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Iris Köster

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Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
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I. Köster

G. Navarro raised in [5] the question whether the character table determines $|\mathrm{N}_G(P)|$ for some Sylow p-subgroup P of G. While the question is still open in general some progress has been recently made, see [3], [6].

In this note we want to analyse the situation for groups with two minimal normal subgroups and mention some applications of Proposition 1. Note that we only considers finite groups as Sylow's Theorems only make sense for groups of finite order. By $\mathrm{Syl}_p(G)$ we denote the set of all Sylow p-subgroups of G. The set of all prime divisors of $|G|$ is denoted by $\pi(G)$. The Sylow p-number $n_p(G)$ is the number of Sylow p-subgroups and $\mathrm{sn}(G) = \{\, n_p(G) \,:\, p \in \pi(G) \,\}$ the set of all Sylow p-numbers of G.

Proposition 1: Let G be a finite group with normal subgroups $M, N \trianglelefteq G$ where $M \cap N = \{1\}$. Then

$$n_p(G) = \frac{n_p(G/M)n_p(G/N)}{n_p(G/MN)}.$$

Proof: Let $P \in \mathrm{Syl}_p(G)$. By Hall's formula [2, Theorem 2.1] we obtain

$$n_p(G) = n_p(G/M)n_p(M)n_p(\mathrm{N}_{PM}(P \cap M))$$

and

$$n_p(G/N) = n_p(G/MN)n_p(MN/N)n_p(\mathrm{N}_{PMN/N}(PN/N \cap MN/N))$$

After rearranging the given equations we need to prove that

$$\begin{aligned} n_p(G) &\overset{!}{=} \frac{n_p(G/M)n_p(G/N)}{n_p(G/MN)} \\ \Leftrightarrow n_p(G/M)n_p(M)n_p(\mathrm{N}_{PM}(P \cap M)) &= n_p(G/M)n_p(M)n_p(\mathrm{N}_{PMN/N}(PN/N \cap MN/N)) \\ \Leftrightarrow n_p(\mathrm{N}_{PM}(P \cap M)) &= n_p(\mathrm{N}_{PMN/N}(PN/N \cap MN/N)) \end{aligned}$$

Now consider the natural surjective group homomorphism $\phi : G \to G/N$ and the restriction $\tilde{\phi} : \mathrm{N}_{PM}(P \cap M) \to \mathrm{N}_{PMN/N}(PN/N \cap MN/N)$. Note that $\tilde{\phi}$ is well-defined.

We want to study the kernel $\ker(\tilde{\phi})$. Let $p \in P$ and $m \in M$ and assume that $pm \in \ker(\tilde{\phi})$, i.e. $pm \in N$. Consider the equation $pm = n$, where $n \in N$. Decompose m (resp. n) in the p-part $m_p \in M$ and the p'-part $m_{p'} \in M$ (resp. n in n_p and $n_{p'}$). This yields

$$\begin{aligned} pm_pm_{p'} &= n_pn_{p'} = n_{p'}n_p, \\ \Leftrightarrow pm_p &= n_{p'}n_pm_{p'}^{-1}, \\ \overset{M\cap N=\{1\}}{\Leftrightarrow} pm_p &= n_{p'}m_{p'}^{-1}n_p, \\ \Leftrightarrow pm_pn_p^{-1} &= n_{p'}m_{p'}^{-1}. \end{aligned}$$

As the order of $pm_pn_p^{-1}$ is a p-power, $n_{p'}m_{p'}^{-1} = 1$ and as $M \cap N = \{1\}$ we immediately get $m_{p'} = n_{p'} = 1$. Therefore we have that $pm = pm_p = n_p$ is a p-element and $\ker(\tilde{\phi})$ is a p-group.

In the last part we need to prove that $\tilde{\phi}$ is surjective.
Assume that $pmN \in \mathrm{N}_{PMN/N}(PN/N \cap MN/N)$, i.e for each $xN \in PN/N \cap MN/N$ there exists $yN \in PN/N \cap MN/N$ such that

$$(pm)^{-1}NxNpmN = m^{-1}p^{-1}xpmN = yN.$$

In particular, as $yN \in PN/N$ there exist a $y \in P$ with $(pm)^{-1}xpm = yn$. Let $pm \in PM$ be an arbitrary preimage of pmN (note that $PM \to PMN/N$ is surjective). Let $x \in P \cap M$. As $xN \in PN/N \cap MN/N$ and $pmN \in \mathrm{N}_{PMN/N}(PN/N \cap MN/N)$ there exists $y \in P$ with $yN \in PN/N \cap MN/N$ such that

$$\exists n \in N : m^{-1}p^{-1}xpm = yn$$
$$\overset{M \trianglelefteq G}{\Leftrightarrow} \underbrace{m^{-1}x^p m}_{\in M} \underbrace{n^{-1}}_{\in N} = y$$

As $y \in P \cap MN$ this yields $y_1 \in P \cap M$ and $y_2 \in P \cap N$ such that $y = y_1y_2$ and thus

$$y_1^{-1}m^{-1}p^{-1}xpm = y_2n \overset{M\cap N=1}{=} 1$$
$$\Rightarrow (pm)^{-1}x(pm) = y_1 \in P \cap M.$$

This yields that $pm \in \mathrm{N}_{PM}(P \cap M)$. Therefore $\tilde{\phi}$ is surjective and as $n_p(\mathrm{N}_{PM}(P \cap M)) = n_p(\mathrm{N}_{PM}(P \cap M)/\ker(\tilde{\phi})) = n_p(\mathrm{N}_{PMN/N}(PN/N \cap MN/N))$ the result holds.

As we are looking for a possible candidate for a counterexample to the claim, that $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$ we obtain the following.

Corollary 2: Suppose G has a minimal nonsolvable normal subgroup N. If G is of minimal order such that $\mathrm{X}(G)$ does not determine $\mathrm{sn}(G)$ then $N \leq G \leq \mathrm{Aut}(N)$.

Proof: By Proposition 1 we can assume that G has exactly one minimal normal subgroup. As $N \cong S^k$ with S simple we see that $\mathrm{C}_G(N) \cap N = 1$. If $\mathrm{C}_G(N) \neq 1$ then there is a minimal normal subgroup N of G contained in $\mathrm{C}_G(N)$, a contradiction to the fact that G has only N as minimal normal subgroup. So $\mathrm{C}_G(N) = 1$.
Consider $\varphi : \mathrm{Inn}(G) \to \mathrm{Aut}(N)$, $\sigma_g \mapsto \sigma_g|_N$. As φ is group homomorphism we know that $G \cong G/\mathrm{Z}(G) \cong \mathrm{Inn}(G) \cong \mathrm{im}(\varphi) \leq \mathrm{Aut}(N)$. Thus the claim holds.

At the end we want to prove that the Sylow numbers of groups, where all Sylow p-subgroups are abelian, are determined by $\mathbb{Z}G$. In particular this holds for non-p-solvable groups:

Proposition 3: Assume G is a finite group such that $P \in \mathrm{Syl}_p(G)$ is abelian for each $p \in \pi(G)$. Then $\mathbb{Z}G$ determines $n_p(G)$ for each p.

Proof of Proposition 3: Let G be a group where all Sylow p-subgroups are abelian for each $p \in \pi(G)$. Note that $\mathbb{Z}G$ determines $\mathrm{Spec}(G)$ and in particular $\mathrm{X}(G)$. Assume that G is a minimal counterexample, i.e. there exists at least one Sylow number $n_p(G)$ not determined by $\mathbb{Z}G$. Using [1] there are subgroups H, K and S such that

(1) $G = HKS$ and $|G| = |H||K||S|$;
(2) $H \trianglelefteq G$ and $K \leq \mathrm{N}_G(S)$;
(3) H and K are solvable and S is semisimple.

Assume that $H = \{1\}$ then $G \cong S \rtimes K$ and by using [4, Theorem 2.1] we see that G is p-solvable for each $p \in \pi(K)$ and $n_p(G)$ determined by [6, Theorem B]. If $p \in \pi(S)$ then $n_p(G) = n_p(S)$ and S is determined up to isomorphism by $\mathrm{X}(G)$.

Assume that $S = \{1\}$ then G is solvable and in particular p-solvable for each $p \in \pi(G)$. Then again $\mathrm{X}(G)$ determines $n_p(G)$. Assume that $K = \{1\}$. By [4, Theorem 2.1] G is p-solvable for each $p \in \pi(H)$, i.e. $n_p(G)$ is determined by $\mathrm{X}(G)$. If $p \in \pi(S)$ then $n_p(G) = n_p(S)|H : \mathrm{C}_H(P)|$ for $P \in \mathrm{Syl}_p(G)$. As $|H|$ is coprime to p we obtain $|x^G \cap P| = |xH^{G/H} \cap PH/H|$ for each conjugacy class x^G of G. As S is determined up to isomorphism and the p-power maps are given we can use [6, Theorem A].

Thus H, K and S are not trivial. If $p \in \pi(K) \cup \pi(H)$ then G is p-solvable and $n_p(G)$ determines. Assume that $p \in \pi(S)$. By Proposition 1 G has only one minimal normal subgroup $X \cong C_q^k \subseteq H$. Consider $\mathrm{C}_G(X)$. If $\mathrm{C}_G(X) \subseteq X$ then G is p-constrained and by the F^*-Theorem we know G is determined up to isomorphism by $\mathbb{Z}G$. Assume $C_G(X) > X$. If $\mathrm{C}_G(X)$ is solvable then the Fitting subgroup $\mathrm{F} := \mathrm{F}(C_G(X))$ is a p-group and $C_G(\mathrm{F}) \subseteq C_{C_G(X)}(F) \subseteq \mathrm{F}$, so G is again p-constrained. Now assume $\mathrm{C}_G(X)$ is not solvable, thus $\mathrm{C}_G(X) \cap \mathrm{E}(G) \neq \{1\}$ where $\mathrm{E}(G)$ denotes the layer of G. By [1] the non-solvable composition factors are isomorphic to J_1, $\mathrm{PSL}_2(2^f)$ and $\mathrm{PSL}_2(q)$ where $q \equiv \pm 3 \mod 8$. The Schur mulitipliers of these groups are either trivial or 2. Thus either G has a minimal normal 2-group or a minimal normal nonsolvable subgroup, a contradiction.

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Iris Köster
Universität Stuttgart
Institut für Geometrie und Topologie
Arbeitsgruppe Gruppen, Gitter und simpliziale Komplexe
Pfaffenwaldring 57
D-70569 Stuttgart
Germany

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