on a suitable convex subset of a normed space can be uniquely represented by a family of bounded continuous functionals. Furthermore, we investigate how regularly these functionals depend on the parameterizing level. Finally, we show how this question relates to the recent problem of property elicitation that simultaneously attracted interest in machine learning, statistical evaluation of forecasts, and finance.

1 Introduction

Suppose we have a normed space $(E, \|\cdot\|_E)$, a non-empty convex subset $B \subset E$ that is contained in some closed affine hyperplane not passing the origin, and a continuous, in general non-linear, functional $\Gamma : B \rightarrow \mathbb{R}$ for which the level sets $\{\Gamma = r\} := \{x \in B : \Gamma(x) = r\}$ are convex for all $r \in \text{im } \Gamma$. Let us denote the interior of the image of Γ by I , that is $I := \overset{\circ}{\Gamma(B)} = \text{im } \Gamma$. In this paper we consider the following questions:

- i) Under which conditions is there a unique family $(z'_r)_{r \in I}$ of (normalized) bounded linear functionals on E such that for all $r \in I$ we have

$$\begin{aligned}\{\Gamma < r\} &= \{z'_r < 0\} \cap B \\ \{\Gamma = r\} &= \{z'_r = 0\} \cap B \\ \{\Gamma > r\} &= \{z'_r > 0\} \cap B ?\end{aligned}$$

- ii) When is the map $r \mapsto z'_r$ measurable or even continuous?

While at first glance these questions seem to be of little practical value they actually lie at the heart of a problem that recently attracted interest in machine learning, statistical evaluation of forecasts, and finance, see [19, 1, 7], [9, 8], and [11, 6, 23, 22], respectively, as well as the various references mentioned in these articles.

Let us briefly explain this problem while generously ignoring all mathematical issues. To this end, let \mathcal{P} be a set of probability measures on Ω , and $\Gamma : \mathcal{P} \rightarrow \mathbb{R}$ be an arbitrary map, which in the following will be called a property on \mathcal{P} . Simple examples of properties of distributions on $\Omega = \mathbb{R}$ are the mean, the median, and the variance, while more complicated properties are the (conditional) value at risk and conditional tail expectation. Now, for some properties including the mean, the median, and others, see [8] for an extensive list, there exists a so-called scoring function $S : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\Gamma(P) = \arg \min_{r \in \mathbb{R}} \mathbb{E}_{Y \sim P} S(r, Y) \tag{1}$$

for all $P \in \mathcal{P}$, i.e. $\Gamma(P)$ is the unique minimizer of the expected scoring function. Such properties, which are called elicitable, have various positive aspects: For example, if P is only approximately known, e.g. by data, then we can replace P by its approximation \hat{P} in (1) to estimate $\Gamma(P)$ by $\Gamma(\hat{P})$. Similarly, if we have two estimates \hat{r}_1 and \hat{r}_2 of $\Gamma(P)$ then we can compare these by comparing the corresponding values $\mathbb{E}_{Y \sim P} S(\hat{r}_1, Y)$ and $\mathbb{E}_{Y \sim P} S(\hat{r}_2, Y)$, or their \hat{P} -approximations if P is unknown, see e.g. [17, 12]. While these observations are rather straightforward they lie, in a conditional i.e. functional form, at the very core of a huge class of machine learning algorithms, namely so-called (regularized) empirical risk minimizers [21, 16, 18].

Elicitable properties are therefore highly desirable, but unfortunately, not every property is elicitable. Indeed, [15], see also [11, 8] showed that for convex \mathcal{P} an elicitable property needs to have convex level sets, and the variance does, for example, not have such level sets. Having convex level sets alone is, however, not sufficient for elicibility, and hence one needs additional assumptions to obtain sufficient conditions. To find such conditions, one key idea, known as Osband's principle [15, 11, 8, 19], is to take the derivative on the right-hand side of (1) to (hopefully) find that $\Gamma(P)$ can be characterized as the only zero of the function $r \mapsto \mathbb{E}_{Y \sim P} S'(r, Y)$. Now observe that the linearity of \mathbb{E} in P makes it possible to write

$$\mathbb{E}_{Y \sim P} S'(r, Y) = \langle S'(r, \cdot), P \rangle, \quad (2)$$

where $\langle z, p \rangle := z(p)$ denotes the evaluation of a linear functional z at some vector p . Clearly, if $z'_r := S'(r, \cdot)$ is interpreted as a functional over $\text{span } \mathcal{P}$ (or its closure), and if these functionals satisfy the equations in our first question, then $\Gamma(P)$ can indeed be characterized as the only zero of the function above. Consequently, this part of Osband's principle will work as soon as we have a positive answer to our first question. However, to construct one (or actually all) scoring functions from S' one needs additional regularity of $r \mapsto S'(r, \cdot)$ such as suitable measurability or even continuity, see [12, 19]. This motivates the second question we deal with in this paper.

We also like to note that two questions above have a simple answer, if Γ is defined on the entire space E by

$$\Gamma(x) := h(\langle z', x \rangle), \quad x \in E, \quad (3)$$

where $z' : E \rightarrow \mathbb{R}$ is a bounded linear functional and $h : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotone map. Indeed, if we pick an $\varphi' \in E'$ with $B \subset \{\varphi' = 1\}$ and assume, for example, that h is strictly increasing, then

$$z'_r = z' - h^{-1}(r)\varphi'$$

defines such a family of separating functionals. However, if B is 'too small', then various φ' are possible, and therefore this construction is, even after renormalization, not unique. Nonetheless, (3) is somewhat archetypal, since for finite dimensional spaces E , every continuous $\Gamma : E \rightarrow \mathbb{R}$ that has convex level sets is of the form (3) for some monotone h , see [20] in combination with Lemma 5.1. Moreover, without h being strictly monotone, we cannot expect a positive answer to our first question, and hence this assumption in (3) was not a restriction, either. In general, however, continuous $\Gamma : B \rightarrow \mathbb{R}$ with convex level sets are *not* of the form (3), not even in three dimensions, since roughly speaking the form (3) is forced by the requirement that the level sets cannot intersect, and hence they need to be parallel if Γ is defined on the full space E . But for smaller B , this is no longer necessary, and it is actually elementary to construct such examples.

The rest of this work is organized as follows: In Section 2 we characterize when we have a separating family in the sense of the first question. Section 3 then investigates measurable dependence on r and Section 4 deals with continuous dependence. In Section 5 we present some auxiliary results on quasi-monotone functions and all proofs can be found in Sections 6 to 8.

2 Existence and Uniqueness of the Separating Family

In this section we give positive answers to the first question raised in the introduction, that is, we show that under some conditions on Γ and B specified below there exists a unique family of separating bounded linear functionals

Let us begin by fixing some notations. Throughout this paper $(E, \|\cdot\|_E)$ is a normed space if not stated otherwise, E' denotes its dual and B_E its closed unit ball. Moreover, for an $A \subset E$ we write $\overset{\circ}{A}^E$ for the interior of A with respect to the norm $\|\cdot\|_E$. If this norm is known from the context we may abbreviate notations by $\overset{\circ}{A} := \overset{\circ}{A}^E$, and for typesetting reasons, we sometimes also write $\text{int } A := \overset{\circ}{A}$. Similarly, \overline{A}^E denotes the closure of A with respect to the norm $\|\cdot\|_E$, and if the latter is known from the context we may again write \overline{A} . Moreover, $\text{span } A$ denotes the linear space spanned by A and cone $A := \{\alpha x : \alpha \geq 0, x \in A\}$ denotes the cone generated by A . In addition, the null space of a linear functional $z' : E \rightarrow \mathbb{R}$ is denoted by $\ker z$, and the restriction of a function $f : A \rightarrow B$ onto $C \subset A$ is denoted by $f|_C$.

With the help of this notations we can now formulate our first set of assumptions that describe the set B . Throughout these assumptions, E denotes a normed space, $B \subset E$ is non-empty and convex, and $H := \text{span } B$.

B1 (Simplex face). There exists a $\varphi' \in E'$ such that $B \subset \{\varphi' = 1\}$.

B2 (Dominating norm). There exists an $x_\star \in B$ such that for $A := -x_\star + B$ and

$$F := \text{span } A$$

there exists a norm $\|\cdot\|_F$ on F with $\|\cdot\|_E \leq \|\cdot\|_F$.

B2* (Non-empty relative interior). Assumption **B2** is satisfied and $0 \in \overset{\circ}{A}^F$.

B3 (Cone decomposition). There exists a constant $K > 0$ such that for all $z \in H$ there exist $z^-, z^+ \in \text{cone } B$ with $z = z^+ - z^-$ and

$$\|z^-\|_E + \|z^+\|_E \leq K \|z\|_E.$$

B4 (Denseness). The space H is dense in E with respect to $\|\cdot\|_E$.

To illustrate these assumptions in view of the elicitation question raised in the introduction, we fix a probability measure μ on some measurable space (Ω, \mathcal{A}) , and consider the set of bounded, integrable probability densities with respect to μ , that is

$$\Delta^{\geq 0} := \{h \in L_\infty(\mu) : h \geq 0, \mathbb{E}_\mu h = 1\}. \quad (4)$$

Our set \mathcal{P} will then be $\mathcal{P} := \{h d\mu : h \in \Delta^{\geq 0}\}$. For $p \in [1, \infty)$, $E := L_p(\mu)$, and $\varphi' := \mathbb{E}_\mu(\cdot)$ we then verify that $\Delta^{\geq 0}$ satisfies **B1**, and for $p = 1$, the norm induced on \mathcal{P} equals the total variation norm. Moreover, we have $H = L_\infty(\mu)$ and therefore **B4** is obviously satisfied. Furthermore, by considering $h = \max\{0, h\} - \max\{0, -h\}$ we obtain **B3** for $K := 2^{1-1/p}$. Consequently, the only task left is to find a suitable $h_\star \in \Delta^{\geq 0}$ and an appropriate norm $\|\cdot\|_F$. Unfortunately, taking $\|\cdot\|_F = \|\cdot\|_E$ won't work in this example, since the elements in

$$-h_\star + \Delta^{\geq 0} = \{h \in L_\infty(\mu) : h \geq -h_\star, \mathbb{E}_\mu h = 0\}$$

are pointwise bounded from below by $-h_\star$ but this cannot be guaranteed in any $\|\cdot\|_E$ -ball in F around the origin. However, for $\|\cdot\|_F := \|\cdot\|_\infty$ and $h_\star := \mathbf{1}_\Omega$ Assumption **B2*** does hold.

The example above illustrates, that the choice of $\|\cdot\|_F$ may give some extra freedom when applying the results of this paper. Unfortunately, however, this freedom comes for an extra price we have to pay at a different condition. Before we can explain the details let us present the following lemma that investigates the spaces H and F in a bit more detail.

Lemma 2.1. *Let **B1** and **B2** be satisfied. Then, the space F satisfies $F \subset \ker \varphi'$. In particular, we have $x_\star \notin F$ and*

$$H = F \oplus \mathbb{R}x_\star.$$

Furthermore, if we equip H with the norm $\|\cdot\|_H$, defined by

$$\|y + \alpha x_\star\|_H := \|y\|_F + \|\alpha x_\star\|_E, \quad y \in F, \alpha \in \mathbb{R},$$

then, we have $\|\cdot\|_E \leq \|\cdot\|_H$ on H , $\|\cdot\|_F = \|\cdot\|_H$ on F . Finally, for all $x_1, x_2 \in B$ we have $x_1 - x_2 \in F$.

Roughly speaking, Lemma 2.1 provides a simple way to extend the norm $\|\cdot\|_F$ to the space $H = \text{span } B$ in which most of our initial geometric arguments take place. In addition, it is a key ingredient in the second of the following set of assumptions on Γ . Throughout these assumptions $B \subset E$ again denotes a non-empty convex subset of the normed space E . Moreover, $\Gamma : B \rightarrow \mathbb{R}$ denotes an arbitrary map and we write $I := \Gamma(B)$. Finally, we assume that **B1** and **B2** are satisfied whenever this is necessary.

- G1 (F-continuous and convex level sets).** The map $\Gamma : B \rightarrow \mathbb{R}$ is $\|\cdot\|_F$ -continuous and its level sets $\{\Gamma = r\}$ are convex for all $r \in \text{im } \Gamma$.
- G1* (E-continuous and convex level sets).** The map $\Gamma : B \rightarrow \mathbb{R}$ is $\|\cdot\|_E$ -continuous and its level sets $\{\Gamma = r\}$ are convex for all $r \in \text{im } \Gamma$.
- G2 (Locally non-constant).** For all $r \in I$, $\varepsilon > 0$, and $x \in \{\Gamma = r\}$, there exist $x^- \in \{\Gamma < r\}$ and $x^+ \in \{\Gamma > r\}$ such that $\|x - x^-\|_H \leq \varepsilon$ and $\|x - x^+\|_H \leq \varepsilon$.
- G3 (Locally non-constant continuous extension).** We have a $\|\cdot\|_E$ -continuous extension $\hat{\Gamma} : \bar{B} \rightarrow \mathbb{R}$ of Γ such that for all $r \in I$, $\varepsilon > 0$, and $x \in \{\hat{\Gamma} = r\}$, there exist $x^- \in \{\hat{\Gamma} < r\}$ and $x^+ \in \{\hat{\Gamma} > r\}$ with $\|x - x^-\|_E \leq \varepsilon$ and $\|x - x^+\|_E \leq \varepsilon$.

By Lemma 2.1 we know that for all $x_1, x_2 \in B$ we have $x_1 - x_2 \in F$ and thus $\|x_1 - x_2\|_F = \|x_1 - x_2\|_H$. Consequently, the assumed $\|\cdot\|_F$ -continuity in **G1** is well-defined and equivalent to $\|\cdot\|_H$ -continuity. Moreover, if **B1** and **B2** are satisfied, then $\|\cdot\|_F$ dominates $\|\cdot\|_E$, and therefore **G1*** implies **G1** in this case.

At first glance the convexity of the level sets and the continuity of Γ are conceptually simple assumptions. When combined, however, they have a significant impact on the shape of Γ and its level sets. To illustrate this let us recall that a function $\Gamma : B \rightarrow \mathbb{R}$ defined on some convex subset $B \subset E$ of a vector space E is called *quasi-convex*, if, for all $r \in \mathbb{R}$, the *sublevel sets* $\{\Gamma \leq r\}$ are convex. It is well-known, see [10] for some historic remarks, and also a simple exercise that Γ is quasi-convex, if and only if

$$\Gamma((1 - \alpha)x + \alpha y) \leq \max\{\Gamma(x), \Gamma(y)\} \quad (5)$$

holds for all $x, y \in B$ and $\alpha \in [0, 1]$. We further say that Γ is *strictly quasi-convex*, if, in addition, this inequality is strict for all $x, y \in B$ with $\Gamma(x) \neq \Gamma(y)$ and all $\alpha \in (0, 1)$. Analogously, Γ is called (*strictly*) *quasi-concave*, if $-\Gamma$ is (strictly) quasi-convex. Finally, Γ is called (*strictly*) *quasi-monotone*, if Γ is both (strictly) quasi-convex and (strictly) quasi-concave. It can be shown, that Γ is quasi-monotone, if and only if Γ is monotone on each segment, see Lemma 5.1, and if Γ is continuous, quasi-monotonicity is also equivalent to the convexity of all level sets, see Lemma 5.2. Consequently, if **G1** or **G1*** is satisfied, then its level sets cannot, for example, form an alveolar partition of B or a triangulation partition, since both would contradict the convexity of the sublevel sets. We refer to [12] for some nice illustrations. *Without* the continuity, however, such partitions would be perfectly fine.

Assumption **G2** essentially states that Γ is not constant on arbitrarily small balls $B \cap \varepsilon B_H$, where the used norm $\|\cdot\|_H$ is typically larger than $\|\cdot\|_E$, that is, the considered balls εB_H are smaller than the balls εB_E . In particular, if a larger norm $\|\cdot\|_F$ is required to ensure **B2***, then in turn this choice leads to a stronger version of **G2**.

Finally, **G3** will be used to extend results for B to \bar{B} . This will particularly useful if the set B is only an auxiliary set in the sense that we are actually interested in \bar{B} , instead. For example, in (4) we only considered bounded densities to ensure **B2***. In general, however, one might be interested in all probability densities, that is, in the set $\bar{\Delta}^{\geq 0}$. Now **G3** essentially states that if we actually have a continuous functional on \bar{B} then we need a weak version of **G2** on $\bar{B} \setminus B$, too.

Before we present our first main results, let us finally introduce the following definition, which formally describes the functionals we seek.

Definition 2.2. *Let E be a normed space, $\Gamma : B \rightarrow \mathbb{R}$ be a map and $I := \Gamma(B)$. Then, a family $(z'_r)_{r \in I}$ of linear maps $z'_r : E \rightarrow \mathbb{R}$ is called a *separating family* for Γ , if for all $r \in I$ we have*

$$\{\Gamma < r\} = \{z'_r < 0\} \cap B \quad (6)$$

$$\{\Gamma = r\} = \{z'_r = 0\} \cap B \quad (7)$$

$$\{\Gamma > r\} = \{z'_r > 0\} \cap B. \quad (8)$$

Note that in the definition above the maps z'_r are not necessarily continuous. In the following, however, all obtained separating families will consist of continuous functionals, but depending on the situation, the continuity will be with respect to either $\|\cdot\|_H$ or $\|\cdot\|_E$.

The following result characterizes the existence of a separating family in H' .

Theorem 2.3. *Let **B1** and **B2*** be satisfied and $\Gamma : B \rightarrow \mathbb{R}$ be a map. We write $I := \Gamma(\overset{\circ}{B})$ and $B_0 := \Gamma^{-1}(I)$. Then the following statements are equivalent:*

- i) Assumptions **G1** and **G2** are satisfied.*
- ii) The map Γ is $\|\cdot\|_F$ -continuous and quasi-monotone. Moreover, $\Gamma|_{B_0}$ is strictly quasi-monotone.*
- iii) There exists a separating family $(z'_r)_{r \in I} \subset H'$ for Γ .*
- iv) There exists a unique separating family $(z'_r)_{r \in I} \subset H'$ for Γ with $\|z'_r\|_{H'} = 1$ for all $r \in I$.*

Theorem 2.3 shows that under the assumptions **B1** and **B2*** on B , the conditions **G1** and **G2** on Γ are both necessary and sufficient for the existence of a separating family in H' . In addition, it shows that the only freedom for choosing this family is the scaling of its members. In combination with Lemma 5.2 we finally see that **G2** can be replaced by the *norm-independent* strict quasi-monotonicity of $\Gamma|_{B_0}$.

Our next goal is to present a similar characterization for separating functionals that are $\|\cdot\|_E$ -continuous. To this end, we write $\|z'\|_{E'} = 1$ for the norm of a functional $z' \in (H, \|\cdot\|_E)'$.

Theorem 2.4. *Let **B1**, **B2***, **B3** be satisfied and $\Gamma : B \rightarrow \mathbb{R}$ be a map. We write $I := \Gamma(\overset{\circ}{B})$ and $B_0 := \Gamma^{-1}(I)$. Then the following statements are equivalent:*

- i) Assumptions **G1*** and **G2** are satisfied.*
- ii) The map Γ is $\|\cdot\|_E$ -continuous and quasi-monotone. Moreover, $\Gamma|_{B_0}$ is strictly quasi-monotone.*
- iii) There exists a separating family $(z'_r)_{r \in I} \subset (H, \|\cdot\|_E)'$ for Γ .*
- iv) There exists a unique separating family $(z'_r)_{r \in I} \subset (H, \|\cdot\|_E)'$ for Γ with $\|z'_r\|_{E'} = 1$ for all $r \in I$.*

Moreover, if condition *iv)* is true and **B4** is also satisfied, then, for all $r \in I$, there exists exactly one $\hat{z}'_r \in E'$ such that $(\hat{z}'_r)|_H = z'_r$ and $\|\hat{z}'_r\|_{E'} = 1$.

When we apply Theorem 2.3 to the example discussed around (4), we see that there is a (unique) family of separating hyperplanes $(\hat{z}'_r)_{r \in I} \subset L_{p'}(\mu)$ for Γ if and only if our property Γ is $\|\cdot\|_p$ -continuous, quasi-monotone, and even strictly quasi-monotone on B_0 . This is, to the best of our knowledge, the first characterization when the part around (2) of Osband's principle does work. Nonetheless, we like to mention that the implications *i) \Rightarrow iv)* of Theorems 2.3 and 2.4 have already been shown in the unreviewed appendix of [19]. However, the remaining implications are new and so is the following third and last result in this section.

Theorem 2.5. *Assume that **B1**, **B2***, **B3**, **B4**, **G1***, **G2**, and **G3** are satisfied, and let $(z'_r)_{r \in I} \subset E'$ be the separating family found in Theorem 2.4. Then, for all $r \in I$, we have*

$$\begin{aligned} \{\hat{\Gamma} < r\} &= \{\hat{z}'_r < 0\} \cap \bar{B} \\ \{\hat{\Gamma} = r\} &= \{\hat{z}'_r = 0\} \cap \bar{B} \\ \{\hat{\Gamma} > r\} &= \{\hat{z}'_r > 0\} \cap \bar{B}. \end{aligned}$$

Theorem 2.5 essentially shows that a separating family for Γ in E' is also a separating family for a continuous extension $\hat{\Gamma}$ satisfying **G3**. Here we note that **G3** can again be replaced by a strict quasi-convexity assumption. Moreover, a family satisfying the three equalities in Theorem 2.5 is a separating family of Γ , and therefore, the implications of Theorem 2.4 apply. In particular, if such a family exists, then Γ needs to be $\|\cdot\|_E$ -continuous and quasi-monotone, and $\Gamma|_{B_0}$ needs to be strictly quasi-monotone. Moreover, by repeating the arguments used in the proof of Theorem 2.3 we see that even $\hat{\Gamma}$ needs to be $\|\cdot\|_E$ -continuous and quasi-monotone.

Finally, let us again have a quick look at the example discussed around (4). Here we see that the part around (2) of Osband's principle works, if, for example, $p = 1$ and Γ is a property on the set of all μ -absolutely continuous probability measures that is continuous with respect to the total variation norm and satisfies the (strict) quasi-monotonicity assumptions discussed above. As far as we know, this is the first such result for probability measures not having a bounded density.

3 Measurable Dependence of the Separating Hyperplanes

In the previous section we have seen that under some conditions on both B and Γ we have a unique family of separating hyperplanes $(\hat{z}'_r)_{r \in I} \subset E'$. Our goal in this section is to investigate under which supplemental assumptions the resulting map $r \mapsto \hat{z}'_r$ is measurable.

To this end, we will consider the following two additional assumptions:

B5 (Completeness and separable dual). The space E is a Banach space and its dual E' is separable.

G4 (Measurability). The pre-image $B_0 := \Gamma^{-1}(I)$ is a Borel measurable subset of E .

Before we discuss these assumptions, we like to present the main result of this section, which shows that the map $r \mapsto \hat{z}'_r$ is measurable provided that **B5** and **G4** hold. To formulate it, we write $\mathcal{B}(X)$ for the Borel σ -algebra of a given topological space X . Moreover, we equip the interval I with the Lebesgue completion $\hat{\mathcal{B}}(I)$ of its Borel σ -algebra $\mathcal{B}(I)$.

Theorem 3.1. *Assume that **B1** to **B5**, as well as **G1***, **G2**, and **G4** are satisfied. Then, the map $Z : I \rightarrow E'$ defined by*

$$Z(r) := \hat{z}'_r,$$

where $\hat{z}'_r \in E'$ are the unique functionals obtained in Theorem 2.4, is measurable with respect to the σ -algebras $\hat{\mathcal{B}}(I)$ and $\mathcal{B}(E')$, and it is also an E' -valued measurable function in the sense of Bochner integration theory with respect to the σ -algebra $\hat{\mathcal{B}}(I)$.

Let us briefly return to our initial example (4) of bounded probability densities. There it can be shown that **G4** is automatically satisfied, and **B5** is satisfied if and only if $1 < p < \infty$ and $E = L_p(\mu)$ is separable. If the remaining assumptions of Theorem 3.1 hold true, too, we thus see that that map $Z : I \rightarrow L_{p'}(\mu)$ is measurable. Unfortunately, however, this may not be the desired property. Indeed, in (4) it seems natural to take $p = 1$ and ask for the measurability of $Z : I \rightarrow L_\infty(\mu)$. Clearly, **B5** is violated in this case, and thus Theorem 3.1 does not provide the desired answer. The following corollary partially addresses this issue.

Corollary 3.2. *Assume that **B1**, **B2***, **B3**, **B4**, **G1*** and **G2** are satisfied for E , $B \subset E$, $\varphi' \in E'$, F , and $\Gamma : B \rightarrow \mathbb{R}$ and let $(\hat{z}'_r)_{r \in I} \subset E'$ be the corresponding family of separating functionals found in Theorem 2.4. In addition, let $E_0 \hookrightarrow E$ be a continuously embedded Banach space with $B \subset E_0$ such that **B2** to **B5**, as well as **G1***, and **G4** are satisfied for E_0 and F . Then we also obtain a family $(\hat{z}'_{0,r})_{r \in I} \subset E'_0$ of separating functionals by Theorem 2.4 and this family is measurable in the sense of Theorem 3.1 with respect to the space E'_0 . Moreover, there exists a measurable map $\alpha : (I, \hat{\mathcal{B}}(I)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that for all $r \in I$ we have $\alpha(r) > 0$ and*

$$(\hat{z}'_r)|_{E_0} = \alpha(r)\hat{z}'_{0,r}. \tag{9}$$

Note that the functionals \hat{z}'_r and $\hat{z}'_{0,r}$ are normalized with respect to the dual norms of $\|\cdot\|_E$ and $\|\cdot\|_{E_0}$, respectively, and therefore, we typically have $\alpha(r) \neq 1$. Moreover, the assumptions ensure that E_0 is dense in E and therefore $\hat{z}'_{0,r}$ can be uniquely extended to a continuous functional on E , namely to $\frac{1}{\alpha(r)}\hat{z}'_r$.

The main message of Corollary 3.2 is that $r \mapsto (\hat{z}'_r)|_{E_0}$ is measurable with respect to E'_0 , that is, even if we use the normalization with respect to E , we still obtain measurability with respect to E'_0 . Applied to our motivating example in front of Corollary 3.2, this means that we obtain a family $(h_r)_{r \in I} \subset L_\infty(\mu)$ that represent the functionals $\hat{z}'_r \in (L_1(\mu))'$ such that $r \mapsto h_r$ is measurable with respect to $L_p(\mu)$ for all $p \in (1, \infty)$. The latter can then be used to conclude that we find a 'version'

$\tilde{h} : I \times \Omega \rightarrow \mathbb{R}$ of this family that is $\hat{\mathcal{B}}(I) \otimes \mathcal{A}$ -measurable, and in turn this measurability can be used to make the second part of Osband's principle work, see [19] for a more elementary but also technically more involved approach. Moreover, our normalization in $(L_1(\mu))' = L_\infty(\mu)$ means that we have $\|h_r\|_\infty = 1$ for all $r \in I$, and the latter is the additional information provided by Corollary 3.2 when compared to Theorem 3.1. Finally, whether $r \mapsto h_r$ is actually measurable with respect to $L_\infty(\mu)$ remains an open question.

4 Continuous Dependence of the Separating Hyperplanes

In this section we investigate even stronger regularity of $r \mapsto \hat{z}'_r$, namely some forms of continuity. To this end, we need the following two additional assumptions:

B6 (Separable Banach space). The space E is a separable Banach space.

G5 (Weak level set continuity). For all $r \in I$ and all sequences $(r_n) \subset I$ with $r_n \rightarrow r$ there exists an $x \in H \setminus \text{span}\{\Gamma = r\}$ such that

$$d(x, \text{span}\{\Gamma = r_n\}) \rightarrow d(x, \text{span}\{\Gamma = r\}), \quad (10)$$

where the distance is measured in the norm $\|\cdot\|_E$.

Note that the separability of E is not really necessary if one works with nets instead of sequences throughout the proofs for this section. However, for the sake of simplicity, we decided to stick with sequences. Also note that **G5** essentially means, see the proof of Theorem 4.1 for details, that $\langle \hat{z}'_{r_n}, x \rangle \rightarrow \langle \hat{z}'_r, x \rangle$ for this particular x . In other words, **G5** asserts that there is at least one $x \notin \ker z'_r$ for which we have some very weak sort of 'continuity'. Here we put continuity in quotation marks since unlike in continuity, **G5** allows x to depend on the chosen sequence (r_n) .

The following result shows that this is already enough to obtain weak*-continuity of $r \mapsto \hat{z}'_r$.

Theorem 4.1. *Let **B1**, **B2***, **B3**, **B4**, **B6**, **G1***, **G2** be satisfied. Moreover, let $Z : I \rightarrow E'$ be defined by*

$$Z(r) := \hat{z}'_r,$$

where $\hat{z}'_r \in E'$ are the unique functionals obtained in Theorem 2.4. Then the following statements are equivalent:

- i) Assumption **G5** be satisfied.
- ii) For all $r \in I$, all sequences $(r_n) \subset I$ with $r_n \rightarrow r$, and all $x \in E$ convergence (10) holds.
- iii) For all $r \in I$, all sequences $(r_n) \subset I$ with $r_n \rightarrow r$, and all $x \in E$ we have $\langle \hat{z}'_{r_n}, x \rangle \rightarrow \langle \hat{z}'_r, x \rangle$.

If E' is a uniformly convex Banach space and the assumptions of Theorem 4.1 are satisfied, then the map $Z : I \rightarrow E'$ is actually norm continuous. Indeed, uniformly convex Banach spaces are reflexive, see [13, Prop. 1.e.3] or [2, p. 196], and thus weak*-continuity equals weak-continuity. Moreover, our normalization guarantees $\|\hat{z}'_r\| = 1$ for all $r \in I$, and therefore, we obtain norm-continuity by [2, p. 198].

The next result shows that **G5** is superfluous, even for norm-continuity, as long as E is finite-dimensional. In a different form it has also been shown in [12].

Corollary 4.2. *Let **B1**, **B2***, **B3**, **B4**, **G1***, **G2** and be satisfied, and E be finite dimensional. Then, the map $Z : I \rightarrow E'$ defined by*

$$Z(r) := \hat{z}'_r,$$

where $\hat{z}'_r \in E'$ are the unique functionals obtained in Theorem 2.4, is norm continuous.

Note that for finite dimensional spaces E , condition **B4** reduces to $H = E$, that is $E = \text{span} B$. Moreover, in the case of the example discussed around (4) a finite dimension of E means that Ω is finite.

Finally, note that condition **G5** does not appear in Corollary 4.2. Since **B6** is automatically satisfied for finite dimensional spaces, we thus conclude by Theorem 4.1 that **G5** always holds in this setting. Whether this is true in more general settings remains an open question.

5 Quasi-Monotonicity

In this section we briefly recall some simple facts about quasi-monotone functions we need throughout the paper. Some of these results may be folklore but since we were not able to find references establishing these results in the needed generality, we added their proofs.

We begin with the following characterization of quasi-monotonicity.

Lemma 5.1. *Let E be a vector space, $X \subset E$ be a convex subset and $\Gamma : X \rightarrow \mathbb{R}$ be a function. Then the following statements are equivalent:*

- i) The function Γ is quasi-monotone.*
- ii) The function $t \mapsto \Gamma(tx_1 + (1-t)x_0)$ defined on $[0, 1]$ is monotone for all $x_0, x_1 \in X$.*

Proof. In the following, we fix some $x_0, x_1 \in X$ and $t \in [0, 1]$, and define $x_t := tx_1 + (1-t)x_0$.

i) \Rightarrow ii). Without loss of generality we may assume $\Gamma(x_0) \leq \Gamma(x_1)$. Then quasi-monotonicity ensures $\Gamma(x_0) \leq \Gamma(x_t) \leq \Gamma(x_1)$. Now let us fix an $s \in [0, t]$. Then x_s is in the segment between x_0 and x_t and hence we obtain by the same reasoning that $\Gamma(x_0) \leq \Gamma(x_s) \leq \Gamma(x_t)$.

ii) \Rightarrow i). By assumption we have $\min\{\Gamma(x_0), \Gamma(x_1)\} \leq \Gamma(x_t) \leq \max\{\Gamma(x_0), \Gamma(x_1)\}$, and this is equivalent to being both quasi-convex and quasi-concave. \square

Our first result shows that for continuous functionals $\Gamma : X \rightarrow \mathbb{R}$, quasi-monotonicity is equivalent to the convexity of all level sets.

Lemma 5.2. *Let E be a topological vector space, $X \subset E$ be a convex subset and $\Gamma : X \rightarrow \mathbb{R}$ be a continuous function. Then the following statements are equivalent:*

- i) For all $r \in \text{im } \Gamma$, the level sets $\{\Gamma = r\}$ are convex.*
- ii) For all $r \in \text{im } \Gamma$, the sets $\{\Gamma < r\}$ and $\{\Gamma > r\}$ are convex.*
- iii) The function Γ is quasi-monotone, i.e. the sets $\{\Gamma \leq r\}$ and $\{\Gamma \geq r\}$ are convex for all $r \in \text{im } \Gamma$.*

Proof. *i) \Rightarrow ii).* By symmetry, it suffices to consider the case $\{\Gamma < r\}$. Let us assume that $\{\Gamma < r\}$ is not convex. Then there exist $x_0, x_1 \in \{\Gamma < r\}$ and an $\alpha \in (0, 1)$ such that for $x_\alpha := (1-\alpha)x_0 + \alpha x_1$ we have $x_\alpha \notin \{\Gamma < r\}$, that is $\Gamma(x_\alpha) \geq r$. Now, we first observe that, for $r_0 := \Gamma(x_0) < r$ and $r_1 := \Gamma(x_1) < r$, we have $r_0 \neq r_1$, since $r_0 = r_1$ would imply $\Gamma(x_\alpha) \in \{\Gamma = r_0\} \subset \{\Gamma < r\}$ by the assumed convexity of the level set $\{\Gamma = r_0\}$. Let us assume without loss of generality that $r_0 < r_1$. Then we have $r_1 \in (\Gamma(x_0), \Gamma(x_\alpha))$, and thus the intermediate value theorem applied to the continuous map $\beta \mapsto \Gamma((1-\beta)x_0 + \beta x_\alpha)$ on $(0, 1)$ yields a $\beta^* \in (0, 1)$ such that for $x^* := (1-\beta^*)x_0 + \beta^* x_\alpha$ we have $\Gamma(x^*) = r_1$. Let us define $\gamma := \frac{(1-\beta^*)\alpha}{1-\beta^*\alpha}$. Then we have $\gamma \in (0, 1)$ and $x_\alpha = (1-\gamma)x^* + \gamma x_1$. By the assumed convexity of $\{\Gamma = r_1\}$, we thus conclude that $\Gamma(x_\alpha) \in \{\Gamma = r_1\} \subset \{\Gamma < r\}$, i.e. we have found a contradiction.

ii) \Rightarrow iii). This follows from $\{\Gamma \geq r\} = \bigcap_{r' < r} \{\Gamma > r'\}$ and $\{\Gamma \leq r\} = \bigcap_{r' > r} \{\Gamma < r'\}$.

iii) \Rightarrow i). This follows from $\{\Gamma = r\} = \{\Gamma \leq r\} \cap \{\Gamma \geq r\}$. \square

Lemma 5.3. *Let E be a topological vector space, $X \subset E$ be a convex subset and $\Gamma : X \rightarrow \mathbb{R}$ be a continuous, quasi-monotone function. Then the image $\text{im } \Gamma$ is an interval and the sets $\{r < \Gamma < s\}$ are convex, open, and non-empty for all $r, s \in \text{im } \Gamma$ with $r < s$.*

Proof. Since X is convex, it is connected, and thus $\Gamma(X)$ is connected by the continuity of Γ . Since the only connected sets in \mathbb{R} are intervals, we conclude that $\Gamma(X)$ is an interval.

Moreover, the sets $\{r < \Gamma < s\} = \{\Gamma > r\} \cap \{\Gamma < s\}$ are open by the continuity of Γ , and Lemma 5.2 shows that they are also convex. To show that they are non-empty, we fix $r, s \in \text{im } \Gamma$ with $r < s$. Then we have $t := (r+s)/2 \in \text{im } \Gamma$ since $\text{im } \Gamma$ is an interval, and thus there is an $x \in X$ with $\Gamma(x) = t$. The construction now gives $x \in \{r < \Gamma < s\}$. \square

Lemma 5.4. *Let E be a normed space, $X \subset E$ be a convex set and $\Gamma : X \rightarrow \mathbb{R}$ be a quasi-monotone function that has a continuous extension $\hat{\Gamma} : \bar{X} \rightarrow \mathbb{R}$. Then $\hat{\Gamma}$ is quasi-monotone and we have*

$$\text{int } \hat{\Gamma}(\bar{X}) = \Gamma(\overset{\circ}{X}).$$

Proof. The quasi-monotonicity of $\hat{\Gamma}$ can be easily established using (5) and the analogue inequality for quasi-concavity. Moreover, since $\hat{\Gamma}$ is an extension of Γ , we obviously have $\Gamma(X) \subset \hat{\Gamma}(\bar{X})$, and thus we find $\Gamma(\overset{\circ}{X}) \subset \text{int } \hat{\Gamma}(\bar{X})$. To show the converse inclusion, we first note that the continuity of $\hat{\Gamma}$ yields $\hat{\Gamma}(\bar{X}) \subset \overline{\hat{\Gamma}(X)} = \overline{\Gamma(X)}$. Therefore, we find

$$\text{int } \hat{\Gamma}(\bar{X}) \subset \text{int } \overline{\Gamma(X)} = \Gamma(\overset{\circ}{X}),$$

where in the last step we used that $\Gamma(X)$ is an interval. \square

Lemma 5.5. *Let E be a normed space, $X \subset E$ be a non-empty set and $\Gamma : X \rightarrow \mathbb{R}$ be a functional that has a continuous extension $\hat{\Gamma} : \bar{X} \rightarrow \mathbb{R}$. Then, for all $r \in \mathbb{R}$, we have*

$$\{\hat{\Gamma} > r\} \subset \overline{\{\Gamma > r\}}^E \quad (11)$$

$$\overline{\{\Gamma \geq r\}}^E \subset \{\hat{\Gamma} \geq r\}. \quad (12)$$

Proof. To prove (11) we fix an $x \in \{\hat{\Gamma} > r\}$ and define $r^* := \hat{\Gamma}(x)$ and $\varepsilon := r^* - r$. Since $\varepsilon > 0$, the continuity of $\hat{\Gamma}$ shows that $\hat{\Gamma}^{-1}((r^* - \varepsilon, \infty))$ is open in \bar{X} with $x \in \hat{\Gamma}^{-1}((r^* - \varepsilon, \infty))$, and thus there exists a $\delta > 0$ such that $(x + \delta B_E) \cap \bar{X} \subset \hat{\Gamma}^{-1}((r^* - \varepsilon, \infty))$. Moreover, $\{\hat{\Gamma} > r\} \subset \bar{X}$ gives a sequence $(x_n) \subset X$ such that $x_n \rightarrow x$. Clearly, we may assume without loss of generality that $\|x - x_n\|_E \leq \delta$ for all $n \geq 1$, and hence we find $\Gamma(x_n) = \hat{\Gamma}(x_n) > r^* - \varepsilon = r$, for all $n \geq 1$, i.e. $(x_n) \subset \{\Gamma > r\}$.

For the proof of (12) we first observe that $\{\Gamma \geq r\} = \{\hat{\Gamma} \geq r\} \cap X$, and thus we find

$$\overline{\{\Gamma \geq r\}}^E = \overline{\{\hat{\Gamma} \geq r\} \cap X}^E \subset \overline{\{\hat{\Gamma} \geq r\}}^E \cap \bar{X} = \{\hat{\Gamma} \geq r\} \cap \bar{X} = \{\hat{\Gamma} \geq r\},$$

where in the second to last step we used both the continuity of $\hat{\Gamma}$ and the fact that \bar{X} is closed. \square

Lemma 5.6. *Let E be a normed space, $X \subset E$ be a convex set and $\Gamma : X \rightarrow \mathbb{R}$ be a functional that has a continuous and strictly quasi-monotone extension $\hat{\Gamma} : \bar{X} \rightarrow \mathbb{R}$. Then, for all $r \in \Gamma(X)$ we have*

$$\{\hat{\Gamma} \geq r\} = \overline{\{\Gamma \geq r\}}^E.$$

Proof. “ \supset ”. This follows from inclusion (12) of Lemma 5.5.

“ \subset ”. Let us fix an $x \in \{\hat{\Gamma} \geq r\}$. Since $\{\hat{\Gamma} \geq r\} \subset \bar{X}$, there then exists a sequence $(x_n) \subset X$ with $x_n \rightarrow x$. Clearly, if $x_n \in \{\Gamma \geq r\}$ for infinitely many n , then there is nothing left to prove, and hence we assume that $x_n \in \{\Gamma < r\}$ for all $n \geq 1$. The continuity of $\hat{\Gamma}$ then yields $\Gamma(x_n) \rightarrow \hat{\Gamma}(x)$, and therefore we conclude that $\hat{\Gamma}(x) \leq r$, that is $\hat{\Gamma}(x) = r$. Let us now fix an $x^+ \in \{\Gamma > r\}$, which exists by Lemma 5.3 and the fact that Γ is quasi-monotone. For $t \in [0, 1]$ we further define $x(t) := (1-t)x + tx^+$. Since \bar{X} inherits its convexity from X and $x \in \bar{X}$, $x^+ \in X$ we then know that $x(t) \in \bar{X}$ for all $t \in [0, 1]$. Moreover, the strict concavity of $\hat{\Gamma}$ ensures $\hat{\Gamma}(x(t)) > \min\{\Gamma(x), \Gamma(x^+)\} = r$ for all $t \in (0, 1)$ and thus we conclude by (11) that

$$x(t) \in \{\hat{\Gamma} > r\} \subset \overline{\{\Gamma > r\}}^E$$

for all $t \in (0, 1)$. For all $n \geq 2$ there thus exist an $x_n^+ \in \{\Gamma > r\}$ with $\|x_n^+ - x(1/n)\|_E \leq 1/n$. Since $\|x(1/n) - x\|_E \leq n^{-1}\|x - x^+\|_E$ we then obtain $x_n^+ \rightarrow x$, which finishes the proof. \square

6 Proofs for Section 2

Proof of Lemma 2.1. Let us fix a $y \in F$. Since $F = \text{span}(-x_* + B)$, there then exists $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $x_1, \dots, x_n \in B$ such that $y = \sum_{i=1}^n \alpha_i(-x_* + x_i)$. By the linearity of φ' , this yields

$$\langle \varphi', y \rangle = \sum_{i=1}^n \alpha_i (\langle \varphi', x_i \rangle - \langle \varphi', x_* \rangle) = 0,$$

where in the last step we used $\langle \varphi', x_i \rangle = 1 = \langle \varphi', x_\star \rangle$. Now $x_\star \notin F$ follows from the just established $F \subset \ker \varphi'$ and $\langle \varphi', x_\star \rangle = 1$. In addition, we immediately obtain $F \cap \mathbb{R}x_\star = \{0\}$, and thus $F \oplus \mathbb{R}x_\star$ is indeed a direct sum. Moreover, the equality $F \oplus \mathbb{R}x_\star = \text{span } B$ follows from

$$\sum_{i=1}^n \alpha_i(-x_\star + x_i) + \alpha_0 x_\star = \sum_{i=1}^n \alpha_i x_i + \left(\alpha_0 - \sum_{i=1}^n \alpha_i \right) x_\star,$$

which holds for all $n \in \mathbb{N}$, $\alpha_0, \dots, \alpha_n \in \mathbb{R}$, and $x_1, \dots, x_n \in B$. Now, $\|\cdot\|_H$ can be constructed in the described way. Here we note, that the definition of $\|\cdot\|_H$ resembles a standard way of defining norms on direct sums, and thus $\|\cdot\|_H$ is indeed a norm. Furthermore, $\|\cdot\|_E \leq \|\cdot\|_H$ immediately follows from the construction of $\|\cdot\|_H$ and the assumed $\|\cdot\|_E \leq \|\cdot\|_F$. Finally, $\|\cdot\|_F = \|\cdot\|_H$ on F is obvious and so is $B - B \subset F$. \square

Our next little lemma shows that the space H can also be generated from F and an *arbitrary* element of B .

Lemma 6.1. *If **B1** and **B2** are satisfied, then we have $F \oplus \mathbb{R}x_0 = H$ for all $x_0 \in B$.*

Proof. By $\varphi'(x_0) = 1$ and the inclusion $F \subset \ker \varphi'$ established in Lemma 2.1, we see that $x_0 \notin F$, and hence $F \cap \mathbb{R}x_0 = \{0\}$.

The inclusion $F \oplus \mathbb{R}x_0 \subset H$ follows from the equality $H = \text{span } B$ established in Lemma 2.1 and

$$\sum_{i=1}^n \alpha_i(-x_\star + x_i) + \alpha_0 x_0 = \sum_{i=0}^n \alpha_i x_i - \sum_{i=1}^n \alpha_i x_\star,$$

which holds for all $n \in \mathbb{N}$, $\alpha_0, \dots, \alpha_n \in \mathbb{R}$, and $x_1, \dots, x_n \in B$.

To prove the converse inclusion, we first note that $-x_\star = (-x_\star + x_0) - x_0 \in F \oplus \mathbb{R}x_0$ implies $\mathbb{R}x_\star \subset F \oplus \mathbb{R}x_0$. Since we also have $F \subset F \oplus \mathbb{R}x_0$, we conclude by Lemma 2.1 that $H = F \oplus \mathbb{R}x_\star \subset F \oplus \mathbb{R}x_0$. \square

Our next lemma shows that the cone decomposition **B3** makes it easier to decide whether a linear functional is continuous.

Lemma 6.2. *Let **B3** be satisfied. Then a linear map $z' : H \rightarrow \mathbb{R}$ is continuous with respect to $\|\cdot\|_E$, if and only if for all sequences $(z_n) \subset \text{cone } B$ with $\|z_n\|_E \rightarrow 0$ we have $\langle z', z_n \rangle \rightarrow 0$.*

Proof. “ \Rightarrow ”: Since $\text{cone } B \subset H$ by the definition of H , this implication is trivial.

“ \Leftarrow ”: By the linearity of z' it suffices to show that z' is $\|\cdot\|_E$ -continuous at 0. To show the latter, we fix a sequence $(z_n) \subset H$ with $\|z_n\|_E \rightarrow 0$. By **B3** there then exist sequences $(z_n^-), (z_n^+) \subset \text{cone } B$ with $z_n = z_n^+ - z_n^-$ and $\|z_n^-\|_E + \|z_n^+\|_E \leq K\|z_n\|_E$. Consequently, we obtain $\|z_n^-\|_E \rightarrow 0$ and $\|z_n^+\|_E \rightarrow 0$, and thus our assumption together with the linearity of z' yields $\langle z', z_n \rangle = \langle z', z_n^+ \rangle - \langle z', z_n^- \rangle \rightarrow 0$. \square

In the following, we almost always need the assumption **B2** to be satisfied. In this case, we sometimes need to consider two metrics on B , namely the metric d_E induced by $\|\cdot\|_E$ and the metric d_F induced by $\|\cdot\|_F$ via translation, that is

$$d_F(x_1, x_2) := \|(-x_\star + x_1) - (-x_\star + x_2)\|_F = \|x_1 - x_2\|_F = \|x_1 - x_2\|_H, \quad x_1, x_2 \in B, \quad (13)$$

where the last identity follows from Lemma 2.1 provided that **B1** also holds. Note that the assumed $\|\cdot\|_E \leq \|\cdot\|_F$ immediately implies $d_E(x_1, x_2) \leq d_F(x_1, x_2)$ for all $x_1, x_2 \in B$, and thus the identity map $\text{id} : (B, d_F) \rightarrow (B, d_E)$ is Lipschitz continuous.

The following result collects some simple properties of the sets $\{\Gamma < r\}$ and $\{\Gamma > r\}$ we wish to separate.

Lemma 6.3. *Let **B2** and **G1** be satisfied. Then $\Gamma(B)$ is an interval, and, for all $r \in \mathring{\Gamma}(B)$, the sets $\{\Gamma < r\}$ and $\{\Gamma > r\}$ are non-empty, convex, and open in B with respect to d_F .*

Proof. Clearly, the sets $\{\Gamma < r\}$ and $\{\Gamma > r\}$ are open with respect to d_F , since Γ is assumed to be continuous with respect to d_F . The remaining assertions follow from the Lemma 5.2 and 5.3. \square

Our next goal is to investigate relative interiors of subsets of A . We begin with a result that shows the richness of \mathring{A}^F .

Lemma 6.4. *Let $\mathbf{B2}^*$ and $\mathbf{G1}$ be satisfied. Then, for all $r \in I$, there exists an $x \in \{\Gamma = r\}$ such that $-x_\star + x \in \mathring{A}^F$.*

Proof. If $x_\star \in \{\Gamma = r\}$ there is nothing to prove, and hence we may assume without loss of generality that $x_\star \in \{\Gamma > r\}$. Let us write $r^\star := \Gamma(x_\star)$. Now, since $r \in I$ and I is an open interval by Lemma 6.3, there exists an $s \in I$ with $s < r$. We fix an $x_0 \in \{\Gamma = s\}$ and, for $\lambda \in [0, 1]$, we consider $x_\lambda := \lambda x_\star + (1 - \lambda)x_0$. Then we have $\Gamma(x_0) = s < r < r^\star = \Gamma(x_\star)$, and thus the intermediate theorem shows that there exists a $\lambda \in (0, 1)$ with $\Gamma(x_\lambda) = r$. Our goal is to show that this x_λ satisfies $-x_\star + x_\lambda \in \mathring{A}^F$. To this end, we recall that $0 \in \mathring{A}^F$, which is ensured by $\mathbf{B2}^*$, gives an $\varepsilon > 0$ such that for all $y \in F$ satisfying $\|y\|_F \leq \varepsilon$ we actually have $y \in A$. Let us write $\delta := \lambda\varepsilon$. Then it suffices to show that, for all $y \in F$ satisfying $\| -x_\star + x_\lambda - y \|_F \leq \delta$, we have $y \in A$. Consequently, let us fix such a $y \in F$. For

$$\tilde{x} := x_\star + \frac{y - (1 - \lambda)(-x_\star + x_0)}{\lambda}$$

we then have $y = \lambda(-x_\star + \tilde{x}) + (1 - \lambda)(-x_\star + x_0)$. By the convexity of A and $-x_\star + x_0 \in A$, it thus suffices to show $-x_\star + \tilde{x} \in A$. However, the latter follows from

$$\begin{aligned} \| -x_\star + \tilde{x} \|_F &= \lambda^{-1} \| y - (1 - \lambda)(-x_\star + x_0) \|_F \\ &= \lambda^{-1} \| y - x_\lambda + x_\star \|_F \\ &\leq \lambda^{-1} \delta, \end{aligned}$$

and thus the assertion is proven. \square

Our last elementary result shows that having non-empty relative interior in A implies a non-empty relative interior in F . This result will later be applied to translates of the open, non-empty sets $\{\Gamma < r\}$ and $\{\Gamma > r\}$.

Lemma 6.5. *Let $\mathbf{B2}^*$ be satisfied, and $K \subset A$ be an arbitrary subset with $\mathring{K}^A \neq \emptyset$, that is K has non-empty relative $\|\cdot\|_F$ -interior in A . Then, for all $y \in \mathring{K}^A$, there exists a $\delta_y \in (0, 1/2]$ such that $(1 - \delta)y \in \mathring{K}^F$ for all $\delta \in (0, \delta_y]$. In particular, we have $\mathring{K}^F \neq \emptyset$.*

Proof. By the assumed $0 \in \mathring{A}^F$, there exists an $\varepsilon_0 \in (0, 1]$ such that $\varepsilon_0 B_F \subset A$. Moreover, the assumption $y \in \mathring{K}^A$ yields an $\varepsilon_1 \in (0, \varepsilon_0]$ such that

$$(y + \varepsilon_1 B_F) \cap A \subset K. \quad (14)$$

We define $\delta_y := \varepsilon_1 / (\varepsilon_1 + \|y\|_F)$. Then, it suffices to show that

$$(1 - \delta)y + \varepsilon_1 \delta B_F \subset K \quad (15)$$

for all $\delta \in (0, \delta_y]$. To show the latter, we fix a $y_1 \in \varepsilon_1 \delta B_F$. An easy estimate then shows that $\| -\delta y + y_1 \|_F \leq \delta \|y\|_F + \|y_1\|_F \leq \delta(\|y\|_F + \varepsilon_1) \leq \varepsilon_1$, and hence we obtain

$$(1 - \delta)y + y_1 = y - \delta y + y_1 \in (y + \varepsilon_1 B_F).$$

By (14) it thus suffices to show $(1 - \delta)y + y_1 \in A$. Now, if $y_1 = 0$, then the latter immediately follows from $(1 - \delta)y + y_1 = (1 - \delta)y + \delta \cdot 0$, the convexity of A , and $0 \in A$. Therefore, it remains to consider the case $y_1 \neq 0$. Then we have

$$\frac{\varepsilon_0}{\|y_1\|_F} y_1 \in \varepsilon_0 B_F \subset A,$$

and $\frac{\|y_1\|_F}{\varepsilon_0} \leq \frac{\varepsilon_1 \delta}{\varepsilon_0} \leq \delta$. Consequently, the convexity of A and $0 \in A$ yield

$$(1 - \delta)y + y_1 = (1 - \delta)y + \frac{\|y_1\|_F}{\varepsilon_0} \left(\frac{\varepsilon_0}{\|y_1\|_F} y_1 \right) + \left(\delta - \frac{\|y_1\|_F}{\varepsilon_0} \right) \cdot 0 \in A,$$

and hence (15) follows. \square

Our next goal is to move towards the proof of Theorem 2.3. This is done in a couple of intermediate results that successively establish more properties of certain, separating functionals. We begin with a somewhat crude separation of convex subsets in A that have an non-empty relative interior.

Lemma 6.6. *Let $\mathbf{B2}^*$ be satisfied, and $K_-, K_+ \subset A$ be two convex sets with $\mathring{K}_\pm^A \neq \emptyset$ and $K_- \cap \mathring{K}_+^F = \emptyset$. Then there exist a $y' \in F'$ and an $s \in \mathbb{R}$ such that*

$$\begin{aligned} K_- &\subset \{y' \leq s\} & \text{and} & & \mathring{K}_-^F &\subset \{y' < s\}, \\ K_+ &\subset \{y' \geq s\} & \text{and} & & \mathring{K}_+^F &\subset \{y' > s\}. \end{aligned}$$

Moreover, if $s \leq 0$, then we actually have $\mathring{K}_-^A \subset \{y' < s\}$, and, if $s \geq 0$, we have $\mathring{K}_+^A \subset \{y' > s\}$.

Proof. By Lemma 6.5 and the assumed $\mathring{K}_\pm^A \neq \emptyset$ we find $\mathring{K}_\pm^F \neq \emptyset$. By a version of the Hahn-Banach separation theorem, see e.g. [14, Thm. 2.2.26], there thus exist a $y' \in F'$ and an $s \in \mathbb{R}$ such that

$$\begin{aligned} K_- &\subset \{y' \leq s\} \\ K_+ &\subset \{y' \geq s\} \\ \mathring{K}_+^F &\subset \{y' > s\}. \end{aligned}$$

Let us first show $\mathring{K}_-^F \subset \{y' < s\}$. To this end, we fix a $y_- \in \mathring{K}_-^F$ and a $y_+ \in \mathring{K}_+^F$. Since \mathring{K}_-^F is open in F , there then exists a $\lambda \in (0, 1)$ such that

$$\lambda y_+ + (1 - \lambda)y_- = y_- + \lambda(y_+ - y_-) \in \mathring{K}_-^F \subset K_-.$$

From the latter and the already obtained inclusions we conclude that

$$s \geq \langle y', \lambda y_+ + (1 - \lambda)y_- \rangle = \lambda \langle y', y_+ \rangle + (1 - \lambda) \langle y', y_- \rangle > \lambda s + (1 - \lambda) \langle y', y_- \rangle.$$

Now, some simple transformations together with $\lambda \in (0, 1)$ yield $\langle y', y_- \rangle < s$, i.e. we have shown $\mathring{K}_-^F \subset \{y' < s\}$.

Let us now show that $s \leq 0$ implies $\mathring{K}_-^A \subset \{y' < s\}$. To this end, we assume that there exists a $y \in \mathring{K}_-^A$ with $\langle y', y \rangle \geq s$. Since $\mathring{K}_-^A \subset K_-$, the already established inclusion $K_- \subset \{y' \leq s\}$ then yields $\langle y', y \rangle = s$. Moreover, by Lemma 6.5 there exists a $\delta > 0$ such that $(1 - \delta)y \in \mathring{K}_-^F$. From the previously established $\mathring{K}_-^F \subset \{y' < s\}$ we thus obtain

$$s > \langle y', (1 - \delta)y \rangle = (1 - \delta)s.$$

Clearly, this yields $\delta s > 0$, and since $\delta > 0$, we find $s > 0$. The remaining implication can be shown analogously. \square

The next result refines the separation of Lemma 6.6 under additional assumptions on the sets that are to be separated. Its assertion, but not its proof, mimics the first part of Step 2 of the proof of Theorem 5 of [12].

Proposition 6.7. *Let $\mathbf{B2}^*$ be satisfied, and $K_-, K_0, K_+ \subset A$ be mutually disjoint, non-empty convex sets with $\mathring{K}_\pm^A = K_\pm$ and $A = K_- \cup K_0 \cup K_+$. Furthermore, assume that, for all $y \in K_0$ and $\varepsilon > 0$, we have $K_- \cap (y + \varepsilon B_F) \neq \emptyset$ and $K_+ \cap (y + \varepsilon B_F) \neq \emptyset$. Then there exist a $y' \in F'$ and an $s \in \mathbb{R}$ such that*

$$\begin{aligned} K_- &= \{y' < s\} \cap A \\ K_0 &= \{y' = s\} \cap A \\ K_+ &= \{y' > s\} \cap A. \end{aligned}$$

Proof. We first observe that we clearly have $\mathring{K}_\pm^A = K_\pm \neq \emptyset$ and $K_- \cap \mathring{K}_+^F \subset K_- \cap K_+ = \emptyset$. Consequently, Lemma 6.6 provides a $y' \in F'$ and an $s \in \mathbb{R}$ that satisfy the inclusions listed in Lemma 6.6.

Our first goal is to show $K_0 = \{y' = s\} \cap A$. To prove $K_0 \subset \{y' = s\} \cap A$, we fix a $y \in K_0$. Since $K_- \cap (y + \varepsilon B_F) \neq \emptyset$ for all $\varepsilon > 0$, we then find a sequence $(y_n) \subset K_-$ such that $y_n \rightarrow y$. By Lemma 6.6 we then obtain

$$\langle y', y \rangle = \lim_{n \rightarrow \infty} \langle y', y_n \rangle \leq s,$$

i.e. $y \in \{y' \leq s\} \cap A$. Using $K_+ \cap (y + \varepsilon B_F) \neq \emptyset$ for all $\varepsilon > 0$, we can analogously show $y \in \{y' \geq s\} \cap A$, and hence we obtain $y \in \{y' = s\} \cap A$.

To show the inclusion $\{y' = s\} \cap A \subset K_0$, we assume without loss of generality that $s \geq 0$. Let us now fix a $y \in A \setminus K_0$, so that our goal becomes to show $y \notin \{y' = s\} \cap A$. Now, if $y \in K_+$, we obtain $\langle y', y \rangle > s$, since we have seen in Lemma 6.6 that $s \geq 0$ implies $K_+ = \mathring{K}_+^A \subset \{y' > s\}$. Therefore, it remains to consider the case $y \in K_-$. Let us fix a $y_1 \in K_+$. Then we have just seen that $\langle y', y_1 \rangle > s$. For $\lambda \in [0, 1]$ we now define $y_\lambda := \lambda y_1 + (1 - \lambda)y$. Now, if there is a $\lambda \in (0, 1)$ with $\langle y', y_\lambda \rangle = s$, we obtain

$$s = \langle y', \lambda y_1 + (1 - \lambda)y \rangle = \lambda \langle y', y_1 \rangle + (1 - \lambda) \langle y', y \rangle > \lambda s + (1 - \lambda) \langle y', y \rangle,$$

that is $\langle y', y \rangle < s$. Consequently, it remains to show the existence of such a $\lambda \in (0, 1)$. Let us assume the converse, that is $y_\lambda \in K_- \cup K_+$ for all $\lambda \in (0, 1)$ by the already established $K_0 \subset \{y' = s\} \cap A$. Since $y_0 = y \in K_-$ and $y_1 \in K_+$, we then have

$$y_\lambda \in K_- \cup K_+ \tag{16}$$

for all $\lambda \in [0, 1]$. Let us now consider the map $\psi : [0, 1] \rightarrow A$ defined by $\psi(\lambda) := y_\lambda$. Clearly, ψ is continuous, and since $K_\pm = \mathring{K}_\pm^A$, the pre-images $\psi^{-1}(K_-)$ and $\psi^{-1}(K_+)$ are open, and, of course, disjoint. Moreover, by $\psi(0) = y_0 = y \in K_-$ and $\psi(1) = y_1 \in K_+$, they are also non-empty, and (16) ensures $\psi^{-1}(K_-) \cup \psi^{-1}(K_+) = [0, 1]$. Consequently, we have found a partition of $[0, 1]$ consisting of two open, non-empty sets, i.e. $[0, 1]$ is not connected. Since this is obviously false, we found a contradiction finishing the proof of $\{y' = s\} \cap A \subset K_0$.

To prove the remaining two equalities, let us again assume without loss of generality that $s \geq 0$. By Lemma 6.6, we then know $K_+ = \mathring{K}_+^A \subset \{y' > s\} \cap A$. Conversely, for $y \in \{y' > s\} \cap A$ we have already shown $y \notin K_0$, and by the inclusion $K_- \subset \{y' \leq s\}$ established in Lemma 6.6 we also know $y \notin K_-$. Since $A = K_- \cup K_0 \cup K_+$, we conclude that $y \in K_+$. Consequently, we have also shown $K_+ = \{y' > s\} \cap A$, and the remaining $K_- = \{y' < s\} \cap A$ now immediately follows. \square

The next result, whose assertion mimics the second part of Step 2 as well as Step 3 of the proof of Theorem 5 in an earlier version of [12], shows the existence of the separating families considered in Theorem 2.3 and Theorem 2.4. The construction idea (17) of z' and the proof of its $\|\cdot\|_E$ -continuity is an abstraction from Lambert's proof. However, the remaining parts of our proof heavily rely on the preceding results of this section and are therefore independent of [12].

Theorem 6.8. *Let **B1**, **B2***, **G1**, and **G2** be satisfied. Then, for all $r \in I$, there exists a $z' \in H'$ such that*

$$\begin{aligned} \{\Gamma < r\} &= \{z' < 0\} \cap B \\ \{\Gamma = r\} &= \{z' = 0\} \cap B \\ \{\Gamma > r\} &= \{z' > 0\} \cap B. \end{aligned}$$

*If, in addition, **B3** and **G1*** are satisfied, then z' is actually continuous with respect to $\|\cdot\|_E$.*

Proof. For some fixed $r \in I$ we consider the sets

$$\begin{aligned} K_- &:= -x_* + \{\Gamma < r\} \\ K_0 &:= -x_* + \{\Gamma = r\} \\ K_+ &:= -x_* + \{\Gamma > r\}. \end{aligned}$$

Our first goal is to show that these sets satisfy the assumptions of Proposition 6.7. To this end, we first observe that $\{\Gamma < r\} \subset B$ immediately implies $K_- \subset -x_* + B = A$, and the same argument can be applied to K_0 and K_+ . Moreover, they are mutually disjoint since the defining level sets are mutually

disjoint, and since $r \in \overset{\circ}{\Gamma}(B)$ they are also non-empty. The equality $A = K_- \cup K_0 \cup K_+$ follows from $B = \{\Gamma < r\} \cup \{\Gamma = r\} \cup \{\Gamma > r\}$, and the convexity of K_- and K_+ is a consequence of the convexity of $\{\Gamma < r\}$ and $\{\Gamma > r\}$ established in Lemma 6.3. The convexity of K_0 follows from **G1**. Moreover, by Lemma 6.3, the set $\{\Gamma < r\}$ is open in B with respect to d_F , and since the metric spaces (B, d_F) and $(A, \|\cdot\|_F)$ are isometrically isomorphic via translation with $-x_*$, we see that K_- is open in A with respect to $\|\cdot\|_F$. This shows $\overset{\circ}{K}_-^A = K_-$, and $\overset{\circ}{K}_+^A = K_+$ can be shown analogously. Finally, observe that for $x \in \{\Gamma = r\}$, $\varepsilon > 0$, and $y := -x_* + x$ we have

$$\begin{aligned} K_- \cap (y + \varepsilon B_F) &= (-x_* + \{\Gamma < r\}) \cap (-x_* + x + \varepsilon B_F) \\ &= (-x_* + \{\Gamma < r\}) \cap (-x_* + x + \varepsilon B_H) \\ &= -x_* + (\{\Gamma < r\} \cap (x + \varepsilon B_H)) \\ &\neq \emptyset, \end{aligned}$$

where in the second step we used the fact $\|\cdot\|_F = \|\cdot\|_H$ on $A \subset F$, see Lemma 2.1, and the last step relies on **G2**. Obviously, $K_+ \cap (y + \varepsilon B_F) \neq \emptyset$ can be shown analogously, and hence, the assumptions of Proposition 6.7 are indeed satisfied.

Now, let $y' \in F'$ and $s \in \mathbb{R}$ be according to Proposition 6.7. Moreover, let $\hat{y}' \in H'$ be the extension of y' to H that is defined by

$$\langle \hat{y}', y + \alpha x_* \rangle := \langle y', y \rangle$$

for all $y + \alpha x_* \in H = F \oplus \mathbb{R}x_*$. Clearly, \hat{y}' is indeed an extension of y' to H and the continuity of \hat{y}' on H follows from

$$|\langle \hat{y}', y + \alpha x_* \rangle| = |\langle y', y \rangle| \leq \|y'\| \cdot \|y\|_F \leq \|y'\| \cdot \|y + \alpha x_*\|_H.$$

With these preparations, we now define a $z' \in H'$ by

$$\langle z', z \rangle := -s \langle \varphi', z \rangle + \langle \hat{y}', z - \langle \varphi', z \rangle x_* \rangle, \quad z \in H. \quad (17)$$

Obviously, z' is linear. Moreover, the restriction φ'_H of φ' to H is continuous with respect to $\|\cdot\|_H$, since Lemma 2.1 ensured $\|\cdot\|_E \leq \|\cdot\|_H$ on H , and consequently we obtain $z' \in H'$.

Let us show that z' is the desired functional. To this end, we first observe that the inclusion $F \subset \ker \varphi'$ established in Lemma 2.1 together with $x_* \in B \subset \{\varphi' = 1\}$ yields $x_* + F \subset \{\varphi' = 1\}$. For $x \in x_* + F \subset H$ this gives

$$\langle z', x \rangle = -s \langle \varphi', x \rangle + \langle \hat{y}', x - \langle \varphi', x \rangle x_* \rangle = -s + \langle \hat{y}', x - x_* \rangle = -s + \langle y', x - x_* \rangle.$$

Moreover, recall that we have $x \in B$ if and only if $-x_* + x \in A$, and hence we obtain

$$\begin{aligned} \{z' = 0\} \cap B &= \{x \in B : \langle y', x - x_* \rangle = s\} \\ &= \{x \in B : -x_* + x \in \{y' = s\}\} \\ &= x_* + \{y \in A : y \in \{y' = s\}\} \\ &= x_* + (\{y' = s\} \cap A) \\ &= x_* + K_0 \\ &= \{\Gamma = r\}. \end{aligned}$$

The remaining equalities $\{\Gamma < r\} = \{z' < 0\} \cap B$ and $\{\Gamma > r\} = \{z' > 0\} \cap B$ can be shown analogously.

Let us finally show that the functional z' found so far is actually continuous with respect to $\|\cdot\|_E$, if **B3** and **G1*** are satisfied. Let us assume the converse. By Lemma 6.2, there then exists a sequence $(z_n) \subset \text{cone } B$ with $\|z_n\|_E \rightarrow 0$ and $\langle z', z_n \rangle \not\rightarrow 0$. Picking a suitable subsequence and scaling it appropriately, we may assume without loss of generality that either $\langle z', z_n \rangle < -1$ for all $n \geq 1$, or $\langle z', z_n \rangle > 1$ for all $n \geq 1$. Let us consider the first case, only, the second case can be treated analogously. We begin by picking an $x_0 \in \{\Gamma > r\} = \{z' > 0\} \cap B$. This yields $\alpha := \langle z', x_0 \rangle > 0$. Moreover, since $(z_n) \subset \text{cone } B$ and $z_n \neq 0$ by the assumed $\langle z', z_n \rangle < -1$, we find sequences $(\alpha_n) \subset (0, \infty)$ and $(x_n) \subset B$ such that $z_n = \alpha_n x_n$ for all $n \geq 1$. Our first goal is to show that $\alpha_n \rightarrow 0$. To this end, we observe that $x_n \in B \subset \{\varphi' = 1\}$ implies $1 = |\langle \varphi', x_n \rangle| \leq \|\varphi'\| \cdot \|x_n\|_E$, and hence we obtain

$$|\alpha_n| \leq |\alpha_n| \cdot \|\varphi'\| \cdot \|x_n\|_E = \|\varphi'\| \cdot \|z_n\|_E \rightarrow 0.$$

For $n \geq 1$, we define $\beta_n := \frac{1}{1+\alpha\alpha_n}$. Our considerations made so far then yield both $\beta_n \rightarrow 1$ and $\beta_n \in (0, 1)$ for all $n \geq 1$. By the definition of α and the assumptions made on (z_n) , this yields

$$\langle z', \beta_n(x_0 + \alpha z_n) \rangle = \beta_n(\alpha + \alpha \langle z', z_n \rangle) < 0 \quad (18)$$

for all $n \geq 1$. On the other hand, $x_0 \in \{\Gamma > r\}$ ensures $\frac{\Gamma(x_0)-r}{2} > 0$, and since $\mathbf{G1}^*$ assumes that Γ is $\|\cdot\|_E$ -continuous, there thus exists a $\delta > 0$ such that, for all $x \in B$ with $\|x - x_0\|_E \leq \delta$, we have

$$|\Gamma(x) - \Gamma(x_0)| \leq \frac{\Gamma(x_0) - r}{2}.$$

For such x , a simple transformation then yields $\Gamma(x) \geq \frac{\Gamma(x_0)+r}{2} > r$, and thus we find

$$\{x \in B : \|x - x_0\|_E \leq \delta\} \subset \{\Gamma > r\} = \{z' > 0\} \cap B.$$

To find a contradiction to (18), it thus suffices to show that

$$\beta_n(x_0 + \alpha z_n) \in \{x \in B : \|x - x_0\|_E \leq \delta\} \quad (19)$$

for all sufficiently large n . To prove this, we first observe that

$$\beta_n(x_0 + \alpha z_n) = \beta_n x_0 + \frac{\alpha\alpha_n}{1 + \alpha\alpha_n} x_n = \beta_n x_0 + (1 - \beta_n)x_n,$$

and since $\beta_n \in (0, 1)$, the convexity of B yields $\beta_n(x_0 + \alpha z_n) \in B$. Finally, we have

$$\|x_0 - \beta_n(x_0 + \alpha z_n)\|_E \leq (1 - \beta_n)\|x_0\|_E + \alpha\beta_n\|z_n\|_E \rightarrow 0$$

since $\beta_n \rightarrow 1$ and $\|z_n\|_E \rightarrow 0$. Consequently, (19) is indeed satisfied for all sufficiently large n , which finishes the proof. \square

Theorem 6.8 has shown the existence of a functional separating the level sets of Γ . Our next and final goal is to show that this functional is unique modulo normalization. To this end, we need the following lemma, which shows that the null space of a separating functional is completely determined by the set $\{\Gamma = r\}$.

Note that the assertion of the Lemmas 6.9 and 6.10 are inspired by Step 3 of the proof of Theorem 5 of [12], but again our proofs are more complicated, since we cannot guarantee $x_\star \in \{\Gamma = r\}$.

Lemma 6.9. *Let $\mathbf{B1}$, $\mathbf{B2}^*$, and $\mathbf{G1}$ be satisfied. Moreover, let $r \in I$ and $z : H \rightarrow \mathbb{R}$ be a linear functional satisfying $\{\Gamma = r\} = B \cap \ker z'$. Then we have $z' \neq 0$ and*

$$\ker z' = \text{span}(\ker z' \cap B) = \text{span}\{\Gamma = r\}.$$

Proof. The second equality is obvious, and since $\ker z'$ is a subspace, the inclusion $\text{span}(\ker z' \cap B) \subset \ker z'$ is also obvious.

To prove the converse inclusion, we fix a $z \in \ker z'$. Moreover, using Lemma 6.4, we fix an $x_0 \in \{\Gamma = r\} = B \cap \ker z'$ satisfying $-x_\star + x_0 \in \mathring{A}^F$. By $z \in \ker z' \subset H$ and Lemma 6.1, which showed $H = F \oplus \mathbb{R}x_0$, there then exist a $y \in F$ and an $\alpha \in \mathbb{R}$ such that $z = y + \alpha x_0$. Obviously, it suffices to show both $\alpha x_0 \in \text{span}(\ker z' \cap B)$ and $y \in \text{span}(\ker z' \cap B)$. Now, $\alpha x_0 \in \text{span}(\ker z' \cap B)$ immediately follows from $x_0 \in \ker z' \cap B$, and for $y = 0$ the second inclusion is trivial. Therefore, let us assume that $y \neq 0$. Since $-x_\star + x_0 \in \mathring{A}^F$, there then exists an $\varepsilon > 0$ such that for all $\check{y} \in F$ with $\| -x_\star + x_0 - \check{y} \|_F \leq \varepsilon$ we have $\check{y} \in A$. Writing $\hat{y} := \frac{\varepsilon}{\|y\|_F} y$, we have $\hat{y} \in F$ by the assumed $y \in F$, and thus also $\tilde{y} := -x_\star + x_0 + \hat{y} \in F$. Moreover, our construction immediately yields $\| -x_\star + x_0 - \tilde{y} \|_F = \varepsilon$, and hence we actually have $\tilde{y} \in A = -x_\star + B$. Consequently, we have found $x_0 + \hat{y} = \tilde{y} + x_\star \in B$. On the other hand, the assumed $x_0 \in \ker z'$ implies $\alpha x_0 \in \ker z'$, and thus we find $y \in \ker z'$ by $z \in \ker z'$ and $z = y + \alpha x_0$. Using both $x_0, y \in \ker z'$, we thus obtain $x_0 + \hat{y} \in \ker z'$, which together with the already established $x_0 + \hat{y} \in B$ shows $x_0 + \hat{y} \in \text{span}(\ker z' \cap B)$. Since $x_0 \in B \cap \ker z'$ by assumption we therefore finally find the desired $y \in \text{span}(\ker z' \cap B)$ by the definition of \hat{y} .

Finally, assume that $z' = 0$. By Lemma 5.3 in combination with $\mathbf{G1}$ and Lemma 5.2 we find an $x \in \{\Gamma < r\}$, and the assumed $z' = 0$ implies $x \in \ker z'$, while $\{\Gamma < r\} \subset B$ implies $x \in B$. This yields $x \in B \cap \ker z' = \{\Gamma = r\}$, which contradicts $x \in \{\Gamma < r\}$. \square

The following lemma shows that, modulo orientation, two normalized separating functionals are equal.

Lemma 6.10. *Let $\mathbf{B1}$, $\mathbf{B2}^*$, and $\mathbf{G1}$ be satisfied. Moreover, let $r \in I$ and $z'_1, z'_2 \in H'$ such that $\{\Gamma = r\} = B \cap \ker z'_1$ and $\{\Gamma = r\} \subset B \cap \ker z'_2$. Then there exists an $\alpha \in \mathbb{R}$ such that $z'_2 = \alpha z'_1$, and if $\{\Gamma = r\} = B \cap \ker z'_2$, we actually have $\alpha \neq 0$.*

Proof. Our assumptions guarantee $B \cap \ker z'_1 \subset B \cap \ker z'_2 \subset \ker z'_2$, and thus Lemma 6.9 yields $\ker z'_1 \subset \ker z'_2$. Moreover, Lemma 6.9 shows $z'_1 \neq 0$, which in turn gives a $z_0 \in H$ with $z_0 \notin \ker z'_1$. For $z \in H$, an easy calculation then shows that

$$z - \frac{\langle z'_1, z \rangle}{\langle z'_1, z_0 \rangle} z_0 \in \ker z'_1 \subset \ker z'_2,$$

and hence we conclude that $\langle z'_2, z \rangle = \frac{\langle z'_1, z \rangle}{\langle z'_1, z_0 \rangle} \langle z'_2, z_0 \rangle$. In other words, for $\alpha := \frac{\langle z'_2, z_0 \rangle}{\langle z'_1, z_0 \rangle}$, we have $z'_2 = \alpha z'_1$. Finally, $\{\Gamma = r\} = B \cap \ker z'_2$ implies $z'_2 \neq 0$ by Lemma 6.9, and hence we conclude that $\alpha \neq 0$. \square

Proof of Theorem 2.3. *i) \Rightarrow iv).* The existence has been proven in the first part of Theorem 6.8. To show the uniqueness, we assume that we have two normalized separating families $(z'_r)_{r \in I} \subset H'$ and $(\tilde{z}'_r)_{r \in I} \subset H'$ for Γ . Moreover, we fix an $r \in I$. Then Lemma 6.10 gives an $\alpha \neq 0$ with $z'_r = \alpha \tilde{z}'_r$. The imposed normalization $\|z'_r\|_{H'} = 1 = \|\tilde{z}'_r\|_{H'}$ implies $|\alpha| = 1$, and the orientation of z'_r and \tilde{z}'_r on $\{\Gamma < r\}$ excludes the case $\alpha = -1$. Thus we have $z'_r = \tilde{z}'_r$.

iv) \Rightarrow iii). Trivial.

iii) \Rightarrow ii). By (6) and (13) we know that $\{\Gamma < r\}$ is relatively open in B with respect to d_F for all $r \in I$. Moreover, for $r < \inf I$ we have $\{\Gamma < r\} = \emptyset$ and for $r > \sup I$ we have $\{\Gamma < r\} = B$. Finally, if $r := \sup I < \infty$, then

$$\{\Gamma < r\} = \bigcup_{n \geq 1} \{\Gamma < r - 1/n\},$$

and therefore $\{\Gamma < r\}$ is relatively open in B with respect to d_F for all $r \in \mathbb{R}$. Consequently, Γ is upper semi-continuous with respect to d_F , and analogously we can show that Γ is lower semi-continuous with respect to d_F . Together, this gives the $\|\cdot\|_F$ -continuity of Γ . Moreover, (7) together with the convexity of B shows that $\{\Gamma = r\}$ is convex for all $r \in I$, and by Lemma 5.2 we conclude that Γ is quasi-monotone. To verify that $\Gamma|_{B_0}$ is strictly quasi-monotone, we fix $x_0, x_1 \in B_0$ and write $x_t := (1-t)x_0 + tx_1$ for $t \in [0, 1]$. Furthermore, we define $r_0 := \Gamma(x_0)$ and $r_1 := \Gamma(x_1)$ and assume without loss of generality that $r_0 \leq r_1$. To check that $\Gamma|_{B_0}$ is quasi-monotone we first observe that the already established quasi-monotonicity of Γ yields $r_0 \leq \Gamma(x_t) \leq r_1$ for all $t \in [0, 1]$. Moreover, we have $r_0, r_1 \in I$, and since I is an interval by Lemma 5.3, we thus find $\Gamma(x_t) \in I$. In other words, we have shown that $x_t \in B_0$ for all $t \in [0, 1]$, and since the latter gives $\Gamma|_{B_0}(x_t) = \Gamma(x_t)$, we obtain the quasi-monotonicity of $\Gamma|_{B_0}$. Let us finally show that $\Gamma|_{B_0}$ is strictly quasi-monotone. To this end, we keep our notation and additionally assume that $r_0 < r_1$. Then, an easy calculation using (7), (8), and $x_1 \in \{\Gamma = r_1\} \subset \{\Gamma > r_0\}$ shows

$$\langle z'_{r_0}, x_t \rangle = (1-t)\langle z'_{r_0}, x_0 \rangle + t\langle z'_{r_0}, x_1 \rangle = t\langle z'_{r_0}, x_1 \rangle > 0$$

for $t \in (0, 1)$ and thus $x_t \in \{z'_{r_0} > 0\} \cap B = \{\Gamma > r_0\}$, that is $\Gamma|_{B_0}(x_t) > r_0$. By considering z'_{r_1} instead, we analogously obtain $\Gamma|_{B_0}(x_t) < r_1$, and hence $\Gamma|_{B_0}$ is indeed strictly quasi-monotone.

ii) \Rightarrow i). Assumption $\mathbf{G1}$ follows from Lemma 5.2. To show that $\mathbf{G2}$ is also satisfied, we fix an $r \in I$ and an $x \in \{\Gamma = r\}$. By Lemma 5.3 there then exist an $s \in I$ with $s > r$ and an $x_1^+ \in \{r < \Gamma < s\}$. For $t \in (0, 1)$ we define $x_t^+ := (1-t)x + tx_1^+$. Then our construction ensures $x, x_1^+ \in B_0$ and hence the strict quasi-concavity of $\Gamma|_{B_0}$ gives $\Gamma(x_t^+) = \Gamma|_{B_0}(x_t^+) > \min\{\Gamma(x), \Gamma(x_1^+)\} = r$, that is $x_t^+ \in \{\Gamma > r\}$. Analogously we find $x_t^- \in \{\Gamma < r\}$, and by choosing sufficiently small t we can verify $\mathbf{G2}$. \square

Proof of Theorem 2.4. *i) \Rightarrow iv).* The existence follows from the second part of Theorem 6.8. Since every $z' \in (H, \|\cdot\|_E)'$ is also an element of H' , the uniqueness can be shown as in the proof of Theorem 2.3.

iv) \Rightarrow iii). Trivial.

iii) \Rightarrow ii). The $\|\cdot\|_E$ -continuity can be shown as in the proof of Theorem 2.3 if d_F and $\|\cdot\|_F$ are replaced by d_E and $\|\cdot\|_E$, respectively. The remaining parts follow from Theorem 2.3 and the already mentioned inclusion $(H, \|\cdot\|_E)' \subset H'$.

ii) \Rightarrow i). This again follows from Theorem 2.3.

Finally, if **B4** is also satisfied, i.e. if H is dense in E , then the existence of the unique extension follows from e.g. [14, Theorem 1.9.1]. Moreover, this theorem also shows that $\|\hat{z}'_r\|_{E'} = \|z'_r\|_{E'} = 1$. \square

Before we can prove Theorem 2.5 we need to establish a simple auxiliary result.

Lemma 6.11. *Assume that **G1*** and **G3** are satisfied. Then for all $r \in I$ we have*

$$\{\hat{\Gamma} \geq r\} = \overline{\{\Gamma \geq r\}}^E.$$

Proof. “ \supset ”. This follows from inclusion (12) of Lemma 5.5.

“ \subset ”. Let us fix an $x \in \{\hat{\Gamma} \geq r\}$. We write $r^* := \Gamma(x)$. By **G3** there then exist $x_n \in \{\hat{\Gamma} > r^*\}$ with $\|x - x_n\|_E \leq 1/n$ for all $n \geq 1$. Now, $r^* \geq r$ together with Lemma 5.5 yields

$$x_n \in \{\hat{\Gamma} > r^*\} \subset \{\hat{\Gamma} > r\} \subset \overline{\{\Gamma > r\}}^E$$

for all $n \geq 1$, and thus we find $x \in \overline{\{\Gamma > r\}}^E$. \square

Proof of Theorem 2.5. Using Lemma 6.11 and Theorem 2.4 we find

$$\{\hat{\Gamma} \geq r\} = \overline{\{\Gamma \geq r\}}^E = \overline{\{z'_r \geq 0\} \cap \overline{B}}^E. \quad (20)$$

Let us define $\Upsilon : B \rightarrow \mathbb{R}$ by $\Upsilon(x) := z'_r(x)$ for $x \in B$. Then $\hat{\Upsilon} := (\hat{z}'_r)|_{\overline{B}}$ is clearly a continuous and strictly quasi-monotone extension of Υ to \overline{B} . Moreover, Theorem 2.4 in combination with Lemma 5.3 shows $\{\Upsilon > 0\} = \{\Gamma > r\} \neq \emptyset$ and $\{\Upsilon < 0\} = \{\Gamma < r\} \neq \emptyset$, and using that $\Upsilon(B)$ is an interval by Lemma 5.3 we conclude that $0 \in \Upsilon(B)$. Consequently, Lemma 5.6 yields

$$\{\hat{z}'_r \geq 0\} \cap \overline{B} = \{\hat{\Upsilon} \geq 0\} = \overline{\{\Upsilon \geq 0\}}^E = \overline{\{z'_r \geq 0\} \cap \overline{B}}^E.$$

Combining this equality with (20) we then find $\{\hat{\Gamma} \geq r\} = \{\hat{z}'_r \geq 0\} \cap \overline{B}$. Analogously, we can prove $\{\hat{\Gamma} \leq r\} = \{\hat{z}'_r \leq 0\} \cap \overline{B}$, and combining the last two equalities we then easily obtain the assertion. \square

7 Proofs for Section 3

To prove Theorem 3.1 we again need a couple of preliminary results. Most of these results consider, in one form or the other, the following function $\Psi : I \rightarrow [0, \infty)$ defined by

$$\Psi(r) := \inf_{z' \in S^+} \sup_{x \in \{\Gamma=r\}} |\langle z', x \rangle|, \quad r \in I, \quad (21)$$

where $S^+ := \{z' \in E' : \|z'_H\|_{E'} = 1 \text{ and } \langle z', x_* \rangle \geq 0\}$.

Our first result shows that the functionals found in Theorem 2.4 are essentially the only minimizers of the outer infimum in (21).

Lemma 7.1. *Assume that **B1**, **B2***, **B3**, **G1***, and **G2** are satisfied. Then, for all $r \in I$, we have $\Psi(r) = 0$, and there exists a $z' \in S^+$ such that*

$$\Psi(r) = \sup_{x \in \{\Gamma=r\}} |\langle z', x \rangle|. \quad (22)$$

Moreover, for every $z' \in S^+$ satisfying (22), we have the following implications

$$\begin{aligned} \Gamma(x_*) < r &\quad \Rightarrow &\quad z'_H &= -z'_r \\ \Gamma(x_*) = r &\quad \Rightarrow &\quad z'_H &= \pm z'_r \\ \Gamma(x_*) > r &\quad \Rightarrow &\quad z'_H &= z'_r, \end{aligned}$$

where (z'_r) is the unique normalized separating family obtained in Theorem 2.4.

Proof. To show the existence of $z' \in S^+$, we assume without loss of generality that $\Gamma(x_*) \geq r$. Then the unique normalized separating functional $z'_r \in (H, \|\cdot\|_E)'$ found in Theorem 2.4 satisfies

$$\sup_{x \in \{\Gamma=r\}} |\langle z'_r, x \rangle| = 0,$$

and since $\Psi(r) \geq 0$, we conclude that

$$\Psi(r) = \sup_{x \in \{\Gamma=r\}} |\langle z'_r, x \rangle| = 0.$$

In addition, $\Gamma(x_*) \geq r$ implies $\langle z'_r, x_* \rangle \geq 0$. Extending z'_r to a bounded linear functional $z' \in E'$ with the help of Hahn-Banach's extension theorem, see e.g. [14, Theorem 1.9.6], then yields $z' \in S^+$, and as a by-product of the proof, we have also established $\Psi(r) = 0$.

To show the implications, we restrict our considerations to the case $\Gamma(x_*) < r$, the remaining two cases can be treated analogously. Then the already established $\Psi(r) = 0$ yields $\langle z', x \rangle = 0$ for all $x \in \{\Gamma = r\}$, that is $\{\Gamma = r\} \subset B \cap \ker z'$. Since $\|z'_r\|_{E'} = 1 = \|z'_H\|_{E'}$, we then conclude by Lemma 6.10 and Theorem 2.4 that $z'_r = -z'_H$ or $z'_r = z'_H$. Assume that the latter is true. Then $\Gamma(x_*) < r$ implies $0 > \langle z'_r, x_* \rangle = \langle z', x_* \rangle \geq 0$, and hence we have found a contradiction. Consequently, we have $z'_r = -z'_H$. \square

Our next goal is to show that there exists a measurable selection of the minimizers of the function Ψ . To this end, we first need to show that the inner supremum is measurable, and to show this, we now consider the functions $\Phi_n : I \times E' \rightarrow \mathbb{R}$, $n \in \mathbb{N} \cup \{\infty\}$ defined by

$$\Phi_n(r, z') := \sup_{x \in \{\Gamma=r\} \cap nB_E} |\langle z', x \rangle|, \quad (r, z') \in I \times E', \quad (23)$$

where $I \subset \mathbb{R}$ is an interval and E is a normed space. The following lemma shows that Φ_n is continuous in the second variable.

Lemma 7.2. *Let E be a normed space, $B \subset E$ be non-empty, and $\Gamma : B \rightarrow \mathbb{R}$ be a continuous map. Then, for all $n \in \mathbb{N}$ and $r \in I$, the map $\Phi_n(r, \cdot) : E' \rightarrow \mathbb{R}$ defined by (23) is continuous.*

Proof. For $z'_1, z'_2 \in E'$ the triangle inequality for suprema yields

$$\begin{aligned} |\Phi_n(r, z'_1) - \Phi_n(r, z'_2)| &= \left| \sup_{x \in \{\Gamma=r\} \cap nB_E} |\langle z'_1, x \rangle| - \sup_{x \in \{\Gamma=r\} \cap nB_E} |\langle z'_2, x \rangle| \right| \\ &\leq \sup_{x \in \{\Gamma=r\} \cap nB_E} |\langle z'_1, x \rangle - \langle z'_2, x \rangle| \\ &\leq \|z'_1 - z'_2\|_{E'} \cdot n. \end{aligned}$$

Now the assertion easily follows. \square

The next lemma shows that the function Φ_n is measurable in the first variable, provided that some technical assumptions are met.

Lemma 7.3. *Let E be a separable Banach space, $B \subset E$ be non-empty, and $\Gamma : B \rightarrow \mathbb{R}$ be a map satisfying **G4**. Then, for all $n \in \mathbb{N}$ and $z' \in E'$, the map $\Phi_n(\cdot, z') : I \rightarrow \mathbb{R}$ defined by (23) is $\hat{\mathcal{B}}(I)$ -measurable.*

Proof. Let us write $B_n := \Gamma^{-1}(I) \cap nB_E$. Note that nB_E is closed and thus $\mathcal{B}(E)$ -measurable. Since $\Gamma^{-1}(I)$ is $\mathcal{B}(E)$ -measurable by **G4**, we conclude that B_n is $\mathcal{B}(E)$ -measurable. Consequently, $\mathbf{1}_{E \setminus B_n} : E \rightarrow \mathbb{R}$ is $\mathcal{B}(E)$ -measurable, and the extension $\hat{\Gamma} : E \rightarrow \mathbb{R}$ defined by

$$\hat{\Gamma}(z) := \begin{cases} \Gamma(z) & \text{if } z \in B_n \\ 0 & \text{otherwise.} \end{cases}$$

is also $\mathcal{B}(E)$ -measurable. Consequently, the map $h : I \times E \rightarrow \mathbb{R}^2$ defined by

$$h(r, z) := (\hat{\Gamma}(z) - r, \mathbf{1}_{E \setminus B_n}(z)), \quad (r, z) \in I \times E$$

is $\mathcal{B}(I) \otimes \mathcal{B}(E)$ -measurable. Moreover, note that the definition of h yields

$$\{z \in E : h(r, z) = 0\} = \{z \in B_n : \Gamma(z) = r\} = \{\Gamma = r\} \cap nB_E.$$

For $F : I \rightarrow 2^E$ defined by

$$F(r) := \{z \in E : h(r, z) \in \{0\}\},$$

we thus find $F(r) = \{\Gamma = r\} \cap nB_E$ for all $r \in I$. Moreover, the graph of F , that is

$$\text{graph } F := \{(r, z) \in I \times E : z \in F(r)\} = \{(r, z) \in I \times E : h(r, z) = 0\}$$

is $\mathcal{B}(I) \otimes \mathcal{B}(E)$ -measurable, and $\xi : I \times E \rightarrow \mathbb{R}$ defined by $\xi(r, z) := |\langle z', z \rangle|$ is continuous and thus $\mathcal{B}(I \times E)$ -measurable. Moreover, we have $\mathcal{B}(I \times E) = \mathcal{B}(I) \otimes \mathcal{B}(E)$ by [3, Lemma 6.4.2] since I and E are both separable, and thus ξ is $\mathcal{B}(I) \otimes \mathcal{B}(E)$ -measurable, too. Since separable Banach spaces are Polish spaces, [4, Lemma III.39 on p. 86] then shows that the map

$$r \mapsto \sup_{z \in F(r)} \xi(r, z)$$

is $\hat{\mathcal{B}}(I)$ -measurable. From the latter we easily obtain the assertion. \square

With the help of the two previous results, the next result now establishes the desired measurability of Φ . Unfortunately, it requires a stronger separability assumption than the preceding lemmas.

Corollary 7.4. *Let E be a Banach space whose dual E' is separable, $B \subset E$ be non-empty, and $\Gamma : B \rightarrow \mathbb{R}$ be a continuous map satisfying **G4**. Then $\Phi_\infty : I \times E' \rightarrow \mathbb{R}$ is $\hat{\mathcal{B}}(I) \otimes \mathcal{B}(E')$ -measurable.*

Proof. Let us first recall, see e.g. [14, Theorem 1.10.7], that dual spaces are always Banach spaces. Consequently, E' is a Polish space. Moreover, the separability of E' implies the separability of E , see e.g. [14, Theorem 1.12.11], and hence the map $\Phi_n(\cdot, z') : I \rightarrow \mathbb{R}$ is $\hat{\mathcal{B}}(I)$ -measurable for all $z' \in E'$ and $n \in \mathbb{N}$ by Lemma 7.3. Since $\Phi_n(r, \cdot) : E' \rightarrow \mathbb{R}$ is continuous for all $r \in I$ and $n \in \mathbb{N}$ by Lemma 7.2, we conclude that Φ_n is a Carathéodory map. Moreover, E' is Polish, and thus Φ_n is $\hat{\mathcal{B}}(I) \otimes \mathcal{B}(E')$ -measurable for all $n \in \mathbb{N}$, see e.g. [4, Lemma III.14 on p. 70]. Finally, we have $\Phi_\infty(r, z') = \lim_{n \rightarrow \infty} \Phi_n(r, z')$ for all $(r, z') \in I \times E'$, and hence Φ_∞ is also $\hat{\mathcal{B}}(I) \otimes \mathcal{B}(E')$ -measurable. \square

The next result shows that we can find the minimizers of the infimum used in the definition of $\Psi : I \rightarrow [0, \infty)$ in a measurable fashion.

Theorem 7.5. *Assume that **B1**, **B2***, **B3**, **B5**, **G1***, **G2**, and **G4** are satisfied. Then there exists a measurable map $\zeta : (I, \hat{\mathcal{B}}(I)) \rightarrow (E', \mathcal{B}(E'))$ such that, for all $r \in I$, we have $\zeta(r) \in S^+$ and*

$$\Psi(r) = \sup_{x \in \{\Gamma=r\}} |\langle \zeta(r), x \rangle|.$$

Proof. Let us first show that S^+ is closed. To this end, we pick a sequence $(z'_n) \subset S^+$ that converges in norm to some $z' \in E'$. Then $\langle z'_n, x_* \rangle \geq 0$ immediately implies $\langle z', x_* \rangle \geq 0$. To show that $\|z'_H\|_{E'} = 1$ we first observe that, for $x \in H$ with $\|x\|_E \leq 1$, we easily find

$$|\langle z', x \rangle| = \lim_{n \rightarrow \infty} |\langle z'_n, x \rangle| \leq 1,$$

and thus $\|z'_H\|_{E'} \leq 1$. To show the converse inequality, we pick, for all $n \geq 1$, an $x_n \in H$ with $\|x_n\|_E \leq 1$ such that $1 - 1/n \leq \langle z'_n, x_n \rangle \leq 1$. Then we obtain

$$|\langle z', x_n \rangle - 1| \leq |\langle z' - z'_n, x_n \rangle| + |\langle z'_n, x_n \rangle - 1| \leq \|z' - z'_n\|_{E'} + 1/n,$$

and since the right hand-side converges to 0, we find $\|z'_H\|_{E'} \geq 1$. Consequently, we have shown $z \in S^+$, and therefore, S^+ is indeed closed. From the latter, we conclude that $\mathbf{1}_{E' \setminus S^+} : E' \rightarrow \mathbb{R}$ is $\mathcal{B}(E')$ -measurable. Moreover, Corollary 7.4 showed that $\Phi_\infty : I \times E' \rightarrow \mathbb{R}$ is $\hat{\mathcal{B}}(I) \otimes \mathcal{B}(E')$ -measurable, and consequently, the map $h : I \times E' \rightarrow \mathbb{R}^2$ defined by

$$h(r, z) := (\mathbf{1}_{E' \setminus S^+}(z'), \Phi_\infty(r, z')), \quad (r, z') \in I \times E',$$

is also $\hat{\mathcal{B}}(I) \otimes \mathcal{B}(E')$ -measurable. We define $F : I \rightarrow 2^{E'}$ by

$$F(r) := \{z' \in E' : h(r, z') = 0\}, \quad r \in I.$$

Note that our construction ensures

$$F(r) = \{z \in S^+ : \Phi_\infty(r, z') = 0\} = \left\{ z' \in S^+ : \Psi(r) = \sup_{x \in \{\Gamma=r\}} |\langle z', x \rangle| \right\}, \quad (24)$$

where in the last step we used the equality $\Psi(r) = 0$ established in Lemma 7.1. Moreover, the latter lemma also showed $F(r) \neq \emptyset$ for all $r \in I$, that is

$$\text{dom } F := \{r \in I : F(r) \neq \emptyset\} = I.$$

Since E' is Polish, Aumann's measurable selection principle, see [18, part ii) of Lemma A.3.18] or [4, Theorem III.22 on p. 74] yields a measurable map $\zeta : (I, \hat{\mathcal{B}}(I)) \rightarrow (E', \mathcal{B}(E'))$ with $\zeta(r) \in F(r)$ for all $r \in I$. Then (24) shows that ζ is the desired map. \square

With these preparations, we can finally prove Theorem 3.1. The basic idea behind this proof is to combine Lemma 7.1 and Theorem 7.5.

Proof of Theorem 3.1. Let us now consider the measurable selection $\zeta : I \rightarrow E'$ from Theorem 7.5. Furthermore, we fix an $r \in I$. If $r > \Gamma(x_*)$, then Lemma 7.1 shows that $\zeta(r)|_H = -z'_r$, and thus $\zeta(r) = -\hat{z}'_r$ by **B4**. Analogously, $r < \Gamma(x_*)$ implies $\zeta(r) = \hat{z}'_r$, and in the case $r = \Gamma(x_*)$ we have either $\zeta(r) = -\hat{z}'_r$ or $\zeta(r) = \hat{z}'_r$. From these relations it is easy to obtain the desired measurability of $Z : (I, \hat{\mathcal{B}}(I)) \rightarrow (E', \mathcal{B}(E'))$.

Since the image $Z(I)$ is separable by the separability of E' , we further see by [5, Theorem 8, page 5] that Z is an E' -valued measurable function in the sense of Bochner integration theory. \square

Proof of Corollary 3.2. We first need to verify the remaining assumptions of Theorem 3.1 for E_0 . To this end, we first observe that $\varphi'_{|E_0} \in E'_0$ and therefore **B1** is satisfied for $\varphi'_{|E_0}$. Moreover, **B2*** and **G2** are independent of E_0 , and hence they are also satisfied. Consequently, we indeed obtain a family $(\hat{z}'_{0,r})_{r \in I} \subset E'_0$ of separating functionals by Theorem 2.4 and this family is measurable in the sense of Theorem 3.1 with respect to the space E'_0 .

Now, Theorem 2.4 yields for both families and all $r \in I$ that

$$\begin{aligned} \{\Gamma = r\} &= \ker(\hat{z}'_r)|_H \cap B \\ \{\Gamma = r\} &= \ker(\hat{z}'_{0,r})|_H \cap B, \end{aligned}$$

and consequently, we obtain an $\alpha(r) \neq 0$ such that

$$(\hat{z}'_r)|_H = \alpha(r)(\hat{z}'_{0,r})|_H \quad (25)$$

by Lemma 6.10. By fixing an $x \in \{\Gamma > r\}$ we further see by Theorem 2.4 that both functionals have the same orientation, and thus we find $\alpha(r) > 0$. In addition, (9) easily follows by the denseness of H in E_0 . To show that α is measurable, we first recall that we have $H \subset E_0 \subset E$, and since H is dense in E , we see that E_0 is dense in E , too. Moreover, E_0 is separable by **B5** and therefore we conclude that E is separable. Consequently, there exists an at most countable $D \subset H$ such that $D \subset B_E$ is dense. Moreover, (25) shows that $(\hat{z}'_{0,r})|_H$ is also continuous with respect to $\|\cdot\|_E$ and therefore, we obtain

$$h(r) := \left\| (\hat{z}'_{0,r})|_H \right\|_{(H, \|\cdot\|_E)'} = \sup_{x \in H \cap B_E} |\langle \hat{z}'_{0,r}, x \rangle| = \sup_{x \in D} |\langle \hat{z}'_{0,r}, x \rangle|.$$

Now, for each $x \in D$, Theorem 3.1 shows that $r \mapsto \langle \hat{z}'_{0,r}, x \rangle$ is measurable with respect to the σ -algebras $\hat{\mathcal{B}}(I)$ and $\mathcal{B}(\mathbb{R})$, and therefore $r \mapsto h(r)$ inherits this measurability. Moreover, using $\|(\hat{z}'_r)|_H\|_{E'} = 1$ and (25) we find $1 = \alpha(r)h(r)$ for all $r \in I$, and from the latter we easily obtain the desired measurability of α . \square

8 Proofs for Section 4

Lemma 8.1. *Let **B1**, **B2***, **G1**, and **G2** be satisfied, and $(z'_r)_{r \in I} \subset H'$ be the unique normalized family of separating functionals obtained in Theorem 2.3. Then, for all $r_0 \in I$ and $z \in \ker z'_{r_0}$, we have*

$$\lim_{r \rightarrow r_0} \langle z'_r, z \rangle = 0.$$

*Moreover, if **B3** and **G1*** are additionally satisfied, then the same holds for the unique functionals $z'_r \in (H, \|\cdot\|_E)'$ obtained in Theorem 2.4.*

Proof. Let us first consider the case $z \in \{\Gamma = r_0\}$. For $\varepsilon > 0$ there then exist $x^- \in \{\Gamma < r_0\}$ and $x^+ \in \{\Gamma > r_0\}$ such that $\|z - x^-\|_H \leq \varepsilon$ and $\|z - x^+\|_H \leq \varepsilon$. Consequently, there exists a $\delta > 0$ such that $[r_0 - \delta, r_0 + \delta] \subset [\Gamma(x^-), \Gamma(x^+)]$. For $\alpha \in [0, 1]$ we define $x_\alpha := (1 - \alpha)x^- + \alpha x^+$. Clearly, this gives $\|z - x_\alpha\|_H \leq \varepsilon$ for all $\alpha \in [0, 1]$. Moreover, the $\|\cdot\|_E$ -continuity of Γ together with the intermediate theorem shows that, for all $r \in (r_0 - \delta, r_0 + \delta)$ there exists an $\alpha_r \in [0, 1]$ such that $\Gamma(x_{\alpha_r}) = r$, that is $x_{\alpha_r} \in \{\Gamma = r\} \subset \ker z'_r$. For $r \in (r_0 - \delta, r_0 + \delta)$, this yields

$$|\langle z'_r, z \rangle| \leq |\langle z'_r, z - x_{\alpha_r} \rangle| + |\langle z'_r, x_{\alpha_r} \rangle| \leq \|z - x_{\alpha_r}\|_H \leq \varepsilon.$$

This shows the assertion for $z \in \{\Gamma = r_0\}$. The general case $z \in \ker z'_{r_0}$ now follows from $\ker z'_{r_0} = \text{span}(\ker z'_{r_0} \cap B) = \text{span}\{\Gamma = r_0\}$ established in Lemma 6.9.

Finally, if **B3** and **G1*** are satisfied and $(z'_r) \subset (H, \|\cdot\|_E)'$ denotes the unique separating functionals obtained by Theorem 2.4 we can literally repeat the first part for $z \in \{\Gamma = r_0\}$ and obtain

$$|\langle z'_r, z \rangle| \leq |\langle z'_r, z - x_{\alpha_r} \rangle| + |\langle z'_r, x_{\alpha_r} \rangle| \leq \|z - x_{\alpha_r}\|_E \leq \|z - x_{\alpha_r}\|_H \leq \varepsilon.$$

by Lemma 2.1. The general case $z \in \ker z'_{r_0}$ again follows by Lemma 6.9. \square

Lemma 8.2. *Let **B1**, **B2***, **B3**, **B4**, **B6**, **G1***, and **G2** be satisfied, and $(\hat{z}'_r)_{r \in I} \subset E'$ be the family of separating functionals obtained in Theorem 2.4. Moreover, let $r \in I$ and $(r_n) \subset I$ with $r_n \rightarrow r$. Then there exist a subsequence (r_{n_k}) of (r_n) and an $\alpha \in [0, 1]$ such that for all $z \in E$ we have*

$$\langle \hat{z}'_{r_{n_k}}, z \rangle \rightarrow \langle \alpha \hat{z}'_r, z \rangle.$$

Proof. Since $(\hat{z}'_{r_n})_{r \in I} \subset B_{E'}$, the sequential Banach-Alaoglu theorem, see e.g. [14, Theorem 2.6.18 in combination with Theorem 2.6.23] and also [14, Exercise 2.73], guarantees that there exist a subsequence $(\hat{z}'_{r_{n_k}})$ and an $z' \in B_{E'}$ such that

$$\langle \hat{z}'_{r_{n_k}}, z \rangle \rightarrow \langle z', z \rangle \tag{26}$$

for all $z \in E$. By Lemma 8.1 we conclude that $\langle z', z \rangle = 0$ for all $z \in \ker \hat{z}'_r$, and thus $\ker(\hat{z}'_r)|_H \subset \ker z'|_H$. Consequently, we have both $\{\Gamma = r\} = B \cap \ker(\hat{z}'_r)|_H$ and $\{\Gamma = r\} \subset B \cap \ker z'|_H$ and therefore Lemma 6.10 gives an $\alpha \in \mathbb{R}$ such that $z'|_H = \alpha \cdot (\hat{z}'_r)|_H$, and thus $z' = \alpha \hat{z}'_r$ by the denseness of H in E . Using $\|z'\| \leq 1 = \|\hat{z}'_r\|$ we find $\alpha \in [-1, 1]$ and (26) gives the desired convergence. Let us finally show that $\alpha \geq 0$. To this end, note that by Lemma 5.3 we find an $r_0 \in I$ with $r_0 > r_n$ for all $n \geq 1$. This obviously gives $r_0 \geq r$. Let us further fix a $z \in \{\Gamma = r_0\}$. Then we have $z \in \{\Gamma > r_n\} = \{z'_n > 0\} \cap B$ and thus $\langle \hat{z}'_{r_{n_k}}, z \rangle > 0$ for all $k \geq 1$. Analogously we conclude from $r_0 \geq r$ that $\langle \hat{z}'_r, z \rangle \geq 0$, and thus $\alpha < 0$ is impossible. \square

Lemma 8.3. *Let E be an arbitrary normed space. Then for all $x \in E$ and all $x' \in E'$ with $\|x'\|_{E'} = 1$ we have*

$$d(x, \ker x') = |\langle x', x \rangle|.$$

Proof. “ \leq ”: Let us fix an $\varepsilon \in (0, 1)$. Since $\|x'\| = 1$, there then exists an $x_0 \in B_E$ with $\langle x', x_0 \rangle \geq 1 - \varepsilon$. Clearly, this gives $x_0 \notin \ker x'$, and an easy calculation then shows

$$z := x - \frac{\langle x', x \rangle}{\langle x', x_0 \rangle} x_0 \in \ker x'.$$

From the latter we then conclude that

$$d(x, \ker x') \leq \|x - z\|_E = \left| \frac{\langle x', x \rangle}{\langle x', x_0 \rangle} \right| \cdot \|x_0\|_E \leq \frac{|\langle x', x \rangle|}{1 - \varepsilon}.$$

Letting $\varepsilon \rightarrow 0$, then gives the desired inequality.

“ \geq ”: If $x \in \ker x'$, there is nothing to prove, and hence we assume without loss of generality that $x \notin \ker x'$. For $\varepsilon > 0$, we now fix an $z \in \ker x'$ such that $\|x - z\|_E \leq d(x, \ker x') + \varepsilon$. Then $x \notin \ker x'$ ensures $x - z \notin \ker x'$, and thus we find

$$d(x, \ker x') + \varepsilon \geq \|x - z\|_E = \left| \frac{\langle x', x \rangle}{\langle x', x - z \rangle} \right| \cdot \|x - z\|_E \geq |\langle x', x \rangle|,$$

where in the last step we used $|\langle x', x - z \rangle| \leq \|x - z\|_E$. Letting $\varepsilon \rightarrow 0$, then gives the desired inequality. \square

Proof of Theorem 4.1. i) \Rightarrow iii). By Lemma 8.2 it suffices to show that independent of the sequence (r_n) and its subsequence we always have $\alpha = 1$. To show the latter let us first assume that (10) is actually satisfied for some $x \in B \setminus \text{span}\{\Gamma = r\}$. Let us assume without loss of generality that $r_0 := \Gamma(x)$ satisfies $r_0 > r$. Then there is an $n_0 \geq 1$ such that $r_0 > r_n$ for all $n \geq n_0$ and thus we find both $x \in \{\Gamma > r_n\} \subset \{z'_{r_n} > 0\}$ and $x \in \{\Gamma > r\} \subset \{z'_r > 0\}$. Combining Lemma 8.3 with Lemma 6.9 and Theorem 2.4 we then find

$$\langle z'_{r_n}, x \rangle = d(x, \ker z'_{r_n}) = d(x, \text{span}\{\Gamma = r_n\}) \rightarrow d(x, \text{span}\{\Gamma = r\}) = d(x, \ker z'_r) = \langle z'_r, x \rangle \quad (27)$$

and since $\langle z'_r, x \rangle > 0$ we conclude that we indeed always have $\alpha = 1$.

In the remaining part of the proof we show that the strong version of (10) used above is implied by **G5**. To this end, let us first assume that (10) is satisfied for some $x \in F \setminus \text{span}\{\Gamma = r\}$. By Lemma 6.4 we fix an $x_r \in \{\Gamma = r\}$ such that $-x_* + x_r \in \mathring{A}^F$. Consequently, there exists an $\varepsilon > 0$ such that for all $y \in F$ with $\|y\|_F \leq \varepsilon$ we have $-x_* + x_r + y \in A = -x_* + B$, that is $x_r + y \in B$. Moreover, we easily find a $\delta > 0$ such that $\|\delta x\|_F \leq \varepsilon$ and thus we obtain $\bar{x} := x_r + \delta x \in B$. In addition, $x \notin \text{span}\{\Gamma = r\}$ together with $x_r \in \text{span}\{\Gamma = r\}$ yields $\bar{x} \notin \text{span}\{\Gamma = r\}$. Let us verify that (10) holds for \bar{x} . To this end, we assume without loss of generality that $\langle z'_r, x \rangle > 0$. Repeating the arguments in (27) we then find $|\langle z'_{r_n}, x \rangle| \rightarrow \langle z'_r, x \rangle$, and by Lemma 8.2 we see that $-\langle z'_{r_{n_k}}, x \rangle \rightarrow \langle z'_r, x \rangle$ for some subsequence (r_{n_k}) is impossible. Therefore, we actually have $\langle z'_{r_n}, x \rangle \rightarrow \langle z'_r, x \rangle$. In addition, Lemma 8.1 shows that $\langle z'_{r_n}, x_r \rangle \rightarrow 0$. With these preparatory considerations we now obtain, analogously to (27), that

$$d(\bar{x}, \text{span}\{\Gamma = r_n\}) = |\langle z'_{r_n}, x_r + \delta x \rangle| \rightarrow |\langle z'_r, x_r + \delta x \rangle| = d(\bar{x}, \text{span}\{\Gamma = r\}),$$

that is $\bar{x} \in B \setminus \text{span}\{\Gamma = r\}$ satisfies (10).

In our last step, we assume that only **G5** is satisfied, i.e. (10) holds for some $x \in H \setminus \text{span}\{\Gamma = r\}$. With the help of the previous step, it then suffices to find an $y \in F \setminus \text{span}\{\Gamma = r\}$ for which (10) holds. To this end, recall that Lemma 6.1 showed $H = F \oplus \mathbb{R}x_r$, where $x_r \in \{\Gamma = r\}$ is again a vector satisfying $-x_* + x_r \in \mathring{A}^F$. Consequently, we have $x = y + \alpha x_r$, for some suitable $y \in F$ and $\alpha \in \mathbb{R}$, and since we have already considered the case $\alpha = 0$ in the previous step, we may assume that $\alpha \neq 0$. Now, we clearly have $y = x - \alpha x_r \in F$, and since $x_r \in \text{span}\{\Gamma = r\}$ but $x \notin \text{span}\{\Gamma = r\}$, we find $y \notin \text{span}\{\Gamma = r\}$, that is $x \in F \setminus \text{span}\{\Gamma = r\}$. Finally, verifying (10) for y is analogous to the previous case.

iii) \Rightarrow ii). This immediately follows from $\ker z' = \text{span}\{\Gamma = r\}$, which has been established in Lemma 6.9, and Lemma 8.3.

ii) \Rightarrow i). This implication is trivial. \square

Proof of Corollary 4.2. Let $r \in I$ and $(r_n) \subset I$ with $r_n \rightarrow r$. Since E is separable, Lemma 8.2 shows that there exist a subsequence (r_{n_k}) of (r_n) and an $\alpha \in [0, 1]$ such that for all $z \in E$ we have

$$\langle z'_{r_{n_k}}, z \rangle \rightarrow \langle \alpha z'_r, z \rangle.$$

Moreover, in finite dimensional spaces, weak*-convergence implies norm-convergence, and hence we obtain $\|z'_{r_{n_k}} - \alpha z'_r\|_{E'} \rightarrow 0$. Since $\|z'_{r_{n_k}}\|_{E'} = 1 = \|z'_r\|_{E'}$, we then find $\alpha = 1$, and since the sequence $(r_n) \subset I$ was chosen arbitrarily, we obtain the assertion by a standard argument. \square

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