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Discontinuous Galerkin Method for an Elliptic Problem with Nonlinear Newton Boundary Conditions in a Polygon

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Abstract

The paper is concerned with the analysis of the discontinuous Galerkin method (DGM) for the numerical solution of an elliptic boundary value problem with a nonlinear Newton boundary condition in a two-dimensional polygonal domain. Using the monotone operator theory it is possible to prove the existence and uniqueness of the exact weak solution and the approximate DG solution. The main attention is paid to the study of error estimates. To this end, the regularity of the weak solution is investigated and it is shown that due to the singular boundary points, the solution looses regularity in a vicinity of these points. It comes out that the error estimation depends essentially on the opening angle of the corner points and the nonlinearity in the boundary term. It also depends on the parameter defining the nonlinear behaviour of the Newton boundary condition. At the end of the paper some computational experiments are presented.

Keywords: elliptic equation, nonlinear Newton boundary condition, monotone operator method, discontinuous Galerkin method, regularity and singular behaviour of the solution, compactness in DG spaces, error estimation.

AMS Subject Classification: 65 N 30, 65 N 15

1 Introduction

In this paper we are concerned with the study of the discontinuous Galerkin method (DGM) for the solution of an elliptic equation with a nonlinear Newton boundary condition in a bounded two-dimensional polygonal domain. Such boundary value problems have applications in science and engineering, see, e.g., [3], [13]. Here we suppose that the nonlinear term has a "polynomial" behaviour, which can be met in the modelling of electrolysis of aluminium with the aid of the stream function. The nonlinear boundary condition describes turbulent flow in a boundary layer ([29]). Similar nonlinearity appears in a radiation heat transfer problem ([26], [23]) or in nonlinear elasticity ([14], [15]).

The paper [8] deals with the mathematical and numerical study of a problem arising in the investigation of the electrolytical producing of aluminium. The problem in [8] is discretized by piecewise linear conforming triangular elements. The solvability of the discrete problem and the convergence of approximate solutions to the exact solution was proved. The paper [9] is devoted to the convergence of conforming linear finite elements using numerical integration applied to the numerical solution of an elliptic boundary value problem with a nonlinear Newton boundary condition. In [10], these results were extended with the aid of the monotone operator theory and error estimates were proved under the assumption that the exact solution is sufficiently regular. The effect of numerical integration was also taken into account.

The subject of the present paper is the analysis of the discontinuous Galerkin method (DGM) applied to the numerical solution of an elliptic boundary value problem with a nonlinear Newton boundary condition in a polygonal domain. The goal is to analyze the discrect problem and error estimates taking into account the actual regularity of the exact solution. In Section 2 the boundary value problem is introduced and a weak solution is defined. Moreover, it is discussed how the Neumann traces on polygonal boundaries are defined. Section 3 is concerned with the derivation of regularity results for the exact weak solution taking into account the singular behaviour near boundary corner points of a linearized boundary value problem. We get the result that only the interior angles of the corner points determine the regularity in $W^{2,q}(\Omega)$. Moreover, we have proved higher regularity in the interior. In Section 4 a discontinuous Galerkin discretization of the problem is introduced and in Section 5 some auxiliary results are treated. Special attention is paid to the compactness in DG spaces and properties of the DG discrete problem. Results from [4] play here an important role. Section 6 is devoted to the analysis of error estimates. It comes out that the error estimation depends essentially on the opening angle of the corner points and the nonlinearity in the boundary term. Finally, in Section 7 results of some numerical experiments are presented.

2 The boundary value problem

By $I\!R$ and $I\!N$ we denote the set of all real numbers and all positive integers, respectively, and set $I\!R^2 = I\!R \times I\!R$. Points of $I\!R^2$ will be usually denoted by $x = (x_1, x_2)$. Let $\Omega \subset I\!R^2$ be a bounded polygonal domain. By $\overline{\Omega}$ and $\partial\Omega$ we denote the closure and the boundary, respectively, of Ω.

We consider the following *boundary value problem*: Find $u : \overline{\Omega} \to I\!R$ such that

$$-\Delta u = f \quad \text{in } \Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial n} + \kappa |u|^{\alpha} u = \varphi \quad \text{on } \partial\Omega, \tag{2.2}$$

where $f : \Omega \to I\!\!R$ and $\varphi : \partial\Omega \to I\!\!R$ are given functions and $\kappa > 0$, $\alpha \geq 0$ are given constants. $\partial/\partial n$ is the derivative in the direction of the unit outward normal to $\partial\Omega$. The classical solution of the above problem can be defined as a function $u \in C^2(\overline{\Omega})$ satisfying (2.1) and (2.2).

In what follows we work with the well-known Lebesgue spaces $L^p(\Omega)$, $L^p(\partial\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, $W^{k,p}(\partial\Omega)$. We set $W_0^{k,p}(\Omega) = \{\varphi \in W^{k,p}(\Omega); \varphi|_{\partial\Omega} = 0\}$, where the restriction $\varphi|_{\partial\Omega}$ is considered in the sense of traces. (See, e.g., [24].) By $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{L^p(\partial\Omega)}$, $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\partial\Omega)}$ we denote the standard norms in $L^p(\Omega)$, $L^p(\partial\Omega)$, $W^{k,p}(\Omega)$ and $W^{k,p}(\partial\Omega)$, respectively. The symbol $|\cdot|_{W^{k,p}(\Omega)}$ stands for the seminorm in $W^{k,p}(\Omega)$. (Similar notation will be used for the Lebesgue and Sobolev spaces over other sets.) If X is a Banach space, then X^* denotes its dual.

Let us assume for the moment that

$$f \in L^2(\Omega), \quad \varphi \in L^2(\partial\Omega). \tag{2.3}$$

In the standard way we can introduce a weak formulation of problem (2.1)–(2.2). To this end, we define the following forms:

$$\begin{aligned}
& b(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \\
& d(u,v) = \kappa \int_{\partial\Omega} |u|^\alpha \, u \, v \, \mathrm{d}S, \\
& L(v) = L^\Omega(v) + L^{\partial\Omega}(v), \\
& L^\Omega(v) = \int_\Omega f \, v \, \mathrm{d}x, \quad L^{\partial\Omega}(v) = \int_{\partial\Omega} \varphi \, v \, \mathrm{d}S, \\
& A(u,v) = b(u,v) + d(u,v), \\
& \qquad u,\, v \in H^1(\Omega).
\end{aligned} \tag{2.4}$$

It is possible to find out that the above forms make sense for functions $u, v \in H^1(\Omega)$.

Definition 1 *We say that a function $u : \Omega \to I\!\!R$ is a weak solution of problem* (2.1)–(2.2)*, if*

$$\begin{aligned}
&\text{a)} \quad u \in H^1(\Omega), \\
&\text{b)} \quad A(u,v) = L(v) \quad \forall\, v \in H^1(\Omega).
\end{aligned} \tag{2.5}$$

This weak formulation can be written as an operator equation. We define a mapping $\mathcal{A} : H^1(\Omega) \to (H^1(\Omega))^*$ and a functional $\psi \in (H^1(\Omega))^*$ such that

$$\langle \mathcal{A}(u), v \rangle = A(u,v), \quad \langle \psi, v \rangle = L(v), \quad u, v \in H^1(\Omega). \tag{2.6}$$

Here $\langle \cdot, \cdot \rangle$ is the duality between $(H^1(\Omega))^*$ and $H^1(\Omega)$. This means that $\langle \psi, v \rangle$ is the value of the continuous linear functional ψ defined on $H^1(\Omega)$ at the element $v \in H^1(\Omega)$. We see that problem 2.5 can be written as the operator equation

$$\mathcal{A}(u) = \psi \tag{2.7}$$

for an unknown $u \in H^1(\Omega)$.

In [10], with the use of the monotone operator theory the following result was proved.

Theorem 1 *Problem* (2.7) *and, thus,* (2.5) *has exactly one solution in* $H^1(\Omega)$.

Remark 1 *Later we will consider*

$$f \in L^q(\Omega), \quad \varphi \in W^{1-\frac{1}{q},q}(\partial\Omega), \tag{2.8}$$

and with the help of regularity results we come back to the classical problem (2.1)–(2.2) in the Sobolev spaces $W^{2,q}(\Omega)$. *Then we understand the Neumann trace* $\frac{\partial u}{\partial n}$ *as an element of the modified trace space* T, *introduced in Theorem 3. See also Remark 2.*

In the following section we will discuss the regularity of the weak solution $u \in H^1(\Omega)$, if the domain Ω is polygonal. We need some important concepts and results.

We will work in standard Sobolev spaces $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ which are well defined on polygons. However, we need the Neumann datum on the boundary which is defined in the classical theory under the assumption that the boundary curve is locally given by $C^{1,1}$-functions. In this smooth case the main idea is to identify the boundary with $\mathbb{R}$ by means of local parametric representations, which requires a certain boundary regularity. For polygonal domains one has to introduce some modified trace spaces, so called *natural* trace spaces or *piecewise defined* trace spaces. We introduce these trace spaces:

Let $\partial\Omega \in C^{0,1}$ be a curved polygon, composed of N simple C^∞-arcs $\Gamma_j, j = 1, \dots, N$. The curve $\overline{\Gamma}_{j+1}$ follows $\overline{\Gamma}_j$, the vertex z_j is the endpoint of Γ_j and the starting point of Γ_{j+1}. The restriction of a suitable smooth function u to Γ_j is denoted by $\gamma_j u$, n_j is the unit outward normal on Γ_j.

Definition 2 *Let be* Ω *a bounded domain whose boundary is a curved polygon. The natural trace space of functions from* $W^{m,p}(\Omega), p \geq 1, m = 1, 2, \dots$ *is formally identified as the quotient space*

$$W^{m-\frac{1}{p},p}(\partial\Omega) \cong W^{m,p}(\Omega)/W_0^{m,p}(\Omega)$$

with the norm

$$\|u\|_{W^{m-\frac{1}{p},p}(\partial\Omega)} = \inf \left\{ \|v\|_{W^{m,p}(\Omega)} : v - u \in W_0^{m,p}(\Omega) \right\}.$$

Thus, we define the trace operator from $W^{m,p}(\Omega)$ into $\prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\partial\Omega),\ l \leq m-1$, as the mapping

$$u \to \left\{ \gamma u, \gamma \frac{\partial u}{\partial n}, \dots, \gamma \frac{\partial^l u}{\partial n^l} \right\}, \quad l \leq m-1,$$

with the help the restriction operator γ to $\partial\Omega$.

In order to describe the behaviour at the corner points z_j, it is meaningfull to consider the traces of functions from $W^{m,p}(\Omega)$ piecewise on Γ_j. We assume that we have for every $\overline{\Gamma}_j$ a parametric representation:

$$x = x^j(t) \quad \text{for } t \in \bar{I}_j = [a_j, b_j] \subset \mathbb{R}.$$

Definition 3 *Let be $s \geq 0$ a real number. It is*

$$W^{s,p}(\Gamma_j) = \left\{\varphi : \varphi(x^j(\cdot)) \in W^{s,p}(I_j)\right\}$$

equipped with the norm

$$\|\varphi\|_{W^{s,p}(\Gamma_j)} = \|\varphi \circ x^j\|_{W^{s,p}(I_j)}.$$

The piecewise defined traces are well defined for elements from $W^{m,p}(\Omega)$, see the following Theorem 1.5.2.1 from [17]:

Theorem 2 *Let Ω be a bounded open subset of $I\!R^2$, whose boundary is a curvilinear polygon. Then for each j, the mapping*

$$u \to \left\{\gamma_j u, \gamma_j \frac{\partial u}{\partial n_j}, \ldots, \gamma_j \frac{\partial^l u}{\partial n_j^l}\right\}, \quad l \leq m-1,$$

which is defined for $u \in C^\infty(\bar{\Omega})$, has a unique extension as an operator from $W^{m,p}(\Omega)$ into $\prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\Gamma_j)$.

The connection between the natural traces in Definition 2 and the piecewise defined traces in Definition 3 was investigated in Theorem 1.5.2.8 in [17] and also described in [21], Theorem 4.2.7. It is clear that the restriction of smooth functions and their derivatives to the boundary $\partial\Omega$ should automatically satisfy compatibility conditions at the vertex points z_j.

Theorem 3 *Let Ω be a bounded open subset of $I\!R^2$, whose boundary is a curvilinear polygon. Then the mapping $u \to \left\{\gamma_j \frac{\partial^l u}{\partial n_j^l}, 1 \leq j \leq N, 0 \leq l \leq m-1\right\}$ is a linear continuous mapping from $W^{m,p}(\Omega)$ onto a subspace $T \subset \prod_{j=1}^{N} \prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\Gamma_j)$. T is defined by the following compatibility conditions in the corner points z_j: Let L be any linear differential operator with coefficients of class C^∞ and of order $d \leq m-\frac{2}{p}$. Denote by $P_{j,k}$ the differential operator tangential to Γ_j such that $L = \sum_{|\alpha| \leq d} a_\alpha D^\alpha = \sum_{k=0}^{d} P_{j,k} \frac{\partial^k}{\partial n_j^k}$ on Γ_j. Then*

- *(a)* $\sum_{k=0}^{d} P_{j,k} \gamma_j \frac{\partial^k u}{\partial n_j^k}(z_j) = \sum_{k=0}^{d} P_{j+1,k} \gamma_{j+1} \frac{\partial^k u}{\partial n_{j+1}^k}(z_j)$ *for* $d < m - \frac{2}{p}$,
- *(b)* $\int_0^{\delta_j} |\sum_{k=0}^{d} P_{j,k} \frac{\partial^k u}{\partial n_j^k}(x^j(t)) - P_{j+1,k} \frac{\partial^k u}{\partial n_{j+1}^k}(x^{j+1}(t))|^2 \frac{dt}{t} < \infty$ *for $d = m-1$ and $p = 2$.*

Let us illustrate this result by some examples: We consider a polygonal domain Ω and restrict to the pieces Γ_1 and Γ_2 of $\partial\Omega$ that meet at the vertex z_1. The outward unit normal vectors are $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, respectively.

1. Let us consider the space $W^{1,p}(\Omega)$. Then for $m = 1, p > 2$ we get $d = 0$ and the differential operators $L = a_0 D^0$ are constant on the pieces Γ_1 and Γ_2. The trace compatibility condition (a) at the corner point z_1 reads:

$$(\gamma_1 u)(z_1) = (\gamma_2 u)(z_1).$$

For $p = 2$ the trace compatibility condition (b) reads :

$$\int_0^{\delta_1} |u(x^1(t)) - u(x^2(t))|^2 \frac{dt}{t} < \infty.$$

For $p < 2$ no compatibility condition occurs.

2. Now we consider the space $W^{2,p}(\Omega)$. For $p < 2$ we have $d = 0$ and the trace compatibility condition (a) reads again

$$(\gamma_1 u)(z_1) = (\gamma_2 u)(z_1).$$

For $p > 2$ we get $d = 0$ and $d = 1$. We investigate the case $d = 1$. The linear differential operators of first order have the form $L = a_0 D^0 + a_1\partial_1 + a_2\partial_2$, where the coefficients $a_i, i = 0, 1, 2$, are arbitrary real numbers. Assume for simplicity that Γ_1 and Γ_2 are straight lines, $z_1 = (0, 0)$, Γ_1 lies on the x_1-axis , the angle between Γ_1 and Γ_2 is ω_0, see Figure 1. Since the boundary curve is oriented clockwise, for the tangential derivative on Γ_1 we have $\partial_{s_1} = -\partial_{x_1}$ and for the outward normal derivative $\partial_{n_1} = -\partial_{x_2}$. On Γ_2 there is the tangent unit vector $s_2 = (\cos\omega_0, \sin\omega_0)^\top$, whereas the outward unit normal vector reads $\boldsymbol{n}_2 = (-\sin\omega_0, \cos\omega_0)^\top$. It follows on Γ_2

$$\partial_{x_1} = \cos\omega_0\partial_{s_2} - \sin\omega_0\partial_{n_2} \tag{2.9}$$

$$\partial_{x_2} = \sin\omega_0\partial_{s_2} + \cos\omega_0\partial_{n_2}. \tag{2.10}$$

$$\begin{aligned} L|_{\Gamma_1} &= a_0 - a_1\partial_{s_1} - a_2\partial_{n_1}, \\ L|_{\Gamma_2} &= a_0 + a_1(\cos\omega_0\partial_{s_2} - \sin\omega_0\partial_{n_2}) + a_2(\sin\omega_0\partial_{s_2} + \cos\omega_0\partial_{n_2}). \end{aligned}$$

Thus, we have

$$\begin{aligned} P_{1,0} &= a_0 - a_1\partial_{s_1}|_{\Gamma_1}, \\ P_{1,1} &= -a_2, \\ P_{2,0} &= a_0 + (a_1\cos\omega_0 + a_2\sin\omega_0)\partial_{s_2}|_{\Gamma_2}, \\ P_{2,1} &= -a_1\sin\omega_0 + a_2\cos\omega_0. \end{aligned}$$

Since a_0, a_1, a_2 are arbitrary we get finally the compatiblity condition at the corner point z_1:

$$\gamma_1 u(z_1) = \gamma_2 u(z_1), \tag{2.11}$$

$$-\partial_{s_1}\gamma_1 u(z_1) = (\cos\omega_0\partial_{s_2}\gamma_2 - \sin\omega_0\gamma_2\partial_{n_2})(z_1), \tag{2.12}$$

$$-\gamma_1\partial_{n_1} u(z_1) = (\sin\omega_0\partial_{s_2}\gamma_2 + \cos\omega_0\gamma_2\partial_{n_2})(z_1). \tag{2.13}$$

For $\omega_0 = \pi$, we get the well known conditions

$$\gamma_1 u(z_1) = \gamma_2 u(z_1), \tag{2.14}$$

$$\partial_{s_1}\gamma_1 u(z_1) = \partial_{s_2}\gamma_2 u(z_1), \tag{2.15}$$

$$\gamma_1 \partial_{n_1} u(z_1) = \gamma_2 \partial_{n_2} u(z_1). \tag{2.16}$$

If a boundary value problem with pure Neumann data is given, than condition (a) from Theorem 3 reads for the special operator $L = \partial_n$ on Γ_1 and Γ_2 for any angle ω_0

$$\gamma_1 \partial_{n_1} u(z_1) = \gamma_2 \partial_{n_2} u(z_1). \tag{2.17}$$

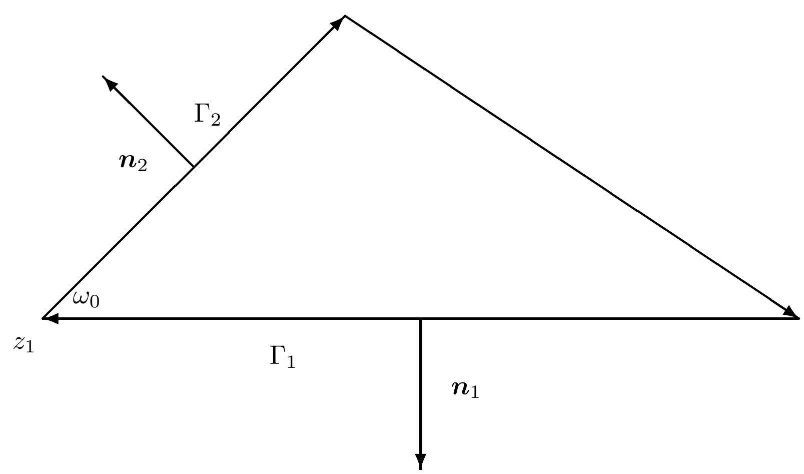


Figure 1: Polygon with directed boundary

Remark 2 *With the help of Theorem 3 we are able to describe the connection between the natural traces and the piecewise defined traces: If the condition (a) or (b) holds, then we can stick together the parts* $\gamma_j \frac{\partial^k u}{\partial n^k}$ *to a trace on the whole boundary* $\partial\Omega$ *denoted by* $\gamma \frac{\partial^k u}{\partial n^k}$. *It holds then*

$$\prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\partial\Omega) = T.$$

Remark 3 *We will consider weak solutions of the boundary value problem (2.1)–(2.2). The Neumann datum is well defined for elements from*

$$E_\Delta(H^1(\Omega); L^p(\Omega)) = \{u \in H^1(\Omega); \Delta u \in L^p(\Omega)\}, \quad p \geq 1,$$

provided with the graph norm $\|u\|_E = \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^p(\Omega)}$. In particular, in [17], Theorem 1.5.3.10, is proved:

Let Ω be a bounded open subset of $I\!R^2$, whose boundary is a curvilinear polygon of class $C^{1,1}$. Then the mapping

$$u \to \gamma_j \frac{\partial u}{\partial n_j}$$

maps $E_\Delta(H^1(\Omega); L^p(\Omega))$ into $H^{-\frac{1}{2}}(\Gamma_j) = \tilde{H}^{\frac{1}{2}}(\Gamma_j)^*$, where $\tilde{H}^{\frac{1}{2}}(\Gamma_j) = \{u \in H^{\frac{1}{2}}(\partial\Omega) : \operatorname{supp} u \subset \bar{\Gamma}_j\}$.

Remark 4 *The above Neumann traces $\gamma_j \frac{\partial u}{\partial n_j} \in H^{-\frac{1}{2}}(\Gamma_j)$ can be defined analogously for elements from $E_\Delta(V; V^*) = \{u \in V = H^1(\Omega); \Delta u \in V^*\}$, equipped with the graph norm. The main point is to show that $C^\infty(\bar{\Omega})$ is dense in $E_\Delta(V; V^*)$.*

3 Regularity

In what follows, at several places, embedding theorems for Sobolev spaces will be applied. We can refer the reader, for example, to the monographs [1], [5] or [7]. It is well known for linear elliptic boundary value problems that the geometry of the domain and the smoothness of the right hand sides determine the regularity of the solution. Sending the nonlinear boundary part in (2.2) to the right hand side, we can use regularity results for the linear problem in polygonal domains.

We start with a weak solution $u \in H^1(\Omega)$ of (2.1)–(2.2), see Definition1, and consider the term $|u|^\alpha u$. It holds:

Lemma 1 *If $u \in H^1(\Omega)$, then $|u|^\alpha u \in W^{1,q}(\Omega)$ with $q = 2 - \varepsilon$, where $\varepsilon > 0$ is a small number.*

Proof. Obviously $|u|^\alpha u$ belongs to $L^r(\Omega)$ for any $1 \le r < \infty$. Indeed,

$$\int_\Omega | \, |u|^\alpha u|^r \, dx = \int_\Omega |u|^{(\alpha+1)r} \, dx < \infty$$

due to the embedding $H^1(\Omega) \subset L^{(\alpha+1)r}(\Omega)$ for all $\alpha \ge 0$, $1 \le r < \infty$. In order to calculate the first weak derivatives of $|u|^\alpha u$, we use the result that

$$\nabla |u| = \operatorname{sign}(u) \nabla u,$$

see [6] Satz 5.20, p. 96. Therefore, by the product rule, we have:

$$\nabla(|u|^\alpha u) = |u|^\alpha \nabla u + \operatorname{sign}(u) \alpha u |u|^{\alpha-1} \nabla u.$$

Thus, using the Hölder inequality for any $s > 1$ we get

$$\begin{aligned} \int_\Omega |\nabla(|u|^\alpha u)|^q \, dx &\le (\alpha+1)^q \int_\Omega |u|^{\alpha q} |\nabla u|^q \, dx \\ &\le (\alpha+1)^q \|u^{\alpha q}\|_{L^s(\Omega)} \| |\nabla u|^q \|_{L^{s'}(\Omega)}. \end{aligned} \tag{3.1}$$

Here $\frac{1}{s} + \frac{1}{s'} = 1$. The factor $\|u^{\alpha q}\|_{L^s(\Omega)}\|$ is finite for large s. The second factor $\||\nabla u|^q\|_{L^{s'}(\Omega)}$ is finite, if $qs' = 2$. Choosing $s' = 1 + \delta$ for a small positive δ, then we get $|u|^\alpha u \in W^{1,q}(\Omega)$, where $q = 2 - \varepsilon$ with $\varepsilon = \frac{2\delta}{1+\delta}$. $\square$

Now we put in (2.2) the nonlinear boundary term to the right hand side and get the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= -\kappa |u|^\alpha u + \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

We discuss the regularity of weak solutions to the linear Neumann problem assuming that $f \in L^q(\Omega)$ and $\varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$. If $u \in H^1(\Omega)$, then, due to Lemma 1, $\kappa|u|^\alpha u \in W^{1-\frac{1}{q},q}(\partial\Omega)$ for $q < 2$.

Let us start with the linear Neumann problem in the polygonal domain Ω.

$$-\Delta u = f \quad \text{in } \Omega, \tag{3.2}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega. \tag{3.3}$$

The regularity of a weak solution from $H^1(\Omega)$ of problem (3.2)–(3.3) was thoroughly investigated in papers [22], [18], [28]. There are to find asymptotic expansions of the weak solution in a neighborhood of a corner point z_i. The solution can be decomposed into singular und more regular terms:

$$u = \sum_i c_i r_i^{\beta_i} f(\omega_i, \beta_i) + u_{regular}$$

where (r_i, ω_i) are the standard polar coordinates around the corner point z_i. The exponents β_i of the singular terms are noninteger and integer eigenvalues of an associate generalized eigenvalue problem in a certain strip in the complex plane. If we ensure that no eigenvalues are in these strips, then no singular terms occur and we get regularity results. We formulate such a result. It is known that for any small $\delta > 0$ the strip $\delta < \mathrm{Re}\beta < \frac{\pi}{\omega_0}$ is free of eigenvalues, where ω_0 is the largest interior angle of the polygonal domain. If $\delta < l - \frac{2}{q} < \frac{\pi}{\omega_0}$, then the following theorem holds. Compare [18] p. 233, Corollary 4.438 and [27] p.373, Corollary 8.3.3.

Theorem 4 *Let $u \in H^1(\Omega)$ be a weak solution of problem (3.2)–(3.3), $f \in W^{l-2,q}(\Omega), g \in W^{l-1-\frac{1}{q},q}(\partial\Omega)$, where $l \geq 2, q > 1, \frac{2}{q} > l - \frac{\pi}{\omega_0}$ and ω_0 is the largest interior angle at boundary corners. Then $u \in W^{l,q}(\Omega)$.*

For $l = 2$ we can prove the following result valid for the solution of the nonlinear boundary value problem.

Theorem 5 *Let $u \in H^1(\Omega)$ be a weak solution of problem (2.1)–(2.2) in the polygonal domain Ω. If $f \in L^q(\Omega), \varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$, where*

$$q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon < 2 \quad \text{for } \omega_0 > \pi, \tag{3.4}$$

$$q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon > 2 \quad \text{for } \frac{\pi}{2} < \omega_0 < \pi, \tag{3.5}$$

$$q \text{ is abitrary} \qquad \text{for } \omega_0 \leq \frac{\pi}{2}, \tag{3.6}$$

and $\varepsilon > 0$ is a small number, then $u \in W^{2,q}(\Omega)$.

Proof.

1. Let $\omega_0 > \pi$. This means that a reentrant corner point occurs. The inequality in Theorem 4 reads $\frac{2}{q} > l - \frac{\pi}{\omega_0}$. It is satisfied for $l = 2$ and $q < 1 + \frac{\pi}{2\omega_0 - \pi}$. Moreover, $q < 2$. Thus, we can put $q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon$ with a small real number $\varepsilon > 0$. Due to Lemma 1, we have $g = -|u|^\alpha u + \varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$ and the assertion follows.

   In what follows, we consider convex polygons:

2. Let $\frac{\pi}{2} < \omega_0 < \pi$. As in the first case we can conclude that $u \in W^{2,\tilde{q}}(\Omega)$ with any $\tilde{q}$ with the property that $\tilde{q} < 2 < 1 + \frac{\pi}{2\omega_0 - \pi}$. Let us choose $\tilde{q} = 2 - \delta$ with an arbitrarily small $\delta > 0$. Therefore, the regularity of the nonlinear boundary term can be improved. We show that $|u|^\alpha u \in W^{1-\frac{1}{q^*},q^*}(\partial\Omega)$ with q^* arbitrarily large. Indeed, the embedding theorem yields that $W^{2,\tilde{q}}(\Omega) \subset C(\bar{\Omega})$ and therefore

$$|u|^\alpha u \in C(\bar{\Omega}) \subset L^{q^*}(\Omega). \tag{3.7}$$

   Due to the embedding $W^{2,\tilde{q}}(\Omega) \subset W^{1,q^*}(\Omega)$, where $q^* = \frac{2\tilde{q}}{2-\tilde{q}} = \frac{2(2-\delta)}{\delta}$ and (3.7) we have

$$\begin{aligned}\int_\Omega |\nabla(|u|^\alpha u)|^{q^*}\, dx &\leq (\alpha+1)^{q^*} \int_\Omega |u|^{\alpha q^*} |\nabla u|^{q^*}\, dx \\ &\leq (\alpha+1)^{q^*} \|u^{\alpha q^*}\|_{C(\bar{\Omega})} \||\nabla u|\|^{q^*}_{L^{q^*}}(\Omega) < \infty.\end{aligned}$$

   Hence, the trace of $|u|^\alpha u$ belongs to the space $W^{1-\frac{1}{q^*},q^*}(\partial\Omega)$ where q^* is arbitrarily large. It follows, that $\varphi - \kappa|u|^\alpha u \in W^{1-\frac{1}{q},q}(\partial\Omega)$. Now, we choose q in such a way that the inequality $\frac{2}{q} > 2 - \frac{\pi}{\omega_0}$ from Theorem 4 is satisfied. This leads to $q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon > 2$, where the positive real number ε is small enough.

3. Let $\omega_0 \leq \frac{\pi}{2}$. Following the considerations of the second case, we get the necessary smoothness of the nonlinear boundary term.The essential inequality $\frac{2}{q} > 2 - \frac{\pi}{\omega_0}$ is satisfied for an arbitrary $q \geq 1$. □

Remark 5 *From* (3.4)–(3.6) *and the fact that* $0 < \omega_0 < 2\pi$ *we see that*

$$\frac{4}{3} < q < \infty. \tag{3.8}$$

Now, we investigate the *interior regularity* of the weak solution.

We consider a domain Ω_0 with a smooth boundary such that $\overline{\Omega}_0 \subset \Omega$. We construct a second smooth subdomain Ω_0' of Ω with $\overline{\Omega}_0 \subset \Omega_0'$ and $\overline{\Omega}_0' \subset \Omega$ and choose a cut-off C^∞-function

$$\begin{aligned} \eta(x) &\equiv 1 \quad \text{for} \quad x \in \Omega_0 \\ \eta(x) &\equiv 0 \quad \text{for} \quad x \in I\!R^2 \setminus \Omega_0' \\ 0 \le \eta(x) &\le 1 \quad \text{else.} \end{aligned}$$

Lemma 2 *Let* $u \in H^1(\Omega)$ *be a weak solution of (2.1)–(2.2) in the polygonal domain* Ω *and let the assumptions of Theorem 5 be satisfied and, moreover,* $f \in W^{1,q}(\Omega)$*. Then* $u \in W^{3,q}(\Omega_0)$.

Proof. Due to Theorem 5, the weak solution belongs to $W^{2,q}(\Omega)$. The function ηu satisfies the following linear boundary value problem in Ω_0':

$$-\Delta(\eta u) = -u\Delta\eta - 2\nabla\eta\nabla u - \eta\Delta u \quad \text{in } \Omega_0', \tag{3.9}$$

$$\eta u = 0 \quad \text{on } \partial\Omega_0'. \tag{3.10}$$

The right hand side of (3.9) belongs to $W^{1,q}(\Omega_0')$ and the standard regularity theorem (cf. [2]) in smooth domains yields that $\eta u \in W^{3,q}(\Omega_0')$. Since $\eta u = u$ in Ω_0 we get $u \in W^{3,q}(\Omega_0)$. $\square$

If the right hand side f is smoother, than we can get higher interior regularity.

Lemma 3 *Let* $u \in H^1(\Omega)$ *be a weak solution of (2.1)–(2.2) in the polygonal domain* Ω *and let the assumptions of Theorem 5 be satisfied. Furthermore, let be* $f \in W^{k,q}(\Omega)$ *for* $k \ge 1$*. Then* $u \in W^{k+2,q}(\Omega_0)$.

Proof. Let us consider an arbitrary C^∞-function ψ with $\psi(x) \equiv 0$ for $x \in I\!R^2 \setminus \Omega_0'$. By induction we can prove: If $f \in W^{k,q}(\Omega)$, then $\psi u \in W^{k+2,q}(\Omega_0')$.

1st step: k=1

Analogously to the proof of lemma 2 it holds:

$$-\Delta(\psi u) = -u\Delta\psi - 2\nabla\psi \cdot \nabla u - \psi\Delta u \quad \text{in } \Omega_0', \tag{3.11}$$

$$\psi u = 0 \quad \text{on } \partial\Omega_0'. \tag{3.12}$$

Since $u \in W^{2,q}(\Omega)$, we have for the different terms of the right hand side of (3.11): $-u\Delta\psi \in W^{2,q}(\Omega_0')$, $\nabla\psi \cdot \nabla u \in W^{1,q}(\Omega_0')$ and $\psi\Delta u \in W^{1,q}(\Omega_0')$. The domain Ω_0' is smooth and therefore the solution ψu of the boundary value problem (3.11), (3.12) belongs to

$W^{3,q}(\Omega_0')$.

2nd step: $k \geq 1$
Assume that for $f \in W^{k,q}(\Omega)$ we get $\psi u \in W^{k+2,q}(\Omega_0')$ for all ψ. Consider $f \in W^{k+1,q}(\Omega)$. Then

$$\begin{aligned} -u\Delta\psi &= -\Delta(\psi)u - 2\nabla\psi \cdot \nabla u - \psi\Delta u \\ &= -\tilde{\psi}u - 2(\psi_1\partial_1 u + \psi_2\partial_2 u) + \psi f, \end{aligned} \tag{3.13}$$

where $\tilde{\psi} = \Delta\psi, \psi_1 = \partial_1\psi, \psi_2 = \partial_2\psi$ are admissible cut-off functions. The assumptions imply that the term $\tilde{\psi}u$ belongs to $W^{k+2,q}(\Omega_0')$ and $\psi f \in W^{k+1,q}(\Omega_0')$. Furthermore, for $i = 1, 2$, we have

$$\psi_i\partial_i u = \partial_i(\psi_i u) - u\partial_i\psi_i \in W^{k+1,q}(\Omega_0').$$

Thus, the right hand side of (3.13) is from $W^{k+1,q}(\Omega_0')$. The classical regularity theory for smooth domains, see [2], implies that the solution ψu of the boundary value problem (3.13), (3.12) belongs to $W^{k+2,q}(\Omega_0')$ for all ψ.
Setting $\psi = \eta$, it follows in Ω_0 that $\eta u = u \in W^{k+2,q}(\Omega_0)$. □

4 Discontinuous Galerkin discretization

In [9] and [10], problem (2.5) was discretized by standard piecewise linear conforming finite elements. In what follows, problem (2.5) will be solved numerically by the discontinuous Galerkin method (DGM) usig piecewise polynomial approximations of degree $r \geq 1$.

Let $\mathcal{T}_h$ be a triangulation of the domain Ω with standard properties. This means that $\mathcal{T}_h$ is formed by a finite number of closed triangles with mutually disjoint interiors. If $K, K' \in \mathcal{T}_h$ are different elements, then we assume that $K \cap K' = \emptyset$ or $K \cap K'$ is a common side of K and K' or $K \cap K'$ is a common vertex of K and K'. Moreover, we assume that the corner points of $\partial\Omega$ are vertices of some elements $K \in \mathcal{T}_h$, adjacent to $\partial\Omega$. The sides of $K \in \mathcal{T}_h$ will be called faces.

In our further considerations we use the following notation. For an element $K \in \mathcal{T}_h$ we set $h_K = \mathrm{diam}(K)$ and $h = \max_{K\in\mathcal{T}_h} h_K$. By ρ_K we denote the radius of the largest circle inscribed into K and by $|K|$ and $|\Omega|$ we denote the two-dimensional Lebesgue measure of K and Ω, respectively.

The symbol $\mathcal{F}_h$ will denote the system of all faces of all elements $K \in \mathcal{T}_h$, where we distinguish the set of all boundary faces

$$\mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}, \tag{4.1}$$

and of all innner faces

$$\mathcal{F}_h^I = \mathcal{F}_h \setminus \mathcal{F}_h^B. \tag{4.2}$$

For each $\Gamma \in \mathcal{F}_h$ we choose a unit vector $\boldsymbol{n}_\Gamma$ orthogonal to Γ. We assume that for $\Gamma \in \mathcal{F}_h^B$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each

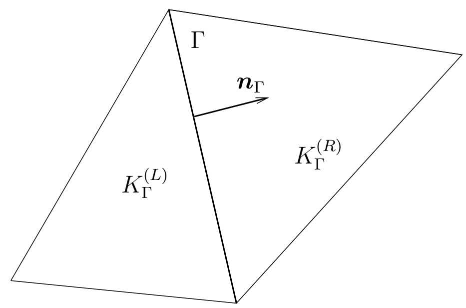


Figure 2: Interior face Γ, elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$ and the orientation of $\boldsymbol{n}_\Gamma$.

face $\Gamma \in \mathcal{F}_h^I$ the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary but fixed. If $\Gamma \in \mathcal{F}_h^I$, then there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$. We use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$ and the inner normal to $\partial K_\Gamma^{(R)}$. If the face $\Gamma \subset \partial\Omega$, then $K_\Gamma^{(L)}$ denotes the element from $\mathcal{T}_h$ adjacent to Γ. See Figure 2.

Over a triangulation $\mathcal{T}_h$, for any integer $s > 0$ and $q \geq 1$ we define the *broken Sobolev spaces*

$$W^{s,q}(\Omega, \mathcal{T}_h) = \{v; v|_K \in W^{s,q}(K)\ \forall\, K \in \mathcal{T}_h\} \tag{4.3}$$

and $H^s(\Omega, \mathcal{T}_h) = W^{s,2}(\Omega, \mathcal{T}_h)$.

For $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h^I$ we introduce the following notation:

$$\begin{aligned} &v|_\Gamma^{(L)} = \text{the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma, \\ &\langle v \rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\right), \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}. \end{aligned} \tag{4.4}$$

The value $[v]_\Gamma$ depends on the orientation of $\boldsymbol{n}_\Gamma$, but the value $[v]_\Gamma \boldsymbol{n}_\Gamma$ is independent of this orientation.

Let $r \geq 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$S_h^r = \{v \in L^2(\Omega); v|_K \in P^r(K),\ \forall K \in \mathcal{T}_h\}, \tag{4.5}$$

where $P^r(K)$ denotes the space of all polynomials on K of degree $\leq r$.

In view of Theorem 1.5 from [16] and Theorem 5, for each $K \in \mathcal{T}_h$ and $\Gamma \in \mathcal{F}_h^I$ we have

$$\begin{aligned} &u|_{\partial\Omega} \in W^{2-1/q,q}(\partial\Omega), \\ &u|_{\partial K} \in W^{2-1/q,q}(\partial K), \quad [u]_\Gamma = 0, \\ &\nabla u \in W^{1,q}(\Omega), \quad \Delta u \in L^q(\Omega),\ |u|^\alpha u|_{\partial\Omega} \in L^p(\partial\Omega)\ \forall p \in [1, \infty). \end{aligned} \tag{4.6}$$

Since $q > \frac{4}{3}$ the embedding theorem yields

$$\nabla u|_{\partial K} \in W^{1-1/q,q}(\partial K) \subset L^2(\partial K). \tag{4.7}$$

Furthermore, the following relations are satisfied for $\Gamma \in \mathcal{F}_h^I$:

$$[\nabla u]_\Gamma = 0, \quad \langle \nabla u \rangle_\Gamma = \nabla u|_\Gamma. \tag{4.8}$$

Hence, the weak solution satisfies the classical boundary value problem (2.1), (2.2) in Sobolev spaces. This allows us to derive the discontinuous Galerkin discretization of problem (2.1)–(2.2). We proceed in a standard way. We multiply equation (2.1) by any $v \in S_h^r$, integrate over every $K \in \mathcal{T}_h$, apply Green's theorem, sum over all $K \in \mathcal{T}_h$, add some expressions vanishing by virtue of (4.6)) and use condition (2.2). We arrive at the following forms which make sense for $u,\, v \in W^{2,q}(\Omega, \mathcal{T}_h)$ with any q satisfying (3.4)–(3.6):

$$\begin{aligned}
b_h(u,v) &= \sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v \,\mathrm{d}x \\
&\quad - \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \left(\boldsymbol{n}_\Gamma \cdot \langle \nabla u \rangle [v] + \theta\, \boldsymbol{n}_\Gamma \cdot \langle \nabla v \rangle\, [u] \right) \mathrm{d}S, && (4.9)\\
d_h(u,v) &= \kappa \sum_{\Gamma\in\mathcal{F}_h^B} \int_\Gamma |u|^\alpha\, uv \,\mathrm{d}S = \kappa \int_{\partial\Omega} |u|^\alpha\, uv \,\mathrm{d}S, && (4.10)\\
J_h(u,v) &= \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \sigma [u]\, [v] \,\mathrm{d}S, && (4.11)\\
a_h(u,v) &= b_h(u,v) + J_h(u,v), && (4.12)\\
A_h(u,v) &= a_h(u,v) + d_h(u,v), && (4.13)\\
L_h(v) &= \int_\Omega f v \,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^B} \int_\Gamma \varphi\, v \,\mathrm{d}S. && (4.14)
\end{aligned}$$

The form J_h represents the so-called interior penalty. The weight σ in (4.11) is defined as

$$\sigma|_\Gamma = \frac{C_W}{h_\Gamma}, \tag{4.15}$$

where h_Γ is the length of the face Γ and $C_W > 0$ is sufficiently large. It will be specified later. In (4.9) the parameter θ is chosen as $\theta = 1,\, 0,\, -1$, which leads to the symmetric, incomplete, nonsymmetric version of the diffusion form, denote by SIPG, IIPG, NIPG, respectively. Now we can introduce the discrete problem.

Definition 4 *We define an approximate solution of problem* (2.1)–(2.2) *as a function* u_h *such that*

$$\begin{aligned}
&a) \quad u_h \in S_h^r, && (4.16)\\
&b) \quad A_h(u_h, v_h) = L_h(v_h) \quad \forall\, v_h \in S_h^r.
\end{aligned}$$

From the properties (4.6) of the exact solution u and the derivation of the discrete problem it follows that

$$A_h(u, v_h) = L_h(v_h) \quad \forall\, v_h \in S_h^r. \tag{4.17}$$

In the broken Sobolev space $H^1(\Omega, \mathcal{T}_h)$ and the space $S_h^r \subset H^1(\Omega, \mathcal{T}_h)$ we use the seminorms

$$|v|_{H^1(\Omega,\mathcal{T}_h)} = \Big(\sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 \,\mathrm{d}x\Big)^{1/2}, \tag{4.18}$$

$$|v|_h = \Big(\sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 \,\mathrm{d}x + J_h(v,v)\Big)^{1/2}, \quad v \in H^1(\Omega, \mathcal{T}_h), \tag{4.19}$$

and the norm

$$|||v||| = \Big(|v|_h^2 + \|v\|_{L^2(\Omega)}^2\Big)^{1/2}, \quad v \in H^1(\Omega, \mathcal{T}_h). \tag{4.20}$$

5 Some auxiliary results

In the error analysis some embedding results valid for the broken Sobolev spaces will be used. They represent an analogy of to the continuous and compact embeddings

$$H^1(\Omega) \hookrightarrow\hookrightarrow L^\gamma(\Omega), \quad H^1(\Omega) \hookrightarrow\hookrightarrow L^\gamma(\partial\Omega)$$

valid for $\gamma \in [1, +\infty)$.

Now we consider a system of triangulations $\{\mathcal{T}_h\}_{h\in(0,\overline{h})}$ with $\overline{h} > 0$ of the domain Ω. In what follows we assume that this system is shape-regular. This means that there exists a constant $C_R > 0$ such that

$$\frac{h_K}{\rho_K} < C_R \quad \forall\, K \in \mathcal{T}_h \;\forall\, h \in (0, \overline{h}). \tag{5.1}$$

By virtue of Theorem 5.2 and Lemma 8, both from [4], the following results hold.

Lemma 4 *Let us consider sequences $\{h_n\}_{n=1}^\infty$, $h_n \in (0, \overline{h})$, and and $\{v_n\}_{n=1}^\infty$, $v_n \in H^1(\Omega, \mathcal{T}_h)$, such that*

$$\sup_{n\in\mathbb{N}} |||v_n||| < +\infty. \tag{5.2}$$

Then there exists a subsequence $\{h_{n_j}\}_{j=1}^\infty$ and $v \in H^1(\Omega)$ such that for each $\gamma \in [1, \infty)$

$$v_{n_j} \to v \quad \text{in } L^\gamma(\Omega), \tag{5.3}$$

$$v_{n_j} \to v \quad \text{in } L^\gamma(\partial\Omega), \tag{5.4}$$

as $j \to \infty$.

As a consequence of Theorem 4.4 from [4] it is possible to obtain an analogy to the embedding $H^1(\Omega) \hookrightarrow L^\gamma(\partial\Omega)$ for $\gamma \in [1,\infty)$.

Lemma 5 *Let $\gamma \in [1,\infty)$. Then there exists a constant $C_1 = C_1(\gamma) > 0$ such that*

$$\|v_h\|_{L^\gamma(\partial\Omega)} \le C_1 |||v_h||| \quad \forall\, v_h \in H^1(\Omega,\mathcal{T}_h),\ \forall\, h \in (0,\overline{h}). \tag{5.5}$$

Lemma 6 *Let $\gamma \in [1,\infty)$. Then there exists a constant $C_2 = C_2(\gamma) > 0$ such that the inequality*

$$|v_{h_j}|^2_{h_j} + \|v_{h_j}\|^\gamma_{L^\gamma(\partial\Omega)} \ge C_2 \quad \forall\, j \in I\!N, \tag{5.6}$$

holds for any sequences $h_j \in (0,\overline{h})$ and $v_{h_j} \in H^1(\Omega,\mathcal{T}_h)$, $j \in I\!N$, $h_j \to 0$ for $j \to \infty$ and $|||v_{h_j}||| = 1$ for all $j \in I\!N$.

Proof. If (5.6) is not valid, then there exist subsequences $h_j \to 0$ and $v_{h_j} \in H^1(\Omega,\mathcal{T}_h)$ (we denote them as before) such that

$$|||v_{h_j}||| = 1, \quad |v_{h_j}|^2_{h_j} + \|v_{h_j}\|^\gamma_{L^\gamma(\partial\Omega)} \le \frac{1}{j}. \tag{5.7}$$

By Lemma 4 there exists $v \in H^1(\Omega)$ such that for $j \to \infty$ we have

$$v_{h_j} \to v \text{ in } L^2(\Omega), \quad v_{h_j} \to v \text{ in } L^\gamma(\partial\Omega) \tag{5.8}$$

and by virtue of (5.7),

$$|v_{h_j}|_{h_j} \to 0, \quad \|v_{h_j}\|_{L^\gamma(\partial\Omega)} \to 0. \tag{5.9}$$

It follows from (5.7) and (5.8) and the definition of the norm $|||\cdot|||$ that

$$1 = |||v_{h_j}|||^2 = |v_{h_j}|^2_{h_j} + \|v_{h_j}\|^2_{L^2(\Omega)} \to \|v\|^2_{L^2(\Omega)}. \tag{5.10}$$

Further, by (5.9) we have $J_h(v_{h_j}, v_{h_j}) \to 0$. Using Lemma 7 and Theorem 5.2, both from [4], we see that

$$\nabla v_{h_j} \rightharpoonup \nabla v \text{ in } L^2(\Omega)^2 \quad \text{(weak convergence)}. \tag{5.11}$$

Moreover, in view of (5.9),

$$\|\nabla v_{h_j}\|_{L^2(\Omega)^2} \to 0. \tag{5.12}$$

By virtue of a well-known result from functional analysis (see, e. g., [20], Theorem 59.1 on page 332) and (5.12) we have

$$\|\nabla v\|_{L^2(\Omega)^2} = \liminf_{j\to\infty} \|\nabla v_{h_j}\|_{L^2(\Omega)^2} = 0. \tag{5.13}$$

Further, (5.8), (5.9) and (5.10) imply that

$$\text{a)} \quad \|v\|_{L^\gamma(\partial\Omega)} = 0, \qquad \text{b)} \quad \|v\|_{L^2(\Omega)} = 1. \tag{5.14}$$

Now, taking into account (5.13) and (5.14) a), we see that $v = \text{const} = 0$ in Ω, which is a contradiction to (5.14) b). $\square$

Corollary 1 *It follows from Lemma 6 that for $\gamma \in [1, \infty)$*

$$(5.15) \qquad |v_h|_h^2 + \|v_h\|^\gamma_{L^\gamma(\partial\Omega)} \geq C_2 \quad \forall\, v_h \in H^1(\Omega, \mathcal{T}_h),\ |||v_h||| = 1,\ \forall\, h \in (0, \overline{h}).$$

Proof. Let $h \in (0, \overline{h})$ and $v_h \in H^1(\Omega, \mathcal{T}_h)$, $|||v_h||| = 1$. Then we can construct sequences h_j and $v_{h_j} \in H^1(\Omega, \mathcal{T}_h)$ such that $h_1 = h$, $v_{h_1} = v_h$, $h_j \to 0$ as $j \to \infty$, and $|||v_{h_j}||| = 1$ for $j \in I\!N$. By (5.6) we have (5.15). $\square$

Remark 6 *Relation* (5.13) *can be proved also in another way. By* (5.12)

$$\|\nabla v_{h_j} - \nabla v_{h_k}\|_{L^2(\Omega)^2} \leq \|\nabla v_{h_j}\|_{L^2(\Omega)^2} + \|\nabla v_{h_k}\|_{L^2(\Omega)^2} \to 0$$

as j, $k \to \infty$ and $\{\nabla v_{h_j}\}_{j=1}^\infty$ is a Cauchy sequence in $L^2(\Omega)^2$. Hence, there exists $w \in L^2(\Omega)^2$ such that $\nabla v_{h_j} \to w$ strongly in $L^2(\Omega)^2$. This and (5.11) *imply that $\|\nabla v\|_{L^2(\Omega)^2} = 0$.*

Important tools in the DGM are the inverse inequality and the multiplicative trace inequality (see [7], Sections 2.5.1 and 2.5.2).

Lemma 7 *There exists a constant $C_I > 0$ such that the inverse inequality holds:*

$$(5.16) \qquad |v_h|_{H^1(K)} \leq C_I h_K^{-1} \|v_h\|_{L^2(K)}, \quad \forall v_h \in P^r(K),\ \forall K \in \mathcal{T}_h,\ \forall h \in (0, \overline{h}),$$

Furthermore the following multiplicative estimates are valid: there exists a constant $C_M > 0$ such that

$$(5.17) \qquad \|v\|^2_{L^2(\partial K)} \leq C_M \left(\|v\|_{L^2(K)} |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \quad \forall v \in H^1(K),\ \forall K \in \mathcal{T}_h,\ \forall h \in (0, \overline{h}),$$

and

$$(5.18) \qquad \|v\|^2_{L^2(\partial K)} \leq C_M \left(\|v\|_{L^{q^*}(K)} |v|_{W^{1,q}(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right),$$
$$\forall v \in W^{1,q}(K),\ \forall K \in \mathcal{T}_h,\ \forall h \in (0, \overline{h}),\ \forall q \in (\frac{4}{3}, 2) \text{ and } q^* > 1 \text{ satisfying } \frac{1}{q^*} + \frac{1}{q} = 1.$$

Proof. It is necessary to prove inequality (5.18). Since $\frac{4}{3} < q < 2$, then $2 < q^* = \frac{q}{q-1} < 4$ and in virtue of the embedding $W^{1,q}(K) \hookrightarrow L^\beta(K)$ with $\beta = \frac{2q}{2-q} > 4$, we have $W^{1,q}(K) \hookrightarrow L^{q^*}(K)$. Moreover, $W^{1-1/q,q}(\partial K) \hookrightarrow L^2(\partial K)$.

Now we start in the same way as in the proof of Lemma 2.19 from [7] and get

$$\rho_K \|v\|^2_{L^2(\partial K)} \leq 2\|v\|^2_{L^2(K)} + 2h_K \int_K |v||\nabla v| \,\mathrm{d}x.$$

This inequality, the Hölder inequality

$$\int_K |v||\nabla v| \,\mathrm{d}x \leq \|v\|_{L^{q^*}(K)} \|\nabla v\|_{L^q(K)^2}$$

and assumption (5.1) yield (5.18). □

In the case when $v \in W^{1,q}(K)$ with $q \geq 2$ we apply the multipliative trace inequality in the form (5.17).

Further, we are concerned with the coercivity of the forms a_h and A_h. We can obtain the following result.

Lemma 8 *(Coercivity of a_h) The inequality*

$$a_h(v_h, v_h) \geq \frac{1}{2}|v_h|_h^2 \quad \forall\, v_h \in S_h^r \quad \forall\, h \in (0, \overline{h}) \tag{5.19}$$

holds provided the constant C_W in (4.15) *from the definition* (4.11) *of the penalty form satisfies the condition*

$$C_W > 0 \quad \textit{for } \theta = -1 \textit{ (NIPG)}, \tag{5.20}$$

$$C_W > 4C_M(1 + C_I) \quad \textit{for } \theta = 1 \textit{ (SIPG)}, \tag{5.21}$$

$$C_W > C_M(1 + C_I) \quad \textit{for } \theta = 0 \textit{ (IIPG)}. \tag{5.22}$$

Proof. a) In the case of the NIPG version, when $\theta = -1$, by (4.12), (4.9) and (4.19) we immediately get

$$a_h(v_h, v_h) = b_h(v_h, v_h) + J_h(v_h, v_h) = |v_h|^2_{H^1(\Omega, \mathcal{T}_h)} + J_h(v_h, v_h) = |v_h|_h^2, \tag{5.23}$$

which implies (5.19).

b) Now we consider the SIPG version, when $\theta = 1$. Let $\delta > 0$. Then from (4.9), (4.15) and the Cauchy and Young inequalities it follows that

$$\begin{aligned} b_h(v_h, v_h) &= |v_h|^2_{H^1(\Omega, \mathcal{T}_h)} - 2 \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \boldsymbol{n}_\Gamma \cdot \langle \nabla v_h \rangle [v_h] \,\mathrm{d}S \\ &\geq |v_h|^2_{H^1(\Omega, \mathcal{T}_h)} - 2 \left\{ \frac{1}{\delta} \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma h_\Gamma (\boldsymbol{n}_\Gamma \cdot \langle \nabla v_h \rangle)^2 \,\mathrm{d}S \right\}^{\frac{1}{2}} \left\{ \delta \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \frac{1}{h_\Gamma} [v_h]^2 \,\mathrm{d}S \right\}^{\frac{1}{2}} \\ &\geq |v_h|^2_{H^1(\Omega, \mathcal{T}_h)} - \omega - \frac{\delta}{C_W} J_h(v_h, v_h), \end{aligned} \tag{5.24}$$

where

$$\omega = \frac{1}{\delta} \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma h_\Gamma |\langle \nabla v_h \rangle|^2 \,\mathrm{d}S. \tag{5.25}$$

Further, from (4.4), the relation

$$\sum_{\Gamma\in\mathcal{F}_h^I} h_\Gamma \int_\Gamma \langle v_h\rangle^2\,\mathrm{d}S \le \sum_{K\in\mathcal{T}_h} h_K \int_{\partial K} |v_h|^2 \mathrm{d}S, \tag{5.26}$$

the multiplicative trace inequality (5.17) and the inverse inequality (5.16), we get

$$\begin{aligned}\omega &\le \frac{1}{\delta}\sum_{K\in\mathcal{T}_h} h_K \|\nabla v_h\|^2_{L^2(\partial K)} \\ &\le \frac{C_M}{\delta}\sum_{K\in\mathcal{T}_h} h_K \left(|v_h|_{H^1(K)}|\nabla v_h|_{H^1(K)} + h_K^{-1}|v_h|^2_{H^1(K)}\right) \\ &\le \frac{C_M(1+C_I)}{\delta}|v_h|^2_{H^1(\Omega,\mathcal{T}_h)}.\end{aligned} \tag{5.27}$$

Now let us choose

$$\delta = 2C_M(1+C_I). \tag{5.28}$$

Then it follows from the condition on C_W in (5.21) and (5.24)–(5.28) that

$$\begin{aligned} b_h(v_h,v_h) &\ge \frac{1}{2}\left(|v_h|^2_{H^1(\Omega,\mathcal{T}_h)} - \frac{4C_M(1+C_I)}{C_W}J_h(v_h,v_h)\right) \\ &\ge \frac{1}{2}\left(|v_h|^2_{H^1(\Omega,\mathcal{T}_h)} - J_h(v_h,v_h)\right).\end{aligned} \tag{5.29}$$

Finally, definition (4.12) of the form a_h and (5.29) imply that

$$\begin{aligned} a_h(v_h,v_h) &= b_h(v_h,v_h) + J_h(v_h,v_h) \\ &\ge \frac{1}{2}\left(|v_h|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(v_h,v_h)\right) = \frac{1}{2}|v_h|_h^2,\end{aligned} \tag{5.30}$$

which is (5.19).

c) For the IIPG version ($\theta = 0$) the proof is similar to the previous case. □

Lemma 9 *(Coercivity of A_h) Let the constant C_W satisfy the conditions from Lemma 8. Then there exists a constant $C_3 > 0$ such that*

$$A_h(v_h,v_h) \ge C_3 |||v_h|||^2 \quad \forall\, v_h \in S_h^r \text{ with } |||v_h||| \ge 1,\ \forall\, h\in(0,\overline{h}). \tag{5.31}$$

Proof. For $v_h \in S_h^r$ we have $A_h(v_h,v_h) = a_h(v_h,v_h) + d_h(v_h,v_h)$. By virtue of (4.13), (4.10) and Lemma 8,

$$A_h(v_h,v_h) \ge \frac{1}{2}|v_h|_h^2 + \kappa\|v_h\|^\gamma_{L^\gamma(\partial\Omega)}, \tag{5.32}$$

where $\gamma = \alpha + 2 \geq 2$. Let $v_h \in S_h^r$ with $|||v_h||| \geq 1$. Then $w_h := v_h/|||v_h||| \in H^1(\Omega, \mathcal{T}_h)$ and $|||w_h||| = 1$. Now by (5.15),

$$|w_h|_h^2 + \|w_h\|_{L^\gamma(\partial\Omega)}^\gamma \geq C_2$$

and, hence, because $2 - \gamma \leq 0$,

$$\begin{aligned} C_2|||v_h|||^2 &\leq |||v_h|||^{2-\gamma} \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma + |v_h|_h^2 \\ &\leq \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma + |v_h|_h^2. \end{aligned}$$

This and (5.32) imply that

$$A_h(v_h, v_h) \geq C_2 \min\left(\frac{1}{2}, \kappa\right) |||v_h|||^2, \tag{5.33}$$

which is (5.31) with $C_3 = C_2 \min\left(\frac{1}{2}, \kappa\right)$. □

A further goal is the proof of the continuity of the form A_h.

Lemma 10 *For $q > \frac{4}{3}$ there exists a constant $C_4 > 0$ such that*

$$\begin{aligned} &|A_h(u, w) - A_h(v, w)| \leq C_4 \left\{(1 + |||u|||^\alpha + |||v|||^\alpha) |||u - v||| + R_h(u - v; q)\right\} |||w|||, \\ &\quad \forall\ u,\ v \in W^{2,q}(\Omega, \mathcal{T}_h),\ \forall w \in S_h^r,\ \forall\, h \in (0, \overline{h}), \end{aligned} \tag{5.34}$$

where

$$R_h(\phi; q) = \left(C_M \sum_{K \in \mathcal{T}_h} h_K |\phi|_{W^{1,q^*}(K)} |\phi|_{W^{2,q}(K)}\right)^{1/2} \tag{5.35}$$

with $\phi \in W^{2,q}(\Omega, T_h)$ and $q^* = q/(q-1)$ for $q \in (4/3, 2)$. If $q \geq 2$, then

$$R_h(\phi; q) = \left(C_M \sum_{K \in \mathcal{T}_h} h_K |\phi|_{H^1(K)} |\phi|_{H^2(K)}\right)^{1/2} \tag{5.36}$$

Proof. It follows from the definition of the form A_h that

$$|A_h(u, w) - A_h(v, w)| \leq |a_h(u - v, w)| + \kappa \left| \int_{\partial\Omega} \left(|u|^\alpha u - |v|^\alpha v\right) w \, \mathrm{d}S \right|. \tag{5.37}$$

First we proceed in a similar way as in [7], Section 2.6. Let $\phi \in W^{2,q}(\Omega, \mathcal{T}_h)$ and $\psi \in S_h^r$. By virtue of (4.9), (4.11), (4.12) and (4.6) we have

$$\begin{aligned} |a_h(\phi, \psi)| \leq & \underbrace{\sum_{K \in \mathcal{T}_h} \int_K |\nabla\phi \cdot \nabla\psi| \, \mathrm{d}x}_{\chi_1} + \underbrace{\sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma |\boldsymbol{n}_\Gamma \cdot \langle\nabla\phi\rangle \, [\psi]| \, \mathrm{d}S\, r}_{\chi_2} \\ & + \underbrace{\sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma |\boldsymbol{n}_\Gamma \cdot \langle\nabla\psi\rangle \, [\phi]| \, \mathrm{d}S}_{\chi_3} + J_h(\phi,\ \psi). \end{aligned} \tag{5.38}$$

The Cauchy inequality and (4.11) imply that

$$\chi_1 \leq |\phi|_{H^1(\Omega,\mathcal{T}_h)}|\psi|_{H^1(\Omega,\mathcal{T}_h)}, \tag{5.39}$$

$$\chi_2 \leq \left(\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma^{-1}\left(\boldsymbol{n}_\Gamma\cdot\langle\nabla\phi\rangle\right)^2\,\mathrm{d}S\right)^{1/2} J_h(\psi,\psi)^{1/2}, \tag{5.40}$$

$$\chi_3 \leq \left(\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma^{-1}\left(\boldsymbol{n}_\Gamma\cdot\langle\nabla\psi\rangle\right)^2\,\mathrm{d}S\right)^{1/2} J_h(\phi,\phi)^{1/2}. \tag{5.41}$$

Moreover,

$$J_h(\phi,\psi) \leq J_h(\phi,\phi)^{1/2}\, J_h(\psi,\psi)^{1/2}. \tag{5.42}$$

Now we estimate the expressions $\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma^{-1}(\boldsymbol{n}_\Gamma\cdot\langle\nabla\vartheta\rangle)^2\mathrm{d}S$ with $\vartheta = \phi \in W^{2,q}(\Omega,\mathcal{T}_h)$ and $\vartheta := \psi \in S_h^r$. In view of (4.15), the multiplicative trace inequality (5.17), the inverse inequality and simple manipulation we get

$$\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma^{-1}\left(\boldsymbol{n}_\Gamma\cdot\langle\nabla\psi\rangle\right)^2\,\mathrm{d}S \leq \frac{C_M}{C_W}(C_I+1)|\psi|^2_{H^1(\Omega,\mathcal{T}_h)}. \tag{5.43}$$

The estimation in the case when $\vartheta := \phi$ is more complicated. In this case we apply the multiplicative trace inequality in the form both (5.17) and (5.18). We find that

$$\begin{aligned}\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma^{-1}\left(\boldsymbol{n}_\Gamma\cdot\langle\nabla\phi\rangle\right)^2\,\mathrm{d}S &\leq C_W^{-1}\sum_{K\in\mathcal{T}_h} h_K\int_{\partial K}|\nabla\phi|^2\,\mathrm{d}S \leq \\ &\leq C_W^{-1}\left(R_h^2(\phi;q)+|\phi|^2_{H^1(\Omega,\mathcal{T}_h}\right).\end{aligned} \tag{5.44}$$

As a result from (5.38)–(5.44) and (4.18)–(4.20) we obtain the inequality

$$\begin{aligned}&|a_h(\phi,\psi)| \\ &\leq \tilde{C}\Big(|\phi|^2_{H^1(\Omega,\mathcal{T}_h)}+J_h(\phi,\phi)+R_h^2(\phi,q)\Big)^{1/2}\Big(|\psi|^2_{H^1(\Omega,\mathcal{T}_h)}+J_h(\psi,\psi)\Big)^{1/2} \\ &\leq \tilde{C}\Big(|||\phi|||^2+R_h^2(\phi;q)\Big)^{1/2}|||\psi|||\end{aligned} \tag{5.45}$$

with a constant $\tilde{C}>0$ independent of ϕ, ψ and h. Hence,

$$|a_h(u,w)-a_h(v,w)| \leq \tilde{C}\Big(|||u-v|||^2+R_h^2(u-v;q)\Big)^{1/2}|||w|||. \tag{5.46}$$

Now we estimate the second term in the right-hand side of (5.37). For η, $\xi \in I\!R$ and $t\in[0,1]$ we set $\beta(t) = |\xi+t(\eta-\xi)|^\alpha(\xi+t(\eta-\xi))$. Then $\beta'(t) = (\alpha+1)(\eta-\xi)|\xi+t(\eta-\xi)|^\alpha$ and, since,

$$\beta(1)-\beta(0) = \int_0^1 \beta'(t)\mathrm{d}t,$$

we have

$$|\eta|^\alpha \eta - |\xi|^\alpha \xi = (\alpha+1)\,(\eta-\xi)\int_0^1 |\xi + t(\eta-\xi)|^\alpha \,\mathrm{d}t.$$

From the convexity it follows that

$$|\xi + t(\eta-\xi)|^\alpha \le |\xi|^\alpha + |\eta|^\alpha, \quad \forall t \in [0,1]. \tag{5.47}$$

Using these relations and the Hölder inequality with parameters $p_i > 1$, $i = 1,2,3$, such that $1/p_1 + 1/p_2 + 1/p_3 = 1$, we get for $w \in S_h^r$

$$\begin{aligned} &\left|\int_{\partial\Omega} \big(|u|^\alpha u - |v|^\alpha v\big) w \,\mathrm{d}S\right| \\ \le\ & (\alpha+1)\int_{\partial\Omega} |u-v|\big(|u|^\alpha + |v|^\alpha\big)\,|w|\,\mathrm{d}s \\ \le\ & (\alpha+1)\,\|u-v\|_{L^{p_1}(\partial\Omega)}\left(\|u\|^\alpha_{L^{p_2\alpha}(\partial\Omega)} + \|v\|^\alpha_{L^{p_2\alpha}(\partial\Omega)}\right)\|w\|_{L^{p_3}(\partial\Omega)}. \end{aligned} \tag{5.48}$$

This inequality and (5.5) imply that

$$\left|\int_{\partial\Omega} (|u|^\alpha u - |v|^\alpha v)\, w \,\mathrm{d}S\right| \le (\alpha+1)|||u-v|||\,(|||u|||^\alpha + |||v|||^\alpha)\,|||w|||. \tag{5.49}$$

Finally, from (5.37), (5.46) and (5.49) we get (5.34). □

Lemma 11 *Let the constant* C_W *satisfy condition* (5.20)*. Then the form* A_h *is uniformly monotone on the space* S_h^r*, i.e., there exists a continuous and increasing function* $\rho : [0,\infty) \to [0,\infty)$ *such that*

$$\begin{aligned} & A_h(u_h,\, u_h - v_h) - A_h(v_h,\, u_h - v_h) \ge \rho(|||u_h - v_h|||) \\ & \forall\, u_h, v_h \in S_h^r \quad \forall\, h \in (0, \overline{h}). \end{aligned} \tag{5.50}$$

Proof. Let u_h, $v_h \in S_h^r$. By (4.9)–(4.13) and (5.19),

$$\begin{aligned} & A_h(u_h,\, u_h - v_h) - A_h(v_h,\, u_h - v_h) \\ =\ & a_h(u_h - v_h,\, u_h - v_h) + d_h(u_h,\, u_h - v_h) - d_h(v_h,\, u_h - v_h) \\ \ge\ & \frac{1}{2}|u_h - v_h|_h^2 + \kappa \int_{\partial\Omega} (|u_h|^\alpha u_h - |v_h|^\alpha v_h)\,(u_h - v_h)\,\mathrm{d}S. \end{aligned} \tag{5.51}$$

Now we shall be concerned with the last term in (5.51). Let $g > 0$ and $\alpha \ge 0$. We define the function $y : I\!R \to I\!R$:

$$y(\xi) = (|\xi + g|^\alpha(\xi + g) - |\xi|^\alpha \xi)g, \quad \xi \in I\!R. \tag{5.52}$$

Then the function $y(\xi)$ is increasing in $(-\frac{g}{2}, +\infty)$ and decreasing in $(-\infty, -\frac{g}{2})$ and

$$\min_{\xi \in I\!R} y(\xi) = y\Big(-\frac{g}{2}\Big) = 2^{-\alpha} g^{\alpha+2}. \tag{5.53}$$

For $\xi, \eta \in I\!R$ let us set $g = |\eta - \xi|$. Then

$$(|\eta|^{\alpha}\eta - |\xi|^{\alpha}\xi)\,(\eta - \xi) = \begin{cases} y(\xi), & \eta \geq \xi \\ y(\eta), & \eta \leq \xi. \end{cases} \tag{5.54}$$

Now (5.53) and (5.54) imply that

$$(|\eta|^{\alpha}\eta - |\xi|^{\alpha}\xi)\,(\eta - \xi) \geq 2^{-\alpha}|\eta - \xi|^{\alpha+2} \tag{5.55}$$

holds for all ξ, $\eta \in I\!R$. This and (5.51) imply that

$$A_h(u_h,\, u_h - v_h) - A_h(v_h,\, u_h - v_h) \geq \frac{1}{2}|u_h - v_h|_h^2 + \kappa\, 2^{-\alpha}\|u_h - v_h\|_{L^{\alpha+2}(\partial\Omega)}^{\alpha+2}. \tag{5.56}$$

Further, we apply Corollary 1. If we assume that $u_h \neq v_h$ and set $w_h = u_h - v_h$, then (5.15) with $\gamma = \alpha + 2$ implies that

$$\frac{1}{2}|w_h|_h^2 + \kappa\, 2^{-\alpha}\|w_h\|_{L^{\alpha+2}(\partial\Omega)}^{\alpha+2}|||w_h|||^{-\alpha} - C_6|||w_h|||^2 \geq 0, \tag{5.57}$$

where $C_6 = C_2 \min\left(\frac{1}{2}, \kappa\, 2^{-\alpha}\right)$. Multiplying (5.57) by $|||w_h|||^{\alpha}$ and subtracting from (5.56), we get

$$A_h(u_h, w_h) - A_h(v_h, w_h) \geq \frac{1}{2}|w_h|_h^2\,(1 - |||w_h|||^{\alpha}) + C_6|||w_h|||^{\alpha+2}. \tag{5.58}$$

If $|||w_h||| \leq 1$, then from (5.58) we get

$$A_h(u_h, w_h) - A_h(v_h, w_h) \geq C_6|||w_h|||^{\alpha+2}. \tag{5.59}$$

Now, if we assume that $|||w_h||| \geq 1$, then $|||w_h|||^{-\alpha} \leq 1$ and, by virtue of (5.56) and (5.57),

$$A_h(u_h, w_h) - A_h(v_h, w_h) \geq C_6|||w_h|||^2. \tag{5.60}$$

Of course, (5.59) and (5.60) also hold for $w_h = 0$, i. e, $u_h = v_h$.

From (5.59) and (5.60) we immediately see that (5.50) holds with

$$\rho(t) = \begin{cases} C_6\, t^{\alpha+2} & \text{for} \quad t \in [0, 1], \\ C_6\, t^2 & \text{for} \quad t \in [1, \infty). \end{cases} \tag{5.61}$$

It is obvious that the function ρ is continuous and increasing. $\square$

6 Error estimation

This section will be devoted to the derivation of error estimates for problem (4.16). First we prove an abstract error estimate.

Theorem 6 *Let conditions* (5.20)-(5.22) *and* (5.1) *be satisfied and let u be the exact weak solution defined by* (2.5)*. Then there exists a function $C_8 = C_8(v_h; u) > 0,\ v_h \in S_h^r$, such that*

$$\tag{6.1} \begin{aligned} &|||u - u_h||| \le \rho_1^{-1}\left(C_8(v_h;u)|||u - v_h||| + C_4 R_h(u - v_h; q)\right) + |||u - v_h|||, \\ &\forall\, v_h \in S_h^r \quad \forall\, h \in (0, \overline{h}), \end{aligned}$$

where u_h is the approximate solution satisfying (4.16)*, the expression R is given in Lemma 10,*

$$\tag{6.2} \rho_1(t) = \rho(t)/t$$

with $\rho(t)$ defined in (5.61) *and ρ_1^{-1} is the inverse to ρ_1.*

P r o o f. Due to the above results, we can proceed in a standard way. Let $h \in (0, \overline{h})$ and $v_h \in S_h^r$ be arbitrary. By virtue of (5.50), (4.16) and (4.17),

$$\begin{aligned} \rho\left(|||u_h - v_h|||\right) &\le A_h(u_h,\, u_h - v_h) - A_h(v_h,\, u_h - v_h) \\ &= L_h(u_h - v_h) - A_h(v_h,\, u_h - v_h) \\ &= A_h(u,\, u_h - v_h) - A_h(v_h,\, u_h - v_h). \end{aligned}$$

Further, Lemma 10 and the relation $|||u||| = \|u\|_{H^1(\Omega)}$ imply that

$$\begin{aligned} \rho\left(|||u_h - v_h|||\right) &\le C_4\left\{(1 + |||u|||^\alpha + |||v_h|||^\alpha)\, |||u - v_h||| + R_h(u - v_h; q)\right\}\, |||u_h - v_h||| \\ &\le C_4\left\{(1 + \|u\|^\alpha_{H^1(\Omega)} + |||v_h|||^\alpha)|||u - v_h||| + R_h(u - v_h; q)\right\}\, |||u_h - v_h|||. \end{aligned}$$

Hence, from this inequality and (6.2) it follows that

$$\tag{6.3} \rho_1\left(|||u_h - v_h|||\right) \le C_8(v_h;u)|||u - v_h||| + C_4 R_h(u - v_h; q),$$

where

$$\tag{6.4} C_8(v_h;u) = C_4(1 + \|u\|^\alpha_{H^1(\Omega)} + |||v_h|||^\alpha).$$

Since the function ρ_1 is increasing in $[0, +\infty)$, there exists its inverse. Then, by (6.3) and the triangle inequality, we have

$$\begin{aligned} |||u - u_h||| &\le |||u - v_h||| + |||u_h - v_h||| \\ &\le \rho_1^{-1}\left(C_8(v_h;u)\, |||u - v_h||| + C_4 R_h(u - v_h; q)\right) + |||u - v_h|||, \end{aligned}$$

what we wanted to prove. $\square$

In what follows, error estimates in terms of h will be analyzed. Again let $r \geq 1$ be an integer. The first step is the definition of a suitable S_h^r-interpolation and the analysis of its approximation properties. To this end, for any measurable subset $\omega \subset \overline{\Omega}$ and $\phi,\ \psi \in L^2(\omega)$ we set

$$(\phi, \psi)_\omega = \int_\omega \phi\psi \,\mathrm{d}x.$$

Now we define the S_h^r-interpolation operator $\pi_h : L^2(\Omega) \to S_h^r$: if $v \in L^2(\Omega)$, then

$$\pi_h v \in S_h^r, \quad (\pi_h v - v,\, v_h)_\Omega = 0 \quad \forall\, v_h \in S_h^r. \tag{6.5}$$

In other words,

$$\begin{aligned} &\pi_h v|_K \in P^r(K) \quad \forall\, K \in \mathcal{T}_h, \\ &(\pi_h v|_K - v|_K,\, v_h)_K = 0 \quad \forall\, v_h \in P^r(K)\ \forall\, K \in \mathcal{T}_h. \end{aligned} \tag{6.6}$$

Using similar techniques as in [5], Theorem 3.1.4, it is possible to prove the approximation properties of the operator π_h (see also [7], Section 2.5).

Lemma 12 *Let $s,\ m \geq 0$ be integers, $\beta,\ \vartheta \in [1,\infty)$ be such that $W^{\mu,\vartheta}(K) \hookrightarrow W^{m,\beta}(K)$ and let us set $\mu = \min(r+1, s)$. Then*

$$\begin{aligned} |v - \pi_h v|_{W^{m,\beta}(K)} \;\leq\; & C_9 |K|^{1/\beta - 1/\vartheta} \frac{h_K^\mu}{\rho_K^m} |v|_{W^{\mu,\vartheta}(K)} \\ & \forall\, v \in W^{s,\vartheta}(K)\ \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0, \overline{h}), \end{aligned} \tag{6.7}$$

where $C_9 > 0$ is a constant independent of v, K, h. Moreover, if (5.1) holds, then

$$\pi \rho_K^2 \leq |K| \leq \frac{\sqrt{3}}{4}\, h_K^2 \tag{6.8}$$

and

$$|v - \pi_h v|_{W^{m,\beta}(K)} \leq C_{10} h_K^{\mu - m + 2(1/\beta - 1/\vartheta)} |v|_{W^{\mu,\vartheta}(K)} \tag{6.9}$$

with C_{10} depending on C_R and C_9 only.

Lemma 13 *Let conditions (5.1) and (5.20)-(5.22) be satisfied and let $u \in H^1(\Omega)$ be the exact solution of problem (2.5). Then there exists a constant $C_{11} > 0$ independent of $h \in (0, \overline{h})$ such that*

$$|||\pi_h u||| \leq C_{11} \|u\|_{H^1(\Omega)}, \quad h \in (0, \overline{h}). \tag{6.10}$$

Moreover, the expression C_8, defined by (6.4), satisfies the inequality

$$C_8(\pi_h u; u) \leq \tilde{C}_8(\|u\|_{H^1(\Omega)}) := C_4 \left(1 + (C_{11}^\alpha + 1)\|u\|_{H^1(\Omega)}^\alpha\right), \quad h \in (0, \overline{h}). \tag{6.11}$$

Proof. By (4.19) and (4.20),

$$\|\|\pi_h u\|\|^2 = \sum_{K\in\mathcal{T}_h} |\pi_h u|^2_{H^1(K)} + J_h(\pi_h u, \pi_h u) + \|\pi_h u\|^2_{L^2(\Omega)}. \tag{6.12}$$

Since π_h is the $L^2(\Omega)$-orthogonal projection onto the space S_h^r, we have

$$\|\pi_h u\|^2_{L^2(\Omega)} \le \|u\|^2_{L^2(\Omega)}. \tag{6.13}$$

Further, the triangle inequality and (6.9) with $m = \mu = 1,\ \beta = \vartheta = 2$, imply that

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} |\pi_h u|^2_{H^1(K)} &\le 2\sum_{K\in\mathcal{T}_h} \left(|\pi_h u - u|^2_{H^1(K)} + |u|^2_{H^1(K)}\right) \\ &\le 2(C_{10}^2+1)|u|^2_{H^1(\Omega)}. \end{aligned} \tag{6.14}$$

Now we estimate the expression $J_h(\pi_h u, \pi_h u)$. It follows from (5.1) that there exists a constant $C_T > 0$ independent of $h \in (0, \overline{h})$ and $K \in \mathcal{T}_h$ such that $C_T\, h_K \le h_\Gamma$ for all $K \in \mathcal{T}_h$ and all $\Gamma \in \mathcal{F}_h$ such that $\Gamma \subset \partial K$. This inequality, the definition (4.11), (4.15) of the form J_h, the multiplicative trace inequality (5.17) and the Young inequality imply that

$$\begin{aligned} J_h(\pi_h u - u, \pi_h u - u) &= \sum_{\Gamma\in\mathcal{F}_h^I} \frac{C_W}{h_\Gamma}\int_\Gamma [u-\pi_h u]^2\,\mathrm{d}S \le \frac{2C_W}{C_T}\sum_{K\in\mathcal{T}_h} h_K^{-1}\|u-\pi_h u\|^2_{L^2(\partial K)} \\ &\le C_{12}\sum_{K\in\mathcal{T}_h}\left(h_K^{-2}\|u-\pi_h u\|^2_{L^2(K)} + |u-\pi_h u|^2_{H^1(K)}\right) \end{aligned} \tag{6.15}$$

where $C_{12} = 2C_W\, C_M/C_T$. Similarly as above, using (6.9) we find that

$$J_h(\pi_h u - u, \pi_h u - u) \le 2C_{10}^2 C_{12} \sum_{K\in\mathcal{T}_h} |u|^2_{H^1(K)} = 2C_{10}^2 C_{12}|u|^2_{H^1(\Omega)}. \tag{6.16}$$

In virtue of the inequalities

$$J_h(\pi_h u, \pi_h u) \le 2J_h(\pi_h u - u, \pi_h u - u) + 2J_h(u,u), \tag{6.17}$$

(6.16) and the relation $J_h(u,u) = 0$ valid due to the fact that $[u]_\Gamma = 0$ for $\Gamma \in \mathcal{F}_h^I$ and $u \in H^1(\Omega)$, we get

$$J_h(\pi_h u, \pi_h u) \le 4C_{10}^2 C_{12}|u|^2_{H^1(\Omega)}. \tag{6.18}$$

Finally, summarizing (6.12), (6.13), (6.14) and (6.18), we get (6.10) with $C_{11} = \left(2\,(C_{10}^2+1) + 4C_{10}^2 C_{12} + 1\right)^{1/2}$. Inequality (6.11) immediately follows from (6.10) and (6.4). □

Lemma 14 *Let $u \in W^{2,q}(\Omega)$, $q \in (4/3, 2)$, $q^* = q/(q-1)$ and $\mu = \min(r+1, 2) = 2$. Then*

$$R_h(u - \pi_h u; q) \leq C_M^{1/2} C_{10} \left(\sum_{K \in \mathcal{T}_h} h_K^{2(\mu - 2/q)} |u|^2_{W^{2,q}(K)} \right)^{1/2}, \quad h \in (0, \overline{h}). \tag{6.19}$$

In the case when $q \geq 2$ the expression $R_h(u - \pi_h u; q)$ satisfies the same estimate.

Proof. Estimate (6.19) is a consequence of (5.35), (5.36) and (6.9). □

Now we prove the error estimate in terms of $h \in (0, \overline{h})$.

Theorem 7 *Let conditions* (5.1) *and* (5.20)-(5.22) *be satisfied. Then, if $u \in W^{2,q}(\Omega)$ is the exact solution of problem* (2.5)*, u_h is the approximate solution defined by* (4.16) *and $4/3 < q \leq 2$ (cf.* (3.4)*), there exist constants C_{13}, $C_{14} > 0$ independent of h and u such that*

$$\begin{aligned} |||u - u_h||| \;&\leq\; \rho_1^{-1} \left(\left(C_{13} \tilde{C}_8(\|u\|_{H^1(\Omega)}) + C_{14} \right) h^{\mu - 2/q} |u|_{W^{2,q}(\Omega)} \right) \\ &\quad + C_{13}\, h^{\mu - 2/q} |u|_{W^{2,q}(\Omega)}, \quad h \in (0, \overline{h}), \end{aligned} \tag{6.20}$$

where $\mu = \min(r+1, 2) = 2$, $\tilde{C}_8(\|u\|_{H^1(\Omega)})$ is defined by (6.11) *and ρ_1 is the function defined by* (5.61) *and* (6.2)*. Furthermore, if the exact solution $u \in W^{2,q}(\Omega)$ with $q > 2$ (cf.* (3.5) - (3.6)*), then there exists a constant C_{13} independent of h and u such that*

$$\begin{aligned} |||u - u_h||| \;&\leq\; \rho_1^{-1} \left(\left(C_{13} \tilde{C}_8(\|u\|_{H^1(\Omega)}) + C_{14} \right) h^{\mu - 1} |u|_{W^{2,q}(\Omega)} \right) \\ &\quad + C_{13}\, h^{\mu - 1} |u|_{W^{2,q}(\Omega)}, \quad h \in (0, \overline{h}). \end{aligned} \tag{6.21}$$

Proof. We proceed in a standard way using the abstract error estimate (6.1), where we set $v_h := \pi_h u$. It is necessary to estimate the expression

$$|||u - \pi_h u|||^2 = \sum_{K \in \mathcal{T}_h} |u - \pi_h u|^2_{H^1(K)} + J_h(u - \pi_h u, u - \pi_h u) + \sum_{K \in \mathcal{T}_h} \|u - \pi_h u\|^2_{L^2(K)}. \tag{6.22}$$

By virtue of (6.9),

$$\|u - \pi_h u\|^2_{L^2(K)} \;\leq\; C_{10}^2 h_K^{2\mu + 2 - 4/q} |u|^2_{W^{2,q}(K)}, \tag{6.23}$$

$$|u - \pi_h u|^2_{H^1(K)} \;\leq\; C_{10}^2 h_K^{2\mu - 4/q} |u|^2_{W^{2,q}(K)}. \tag{6.24}$$

Now, (6.15), (6.23) and (6.24) imply that

$$|||u - \pi_h u|||^2 \leq C_{15} \sum_{K \in \mathcal{T}_h} h_K^{2\mu - 4/q} |u|^2_{W^{2,q}(K)}, \tag{6.25}$$

where $C_{15} = C_{10}^2(1 + \overline{h}^2 + 2C_{12})$. Let us remind that

$$|v|^2_{W^{2,q}(K)} = \left(\int_K |D^2 v|^q \mathrm{d}x\right)^{2/q},$$

where

$$|D^2 v|^q = \sum_{i,j=1}^{2} \left|\frac{\partial^2 v}{\partial x_i \partial x_j}\right|^q.$$

a) Now let us assume that $1 < q \leq 2$. Then $2/q \geq 1$ and we have

$$(6.26)\quad \sum_{K\in\mathcal{T}_h} |v|^2_{W^{2,q}(K)} = \sum_{K\in\mathcal{T}_h} \left(\int_K |D^2 v|^q \mathrm{d}x\right)^{2/q} \leq \left(\sum_{K\in\mathcal{T}_h} \int_K |D^2 v|^q \mathrm{d}x\right)^{2/q} = |v|^2_{W^{2,q}(\Omega)}.$$

This is a consequence of the inequality $\sum_{i=1}^n |a_i|^\beta \leq \left(\sum_{i=1}^n |a_i|\right)^\beta$ valid for $a_i \in I\!R$, $i = 1, \ldots, n$, and $\beta \geq 1$, following from Jensen's inequality (see, e. g., [19] , (1.4.1) and Theorem 19).

Now, summarizing the abstract error estimate (6.1), where we set $v_h := \pi_h u$, and use relations (6.10), (6.11), (6.19) and (6.26), we arrive at the error estimate (6.20) with $C_{13} := C_{15}^{1/2}$ and $C_{14} := C_4 C_{10} C_M^{1/2}$.

(b) Further, let us consider the case when $q > 2$. Applying the Hölder inequality to right-hand side of (6.25), we get

$$(6.27)\qquad |||u - \pi_h u|||^2 \leq C_{13} \left(\sum_{K\in\mathcal{T}_h} \left(h_K^{2\mu - 4/q}\right)^\gamma\right)^{1/\gamma} \left(\sum_{K\in\mathcal{T}_h} |u|^q_{W^{2,q}(K)}\right)^{2/q},$$

with γ such that $1/(q/2) + 1/\gamma = 1$, i. e., $\gamma = q/(q-2)$. We can write

$$(6.28)\qquad \sum_{K\in\mathcal{T}_h} \left(h_K^{2\mu-4/q}\right)^\gamma \leq \left(\sum_{K\in\mathcal{T}_h} h_K^2\right) h^{(2\mu-4/q)\frac{q}{q-2}-2}$$

and take into account that

$$(6.29)\qquad (2\mu - 4/q)\,\frac{q}{q-2} - 2 = \frac{2(\mu-1)q}{q-2}.$$

Now, by (5.1) and (6.8),

$$(6.30)\qquad \sum_{K\in\mathcal{T}_h} h_K^2 \leq \frac{C_R^2}{\pi} \sum_{K\in\mathcal{T}_h} |K| = \frac{C_R^2}{\pi}|\Omega|.$$

Finally, in virtue of (6.27) - (6.30) and (6.1), where $v_h := \pi_h u$, we get the estimate

$$(6.31)\qquad |||u - \pi_h u|||^2 \leq C_{15} \left(\frac{C_R^2}{\pi}|\Omega|\right)^{\frac{q-2}{2}} h^{2(\mu-1)} |u|^2_{W^{2,q}(\Omega)}.$$

In a similar way, by (6.19), we find that

$$R_h(u-\pi_h u;q) \le C_{10}\left(\frac{C_R^2}{\pi}|\Omega|\right)^{\frac{q-2}{4}} h^{\mu-1}|u|_{W^{2,q}(\Omega)}. \tag{6.32}$$

These estimates lead to (6.21) with $C_{13}=C_{15}^{1/2}\left(\frac{C_R^2}{\pi}|\Omega|\right)^{\frac{q-2}{4}}$ and $C_{14}=C_{10}C_M^{1/2}\left(\frac{C_R^2}{\pi}|\Omega|\right)^{\frac{q-2}{4}}$.
□

Remark 7 *If the data f and φ of problem* (2.1)–(2.2) *are such that the exact weak solution $u\in H^s(\Omega)$ with $s>2$ (in spite of singular corners on $\partial\Omega$), then in virtue of* (6.9), (6.15), (6.11), (5.36) *and* (6.1), *we obtain the error estimate*

$$\begin{aligned} |||u-u_h||| \;&\le\; \rho_1^{-1}\left(\left(C_{13}\tilde C_8(\|u\|_{H^1(\Omega)})+C_{14}\right) h^{\mu-1}|u|_{H^\mu(\Omega)}\right) \\ &\quad + C_{13}\, h^{\mu-1}|u|_{H^\mu(\Omega)}, \quad h\in(0,\overline{h}), \end{aligned} \tag{6.33}$$

where $\mu=\min(r+1,s)$.

Remark 8 *It follows from* (6.20), (6.21), (6.33), (5.61) *and* (6.2) *that there exist constants $C^*,\ C^{**}>0$ such that*

$$|||u-u_h||| \le C^* h^{\frac{\mu-\delta}{1+\alpha}} + C^{**}h^{\mu-\delta}, \quad h\in(0,\min(1,\overline{h})), \tag{6.34}$$

where we have

$$\begin{aligned} &a)\ \delta=2/q,\ \mu=2, \quad \text{provided } u\in W^{2,q}(\Omega),\ q\in(4/3,2],\\ &b)\ \delta=1,\ \mu=2, \quad \text{provided } u\in W^{2,q}(\Omega),\ q>2,\\ &c)\ \delta=1,\ \mu=\min(r+1,s), \quad \text{provided } u\in H^s(\Omega),\ s>2. \end{aligned} \tag{6.35}$$

Remark 9 *It follows from the above results that the order of convergence of the DG method applied to problem* (2.1)-(2.2) *depends on the polynomial degree of the approximate solution and the regularity of the exact solution (as in other finite element techniques). However, due to the corner singularities, the regularity is low – by Theorem 5, $u\in W^{2,q}(\Omega)$. By Lemma 3, in an interior subdomain $\Omega_0\subset\overline{\Omega}_0\subset\Omega$, we have $u\in W^{k+2,q}(\Omega_0)$, where q is defined by* (3.4)-(3.6) *and k corresponds to the regularity. This could allow us to improve the error estimate by a suitable mesh refinement in $\Omega\setminus\Omega_0$. Let us sketch roughly the main idea.*

We consider the situation when $u\in W^{2,q}(\Omega)$ and $u|_{\Omega_0}\in W^{k+2,q}(\Omega_0)$ with $k>0$. By h we denote the maximal size of the mesh in $\overline{\Omega}_0$, whereas $\tilde h$ is the size of the refined mesh in $\Omega\setminus\Omega_0$. By virtue of (6.9) *we have*

$$|u-\pi_{\tilde h}u|_{H^1(K)} \le C_{10}\tilde h^{2(1-1/q)}|u|_{W^{2,q}(K)}, \tag{6.36}$$

for $K\in\mathcal{T}_h,\ K\subset\overline{\Omega}\setminus\Omega_0$ and

$$|u-\pi_h u|_{H^1(K)} \le C_{10}h^{\mu-2/q}|u|_{W^{\mu,q}(K)}, \tag{6.37}$$

for $K \subset \overline{\Omega}_0$ and $\mu = \min(r+1, k+2)$. Hence, the order $O(h^{\mu-2/q})$ of accuracy will be valid in the whole domain Ω, if the mesh is refined near the boundary $\partial\Omega$ in such a way that

$$\tilde{h} \approx h^{\frac{\mu-2/q}{2(1-1/q)}}. \tag{6.38}$$

The analysis of this approach and the construction of a possible local mesh refinement near the boundary under a special consideration of the corner points will be the subject of a further work.

7 Numerical experiments

In this section, we document the derived error estimates formulated in Remark 8 by two numerical examples. Mainly, we explore the reduction of the order of convergence caused either by the nonlinearity of the solved problem or the low regularity of the exact solution. Problems with low regular solutions are particularly interesting since in practical applications of problem (2.1)-(2.2) the solution is rarely smooth.

In both experiments we discretize the problem by the SIPG variant of the DG method, which achieves the optimal orders of convergence $r+1$ and r in the $\|\cdot\|_{L^2(\Omega)}$ and $\|\!|\cdot|\!\|$, respectively, for sufficiently regular linear problems. We use uniform triangular meshes with element diameters $h_l = h_0/2^l$, $l = 0, 1, \ldots, 5$. Denoting the error of the discrete solution by $e_h = u - u_h$, we compute the experimental order of convergence (EOC) by

$$EOC = \frac{\log e_{h_{l+1}} - \log e_{h_l}}{\log h_{l+1} - \log h_l}, \qquad l = 0, 1, \ldots. \tag{7.39}$$

The discrete problem (4.16) represents a nonlinear system for $\alpha > 0$. We solved this problem by the damped Newton method with tolerance on the residual 10^{-9}.

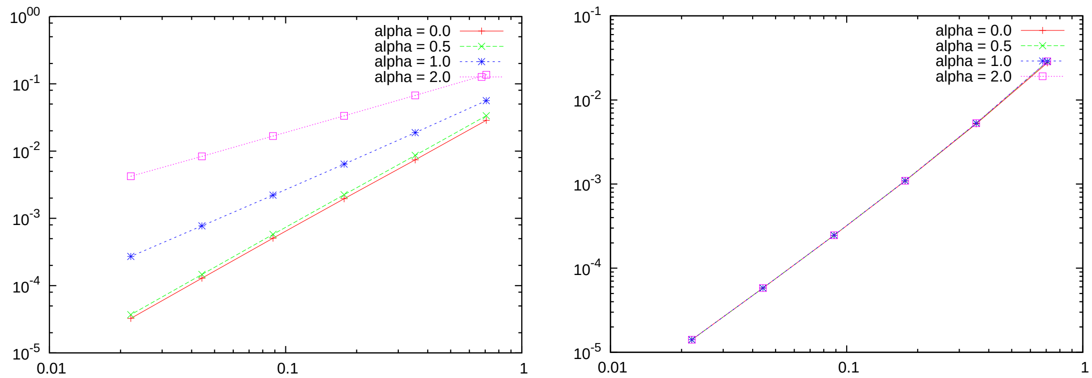
Remark 10 *One must proceed with caution when choosing the initial approximation u_h^0 for the Newton solver. If we choose $u_h^0 = 0$, which is often used when no additional information about the solution is known, then $|u_h^0|^\alpha u_h^0 = 0$ and the first step of the Newton method is equivalent to the problem with Neumann boundary condition on the whole boundary $\partial\Omega$. Since the solution of this problem is not unique, the corresponding matrix is singular and the computation breaks down.*

We carried out the numerical experiments using the FEniCS Project software [11].

7.1 Example 1: Regular problem

In the first experiment, we consider the problem (2.1), (2.2) on the unit square domain $\Omega = (0,1)^2$. The data φ and f are chosen such that the exact solution has the form

$$u(x_1, x_2) = x_1(1-x_1)x_2(1-x_2). \tag{7.40}$$



(a) discretization error measured in $\|\!|\cdot|\!\|$. (b) discretization error measured in $|\cdot|_{H^1(\Omega)}$.

Figure 3: Example 1 – EOC for piecewise quadratic DG method, $\|\!|\cdot|\!\|$ (left), $|\cdot|_{H^1(\Omega)}$ (right).

This function belongs to $H^k(\Omega)$ for arbitrary $k \in I\!N$. Therefore, according to the estimate (6.34) we expect $\|\!|e_h|\!\| \approx O\left(h^{\frac{r}{1+\alpha}}\right)$.

We discretized the problem with piecewise quadratic SIPG method, i.e., $r = 2$. In Table 1, we present the convergence history of the error computed on six uniformly refined triangular meshes for four choices of the nonlinearity parameter $\alpha = 0.0, 0.5, 1.0, 2.0$. By N_{hr} we denote the number of degrees of freedom (DOF) of the resulting discrete problem, h denotes $\max_{K\in\mathcal{T}_h} h_K$, $iter_{nl}$ denotes the number of Newton iterations. In the subsequent columns we list $L^2(\Omega)$-norm, $H^1(\Omega)$-seminorm and the energy norm, defined by (4.20), of the error and their corresponding experimental orders of convergence (EOC).

For the choice $\alpha = 0.0$, the problem is linear. Therefore, only one Newton iteration is needed and the order of convergence of the error measured both in $L^2(\Omega)$-norm and DG-norm are very close to the optimal order 3 and 2, respectively. With increasing α the nonlinearity of the problem becomes more significant, which causes the increasing number of iterations of the nonlinear Newton solver.

Regarding the errors, it seems that the nonlinearity of the problem mostly influences the L^2-norm of the error. On the other hand, the $H^1(\Omega)$-seminorm is almost identical for all choices of α, see Figure 3b (b). In fact, the $L^2(\Omega)$-norm considerably dominates over other norms on fine meshes for $\alpha > 0$ and hence it determines also the behaviour of the error $\|\!|e_h|\!\|$. In this case, the order of convergence decreases with growing parameter of the nonlinearity α as stated by the theoretical estimates. Only, due to the domination of the L^2-error it behaves like $O(h^{\frac{r+1}{1+\alpha}})$.

7.2 Example 2: Irregular solution on domains with one reentrant corner

As shown in previous sections, reentrant corners in the computational domain are sources of singularities in the solution. The second experiment is a variation of the well-known

$\alpha = 0.0$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	0.707	1	0.00282918	–	0.02759772	–	0.02862108	–
192	0.354	1	0.00035946	2.98	0.00520439	2.41	0.00739221	1.95
768	0.177	1	0.00004543	2.98	0.00109006	2.26	0.00195487	1.92
3072	0.088	1	0.00000571	2.99	0.00024576	2.15	0.00050617	1.95
12288	0.044	1	0.00000072	3.00	0.00005805	2.08	0.00012895	1.97
49152	0.022	1	0.00000009	3.00	0.00001409	2.04	0.00003255	1.99
$\alpha = 0.5$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	0.707	8	0.01761720	–	0.02858000	–	0.03353084	–
192	0.354	8	0.00445344	1.98	0.00528623	2.43	0.00861770	1.96
768	0.177	10	0.00111450	2.00	0.00109594	2.27	0.00224940	1.94
3072	0.088	12	0.00027862	2.00	0.00024616	2.15	0.00057773	1.96
12288	0.044	12	0.00006980	2.00	0.00005808	2.08	0.00014662	1.98
49152	0.022	12	0.00001758	1.99	0.00001409	2.04	0.00003700	1.99
$\alpha = 1.0$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	0.707	13	0.04855046	–	0.02873104	–	0.05626166	–
192	0.354	10	0.01724715	1.49	0.00529285	2.44	0.01875396	1.58
768	0.177	18	0.00609945	1.50	0.00109619	2.27	0.00640441	1.55
3072	0.088	13	0.00215647	1.50	0.00024616	2.15	0.00221504	1.53
12288	0.044	20	0.00076253	1.50	0.00005808	2.08	0.00077335	1.52
49152	0.022	15	0.00026982	1.50	0.00001409	2.04	0.00027178	1.51
$\alpha = 2.0$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	0.707	19	0.13328944	–	0.02879852	–	0.13621805	–
192	0.354	12	0.06676208	1.00	0.00529499	2.44	0.06716179	1.02
768	0.177	16	0.03338298	1.00	0.00109625	2.27	0.03343968	1.01
3072	0.088	26	0.01669148	1.00	0.00024617	2.15	0.01669912	1.00
12288	0.044	19	0.00834577	1.00	0.00005808	2.08	0.00834677	1.00
49152	0.022	17	0.00422103	0.98	0.00001409	2.04	0.00422116	0.98

Table 1: Example 1 – number of Newton iterations, discretization errors and convergence rates for $\alpha = 0.0, 0.5, 1.0, 2.0$.

test case, see e.g. [31]. We consider problem (2.1)–(2.2) in domains with the corner angle $\omega > 180°$. We prescribe the data of the problem such that the exact solution is defined by

$$u = \mathbf{r}^{\beta} \cos(\beta\theta), \tag{7.41}$$

where $\mathbf{r} = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(\frac{x_2}{x_1})$ and $\beta = \frac{180}{\omega}$.The angle of the reentrant corner ω determines the parameter β and also the strength of the singularity – the exact solution $u \in H^{1+\beta-\varepsilon}(\Omega)$ for sufficiently small $\varepsilon > 0$. We can examine the dependence of the order of convergence on the polynomial degree r, parameter α and also on the size of the angle ω. Here we set $r = 1$.

Figure 4 shows the exact solutions of the reentrant corner problem for various choices of the largest angle $\omega = 225°, 270°, 315°, 359°$. Table 2 shows the dependence of the order of convergence on the angle ω for $\alpha = 1.0$. In Figure 5 we see the dependence of the order of convergence on the angle ω (left) and parameter α (right). In agreement with the theory (see Remark 8 and Theorem 5) we observe that with increasing ω the order of convergence decreases from the value $EOC = 0.8$ for $\omega = 225°$ to $EOC = 0.5$ for $\omega = 359°$. On the other hand, changing the parameter of the nonlinearity α does not influence the discretization error in this case. This means that in this case the derived error estimates are not sharp for the varying parameter α. On the basis of both examples, it seems that this is caused by the nonzero values of the exact solution u on the boundary of Ω. A deeper investigation of this phenomenon will require further analysis.

Conclusion

The presented paper is concerned with the numerical solution of an elliptic problem in a polygonal domain equipped by a nonlinear Newton boundary condition with a polynomial nonlinearity. The paper contains the analysis of the regularity of the exact weak solution. Then the problem is discretized by the discontinuous Galerkin method and error estimates are derived. Presented numerical experiments show that the derived theoretical results describe the "worst scenario" and in some cases the experimental order of convergence is better than in derived estimates.

There are several subjects for future work:

- further analysis of the influence of the nonlinearity on the order of convergence of the method,
- influence of a suitable mesh refinement in the vicinity of the boundary corner points of the computational domain,
- analysis of the effect of the numerical integration,
- extension of the results to 3D and/or nonstationary problems.

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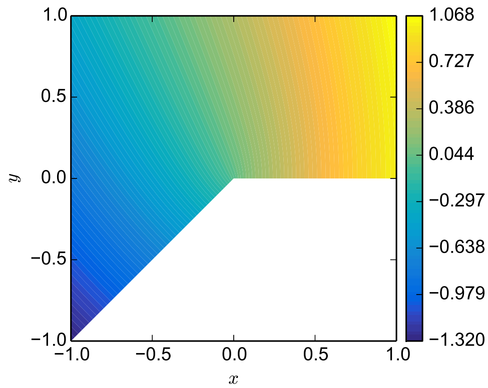
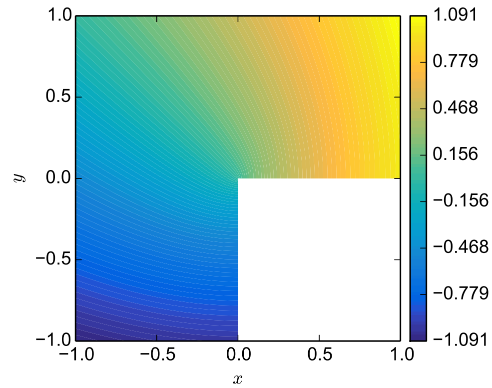
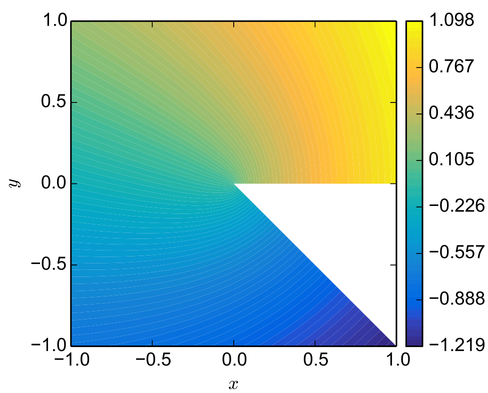


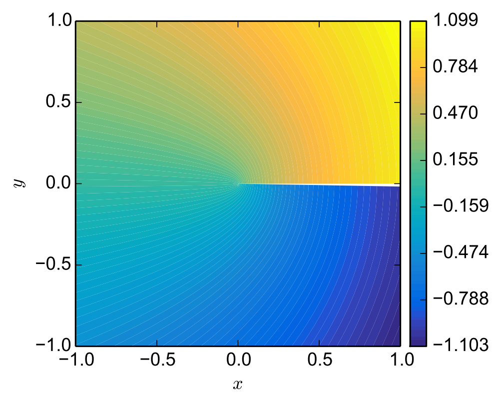
(a) $\omega = 225°$



(b) $\omega = 270°$



(c) $\omega = 315°$

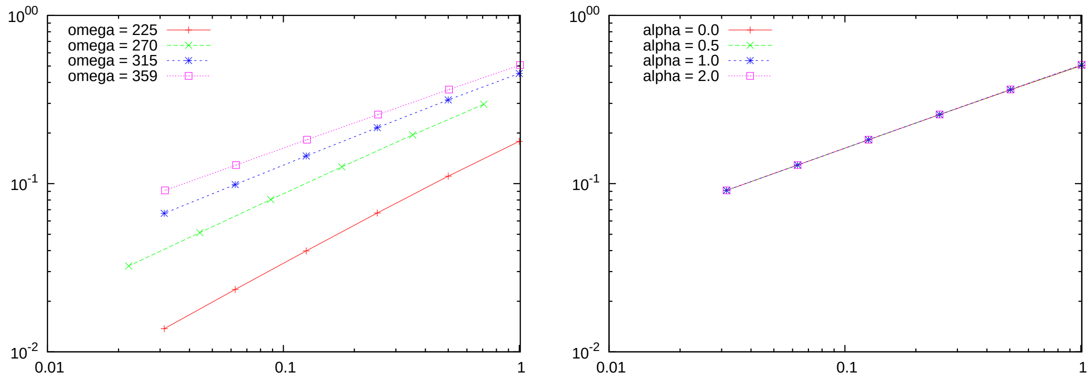


(d) $\omega = 359°$

Figure 4: Example 2 – the solution of the reentrant corner problem with various sizes of ω.

$\omega = 225^\circ$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	1.000	7	0.02071479	–	0.07997921	–	0.17845227	–
192	0.500	7	0.00667863	1.63	0.04784060	0.74	0.11073783	0.69
768	0.250	7	0.00217060	1.62	0.02822007	0.76	0.06694848	0.73
3072	0.125	7	0.00070830	1.62	0.01646232	0.78	0.03985072	0.75
12288	0.063	7	0.00023141	1.61	0.00953982	0.79	0.02348059	0.76
49152	0.031	7	0.00007566	1.61	0.00550860	0.79	0.01373875	0.77
$\omega = 270^\circ$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
72	0.707	7	0.02906954	–	0.15423085	–	0.29636584	–
288	0.354	7	0.01057678	1.46	0.09908832	0.64	0.19498212	0.60
1152	0.177	7	0.00394771	1.42	0.06340797	0.64	0.12584729	0.63
4608	0.088	7	0.00150541	1.39	0.04032541	0.65	0.08043361	0.65
18432	0.044	7	0.00058245	1.37	0.02553298	0.66	0.05112271	0.65
73728	0.022	7	0.00022745	1.36	0.01612626	0.66	0.03238437	0.66
$\omega = 315^\circ$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
72	1.000	7	0.05496242	–	0.26728239	–	0.45190308	–
288	0.500	7	0.02107635	1.38	0.18501414	0.53	0.31448433	0.52
1152	0.250	7	0.00846933	1.32	0.12660408	0.55	0.21516078	0.55
4608	0.125	7	0.00355786	1.25	0.08604547	0.56	0.14602555	0.56
18432	0.063	7	0.00154245	1.21	0.05822487	0.56	0.09871204	0.56
73728	0.031	7	0.00068179	1.18	0.03929666	0.57	0.06659159	0.57
$\omega = 359^\circ$								
N_{hr}	h	$iter_{nl}$	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
120	1.008	7	0.03266120	–	0.36414071	–	0.50740536	–
480	0.504	7	0.01397334	1.22	0.26057790	0.48	0.36287525	0.48
1920	0.252	7	0.00631178	1.15	0.18559718	0.49	0.25769754	0.49
7680	0.126	7	0.00299006	1.08	0.13174767	0.49	0.18248317	0.50
30720	0.063	7	0.00145686	1.04	0.09332023	0.50	0.12905472	0.50
122880	0.031	7	0.00071966	1.02	0.06601860	0.50	0.09121616	0.50

Table 2: Example 2 – number of Newton iterations, discretization errors and convergence rates for $\omega = 215^\circ, 270^\circ, 315^\circ, 359^\circ$ and $\alpha = 1.0$.



(a) $\omega = 225°, 270°, 315°, 359°,\ \alpha = 1.0$ (b) $\omega = 359°,\ \alpha = 0.0, 0.5, 1.0, 2.0$

Figure 5: Example 2 – dependence of the error measured in $\|\!|\cdot|\!\|$ on the parameters ω and α.

	$\alpha = 0.0$			$\alpha = 0.5$			$\alpha = 2.0$		
h	$iter_{nl}$	$\|\!|e_h|\!\|$	EOC	$iter_{nl}$	$\|\!|e_h|\!\|$	EOC	$iter_{nl}$	$\|\!|e_h|\!\|$	EOC
1.008	1	0.50321304	–	6	0.50565663	–	7	0.50904831	–
0.504	1	0.36122663	0.48	5	0.36228812	0.48	7	0.36323090	0.49
0.252	1	0.25711420	0.49	5	0.25752823	0.49	7	0.25774209	0.49
0.126	1	0.18229182	0.50	5	0.18244119	0.50	7	0.18247647	0.50
0.063	1	0.12899560	0.50	5	0.12904647	0.50	7	0.12904679	0.50
0.031	1	0.09119892	0.50	5	0.09121544	0.50	7	0.09121205	0.50

Table 3: Example 2 – number of Newton iterations, discretization errors and convergence rates for $\alpha = 0.0, 0.5, 2.0$ and $\omega = 359°$.

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