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Abstract

We characterize the forms that occur as restrictions of norm forms of octonion fields. The results are applied to forms of types E_6, E_7, and E_8, and to positive definite forms over fields that allow a unique octonion field.

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1 Introduction

Let F be a commutative field. An *octonion field* over F is a non-split composition algebra of dimension 8 over F. Thus there exists an anisotropic multiplicative form (the norm of the algebra) with non-degenerate polar form. It is known that such an algebra is not associative (but alternative). A good source for general properties of composition algebras is [9].

In [1], the group Λ_V generated by all left multiplications by non-zero elements is studied for various subspaces V of a given octonion field. In that paper, it is proved that Λ_V has a representation by similitudes of V (with respect to the restriction of the norm to V), and these representations are used to study exceptional homomorphisms between classical groups (in [1, 6.1, 6.3, or 6.5]). It is thus natural to ask which forms do occur as restrictions of norm forms of octonion fields. We answer this question in the present paper (in 3.1 below). In fact, not every quadratic form in five, six or seven variables can be interpreted as a restriction of the norm of some octonion field. We give abstract characterizations of the forms in question (namely, forms of types E_6, E_7, and E_8), in terms of concepts that play their role in the theory of spherical Tits buildings, cf. [10, Ch. 12].

1.1 Properties of composition algebras. *Let C be a composition algebra over F, and let N_C be its norm. The polar form will be denoted by $(x|y) := N_C(x+y) - N_C(x) - N_C(y)$.*

- **(a)** *The map $\kappa\colon C \to C\colon x \mapsto \overline{x} := (x|1)1 - x$ is an involutory anti-automorphism, called the* standard involution *of C. The norm and its polar form can be recovered from the standard involution as $N_C(x) = x\overline{x} = \overline{x}x$ and $(x|y) = x\overline{y} + y\overline{x}$.*
  In particular, we have the hyperplane $\operatorname{Pu} C := 1^\perp = \{x \in C \mid \overline{x} = -x\}$ of pure elements.
- **(b)** The Cayley–Dickson doubling process (cf. [9, 1.5.3]): *If B is a subalgebra of C with $\dim_F C = 2\dim_F B$ and such that $B^\perp \cap B = \{0\}$ then $B^\perp = Bw = wB$ holds for each $w \in B^\perp$ with $N_C(w) \neq 0$, and the multiplication in $C = B \oplus B^\perp$ is given by $(x + yw)(u + vw) = (xu - N_C(w)\overline{v}y) + (vx + y\overline{u})w$.*

Conversely, given any[1] *associative composition algebra B over F with norm N_B and any $\gamma \in F^\times$ one uses that formula to define a composition algebra $\mathbb{D}^\gamma(B)$ on the direct sum $B \oplus wB$ using the multiplication rule as above with the scalar γ playing the role of $N(w)$; the norm of $\mathbb{D}^\gamma(B)$ is then the orthogonal sum $N_{\mathbb{D}^\gamma(B)} = N_B \ominus \gamma N_B$.*

2 Norm splittings

2.1 Definition. Let q be an anisotropic quadratic form on a vector space of dimension $2d$ over F. We say that q has a *norm splitting (involving K/F)* if there exist a separable quadratic field extension K/F with norm $N_{K/F}$ and scalars $\alpha_1, \dots, \alpha_d \in F^\times$ such that q is equivalent to the orthogonal sum $\alpha_1 N_{K/F} \ominus \cdots \ominus \alpha_d N_{K/F}$.

The notion of norm spliting is used to single out various forms that play their roles in the theory of spherical buildings (see [10, Ch. 12]).

2.2 Definitions. Let q be an anisotropic quadratic form in n variables over some field F.

(a) The form q is *of type* E_6 if $n = 6$ and q has a norm splitting. See [10, Ch. 12] and [3].

(b) The form q is of *type* E_7 if $n = 8$ and q has a norm splitting but is *not* similar to the norm form of any octonion algebra. See [10, 12.31].

In terms of the norm splitting $\alpha_1 N_{K/F} \ominus \alpha_2 N_{K/F} \ominus \alpha_3 N_{K/F} \ominus \alpha_4 N_{K/F}$, the forms of type E_7 are characterized by $\alpha_1\alpha_2\alpha_3\alpha_4 \notin N_{K/F}(K)$. (See 3.1 (f) below.)

(c) The form q is *of type* E_8 if $n = 12$ and q has a norm splitting $\alpha_1 N_{K/F} \ominus \alpha_2 N_{K/F} \ominus \alpha_3 N_{K/F} \ominus \alpha_4 N_{K/F} \ominus \alpha_5 N_{K/F} \ominus \alpha_6 N_{K/F}$ such that $-\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 \in N_{K/F}(K)$.

2.3 Theorem. *The extension field involved in the norm splitting for a form of type E_6 is unique (up to isomorphism as an algebra over F).*

Proof. The even Clifford algebra for a form q is isomorphic to the full matrix algebra $K^{4\times 4}$ if q has a norm splitting involving K/F (see [3, 5.3], cf. [10, 12.43 (i)]), and the algebra $K^{4\times 4}$ determines K/F, up to isomorphism. The latter claim follows since K is (isomorphic to) the endomorphism ring of K^4 considered as a simple module for $K^{4\times 4}$ (e.g., see [2, 26.4]), or from an application of projective geometry, e.g., see [7, 4.7][2]. □

For quadratic forms of type E_7, the field extension involved in the norm splitting is not determined uniquely, in general, as the following example shows.

2.4 Example. Consider the quadratic extensions $K := \mathbb{Q}(\sqrt{-1})$ and $L := \mathbb{Q}(\sqrt{-3})$ of the field $\mathbb{Q}$ of rational numbers; these are composition algebras obtained as doubles $\mathbb{D}^1(\mathbb{Q})$ and $\mathbb{D}^3(\mathbb{Q})$, respectively (cf. 1.1 (b)). Let $i \in K$ be a square root of -1, and let $N := N_{K/\mathbb{Q}}$ and $M := N_{L/\mathbb{Q}}$ be the corresponding norm forms. We note that N and M are not similar because the discriminants differ. We now show that the orthogonal sum $N \ominus N$ is equivalent to $M \ominus 2M$. Doubling K (cf. 1.1 (b)) we obtain $H_1 := \mathbb{D}^1(K) = K \oplus jK$ with $N_{H_1}(j) = 1$. This is a quaternion field, the norm form is equivalent to $N \ominus N$. The element $y := i + j + ij \in \mathrm{Pu}(H_1)$

[1] Associativity is needed in order to make the new norm multiplicative. Note that a composition algebra in characteristic two necessarily has non-trivial standard involution because the polar form of the norm is required to be non-degenerate. If $\operatorname{char} F \neq 2$ we can (and will) start with F itself, made a composition algebra by the form $N(x) = x^2$ on F.

[2] There is a confusing systematic typo in [7]: $\mathcal{L}KV$ should be replaced by $\mathrm{GL}_K(V)$.

has norm 3. The subalgebra $\mathbb{Q}+\mathbb{Q}y$ of H_1 is isomorphic to the field L, and $i-j$ is an element of norm 2 in $(\mathbb{Q}+\mathbb{Q}y)^\perp$. So $\mathbb{Q}(y) \cong L$ and $H_1 \cong \mathbb{D}^2(L)$ yield that $N \oplus N$ and $M \oplus 2M$ are both equivalent to N_{H_1}.

The form $N \oplus 3N$ is equivalent to the norm form of a quaternion field $H_3 := \mathbb{D}^3(K) = K \oplus wK$ where $N_{H_3}(w) = 3$. In $\mathrm{Pu}(H_3)$ we find an element $2i+w$ of norm $2^2+3\cdot 1^2 = 7$, while there is no such element in $\mathrm{Pu}(H_1)$ (by Legendre's three-square theorem, see [5, Ex. 3.12]). This shows that the two forms $N \oplus N$ and $N \oplus 3N$ are not equivalent, and also not similar. (We could also split off the summand N from both forms, and then argue that N and $3N$ are not equivalent — it is easy to see that $3 \notin N(K)$; i.e., the prime number 3 is not a sum of two squares of rational numbers.)

The form $q\colon \mathbb{Q}^8 \to \mathbb{Q}\colon x = (x_1,\dots,x_8) \mapsto x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+3x_7^2+3x_8^2$ obviously has a norm splitting $N \oplus N \oplus N \oplus 3N$. The form q is not similar to the norm form of an octonion field. In fact, if q were similar to the norm form N_C of some octonion algebra C then $1 \in q(\mathbb{Q}^8)$ yields that q is equivalent to N_C. Then we see that C contains a quaternion subfield H isomorphic to H_1 (with norm form $N \oplus N$), and the restriction $N_C|_{H^\perp}$ is equivalent to $N \oplus 3N$ by Witt's cancellation law. However, as $H^\perp$ contains an element of norm 1, doubling yields $C \cong \mathbb{D}^1(H)$ so that $N_C|_{H^\perp}$ is equivalent to $N \oplus N$. This is impossible by our remark in the preceding paragraph. (See also 3.1 (f) below.)

So q is a form of type E_7. This form has norm splittings involving two inequivalent quadratic extensions: For instance, using the equivalence of $N \oplus N$ with $M \oplus 2M$ (as noted above) and the obvious equivalence of $N \oplus 3N$ with $M \oplus M$ we find that $M \oplus M \oplus M \oplus 2M$ is a norm splitting for q, as well.

3 Subforms of norm forms

Not every quadratic form in five, six or seven variables can be interpreted as a restriction of the norm of some octonion field, as in [1, 6.1, 6.3, or 6.5]).

3.1 Theorem. *Let $q\colon V \to F$ be an anisotropic quadratic form.*

- **(a)** *If $\dim V = 3$ then q is similar to the restriction of the norm of a quaternion field H to $\mathrm{Pu}\,H$ if, and only if, the polar form is not zero. (This condition is trivially satisfied if $\mathrm{char}\,F \neq 2$.)*
- **(b)** *If $\dim V = 4$ then q is similar to the norm of a quaternion field if, and only if, it has a norm splitting.*
- **(c)** *If $\dim V = 5$ then there exists an octonion field $\mathbb{O}$ and a 3-dimensional subspace $W < \mathbb{O}$ with $1 \in W \nleq W^\perp$ such that q is similar to the restriction $N_{\mathbb{O}}|_{W^\perp}$ if, and only if, there exists a hyperplane $U < V$ such that $q|_U$ has a norm splitting.*
- **(d)** *If $\dim V = 6$ then there exists an octonion field $\mathbb{O}$ and a 2-dimensional subspace $L < \mathbb{O}$ with $1 \in L \nleq L^\perp$ such that q is similar to $N_{\mathbb{O}}|_{L^\perp}$ if, and only if, the form q has a norm splitting.*
- **(e)** *If $\dim V = 7$ then there exists an octonion field $\mathbb{O}$ such that q is similar to $N_{\mathbb{O}}|_{\mathrm{Pu}\,\mathbb{O}}$ if, and only if, there exists a hyperplane $U < V$ and $y \in U^\perp$ such that $q|_U$ has a norm splitting $\alpha_1 N_{K/F} \oplus \alpha_2 N_{K/F} \oplus \alpha_3 N_{K/F}$ and $q(y) \in \alpha_1\alpha_2\alpha_3\, N_{K/F}(K^\times)$.*
- **(f)** *If $\dim V = 8$ then there exists an octonion field $\mathbb{O}$ such that q is similar to $N_{\mathbb{O}}$ if, and only if, the form q has a norm splitting $\alpha_1 N_{K/F} \oplus \alpha_2 N_{K/F} \oplus \alpha_3 N_{K/F} \oplus \alpha_4 N_{K/F}$ with $\alpha_1\alpha_2\alpha_3\alpha_4 \in N_{K/F}(K^\times)$. Every norm splitting $\beta_1 N_{L/F} \oplus \beta_2 N_{L/F} \oplus \beta_3 N_{L/F} \oplus \beta_4 N_{L/F}$ of q then has the property $\beta_1\beta_2\beta_3\beta_4 \in N_{L/F}(L^\times)$.*

Proof. We use the doubling process for composition algebras, cf. 1.1 (b). The proof of assertion (a) is postponed, we treat assertion (b) first.

Assume $\dim V = 4$ and $q = \alpha_1 N_{K/F} \oplus \alpha_2 N_{K/F}$. Then $\alpha_1^{-1} q$ is equivalent to the norm of the quaternion algebra $H := \mathbb{D}^{\alpha_1^{-1}\alpha_2}(K)$ obtained by doubling K (considered as a two-dimensional composition algebra over F). As q is anisotropic, that quaternion algebra is in fact a quaternion field. Conversely, every quaternion field is obtained by doubling any separable 2-dimensional subalgebra K, and that procedure gives a splitting for the norm. This completes the proof of assertion (b).

In order to prove assertion (a), we note first that the restriction of the norm to the space of pure quaternions has the required properties. Now assume $\dim V = 3$ and that there exists a two-dimensional subspace M such that the polar form is non-degenerate on M. Then there exists a separable quadratic extension K/F such that $q|_M = \alpha_1 N_{K/F}$ holds for some $\alpha_1 \in F^\times$. Pick $y \in M^\perp \smallsetminus \{0\}$, then q is the restriction of $\alpha_1 N_{K/F} \oplus q(y) N_{K/F}$ to a suitable hyperplane. By assertion (b) we thus know that there exists a quaternion field H such that q is similar to the restriction of N_H to some hyperplane $W < H$. Pick $a \in W^\perp \smallsetminus \{0\}$; then $\operatorname{Pu} H = a^{-1}W$ yields that q is in fact similar to $N_H|_{\operatorname{Pu} H}$. This completes the proof of assertion (a).

Assume $\dim V = 5$, and that there exists a hyperplane $U < V$ such that $q|_U = \alpha_1 N_{K/F} \oplus \alpha_2 N_{K/F}$. From assertion (b) we know that there exists a quaternion field H with $N_H = \alpha_1^{-1} q$. Pick any $y \in U^\perp \smallsetminus \{0\}$ and construct the composition algebra $\mathbb{O} := \mathbb{D}^{\alpha_1^{-1}q(y)}(H) = H \oplus wH$ with $N(w) = \alpha_1^{-1} q(y)$. Then the restriction of $N_{\mathbb{O}}$ to $X := H \oplus Fw < \mathbb{O}$ is similar to q, and thus anisotropic. As the norm of a split composition algebra is hyperbolic, the existence of a five-dimensional anisotropic subspace yields that $\mathbb{O}$ is not split, and thus an octonion field. We pick $a \in X^\perp \smallsetminus \{0\}$, then $W := a^{-1}X^\perp$ contains 1, and $W \not\leq W^\perp$ because $W < a^{-1}H^\perp$ and the polar form on $H^\perp = wH$ is not degenerate. Conversely, starting from a subspace W with the required properties in any octonion field we recover that octonion field by doubling the quaternion field generated by W, and see that the restriction of the norm to $W^\perp$ has the required properties. This completes the proof of assertion (c).

Assume $\dim V = 6$ and $q = \alpha_1 N_{K/F} \oplus \alpha_2 N_{K/F} \oplus \alpha_3 N_{K/F}$. We already know that there is a quaternion field H with norm equivalent to $N_{K/F} \oplus \alpha_1^{-1}\alpha_2 N_{K/F}$. Doubling yields an octonion algebra $\mathbb{O} := \mathbb{D}^{\alpha_1^{-1}\alpha_3}(H) = H \oplus wH$ with $N_{\mathbb{O}}(w) = \alpha_1^{-1}\alpha_3$. That algebra is an octonion field because it contains an anisotropic subspace Y of dimension 6. We pick $a \in Y^\perp \smallsetminus \{0\}$, then $L := a^{-1}Y^\perp$ is a two-dimensional subalgebra which is separable because the restriction of the polar form to $Y^\perp$ is not degenerate. So q is similar to $N_{\mathbb{O}}|_{L^\perp}$, as required. Again, in any octonion field with any separable 2-dimensional subalgebra L we recover the octonion field by repeated doubling and see that the restriction of the norm to $L^\perp$ has a splitting as claimed in assertion (d).

Assume $\dim V = 7$, that there exists a hyperplane $U < V$ such that $q|_U = \alpha_1 N_{K/F} \oplus \alpha_2 N_{K/F} \oplus \alpha_3 N_{K/F}$, and $y \in U^\perp$ with $q(y) \in \alpha_1\alpha_2\alpha_3 N_{K/F}(K^\times)$. From the previous paragraph we know that there exists an octonion field $\mathbb{O}$ with a two-dimensional separable subalgebra L, a quaternion subfield H, and a similitude ψ from U onto $Y = aL^\perp$, for a suitable $a \in \mathbb{O}$. The multiplier of ψ is α_1. The construction of Y yields that there exists $b \in L^\perp \cap H$ with $N_{\mathbb{O}}(b) = \alpha_1^{-1}\alpha_2$, and we find that wb is an element of $Y^\perp$ with $N_{\mathbb{O}}(wb) = \alpha_1^{-1}\alpha_2\alpha_3$. Our assumption on $q(y)$ now yields that the similitude ψ extends to a similitude $\tilde{\psi}$ from V onto $T := aL^\perp \oplus Fwb$ with $\tilde{\psi}(y) \in Lwb$. Pick $c \in T^\perp \smallsetminus \{0\}$, then $c^{-1}T = 1^\perp = \operatorname{Pu}\mathbb{O}$, and $N_{\mathbb{O}}|_{\operatorname{Pu}\mathbb{O}}$ is similar to q, as claimed.

It remains to note that, in any octonion field, the restriction of the norm to the space of pure elements has the required properties. To this end, we remark first that these properties are preserved under similitudes because $(\mu\alpha_1)(\mu\alpha_2)(\mu\alpha_3) = \mu^3(\alpha_1\alpha_2\alpha_3)$ and $\mu\alpha_1\alpha_2\alpha_3$ lie in the same square class. Pick $u \in \mathbb{O}$ with $T_{\mathbb{O}}(u) = 1$; then $L := F + Fu$ is a separable quadratic extension of F, and $\mathrm{Pu}\,\mathbb{O} = \mathrm{Pu}\,L \oplus L^\perp$. There are scalars $\beta_1, \beta_2 \in F$ such that $N_{\mathbb{O}}|_{L^\perp}$ has a norm splitting $\beta_1 N_{L/F} \ominus \beta_2 N_{L/F} \ominus \beta_1\beta_2 N_{L/F}$. Now the restriction of $N_{\mathbb{O}}$ to the hyperplane $S := F \oplus L^\perp$ has the required properties, and so has the hyperplane $\mathrm{Pu}\,\mathbb{O} = a^{-1}S$, where $a \in S^\perp \smallsetminus \{0\}$. This completes the proof of assertion (e).

The last assertion (f) follows from the doubling process. □

At the end of the proof of 3.1 (e), it is tempting to use the hyperplane $K^\perp$ of $\mathrm{Pu}\,\mathbb{O}$, with the norm splitting $\beta_1 N_{K/F} \ominus \beta_2 N_{K/F} \ominus \beta_1\beta_2 N_{K/F}$. However, the intersection $\mathrm{Pu}\,\mathbb{O} \cap (K^\perp)^\perp = \mathrm{Pu}\,K$ is, in general, not spanned by an element of norm $\beta_1\beta_2(\beta_1\beta_2)$. In fact, the existence of such an element in $\mathrm{Pu}\,K$ means $K \cong F[X]/(X^2+1)$ if $\mathrm{char}\,F \neq 2$, and means $K \cong F[X]/(X^2+X+1)$ if $\mathrm{char}\,F = 2$. So we have to use a different norm splitting for the hyperplane $K^\perp$, or indeed a different hyperplane with norm splitting. The proof above circumvents this problem.

Recall that a quadratic form of type E_6 is defined as an anisotropic form in 6 variables with a norm splitting. In 3.1 (d) we have thus obtained a characterization of that class of forms (cf. [10, 12.33]):

3.2 Corollary. *A form of type E_6 exists over F if, and only if, there exists an octonion field over F.*

In 3.1 (d) we have noted that every form of type E_6 is embedded into the norm form of an octonion field over F. In fact, that embedding is essentially unique.

3.3 Theorem. *Let $q\colon U \to F$ be a form of type E_6, and let C, D be octonion algebras with vector subspaces $V < C$ and $W < D$, respectively, such that q is similar to both restrictions $N_C|_V$ and $N_D|_W$. Then C and D are isomorphic as F-algebras.*

If there are subalgebras $L < C$ and $M < D$ such that $V = L^\perp$ and $W = M^\perp$, respectively, then the isomorphism can be chosen in such a way that it maps L to M.

Proof. Let $q = \alpha_1 N_{K/F} \ominus \alpha_2 N_{K/F} \ominus \alpha_3 N_{K/F}$ be a norm splitting. Pick $v \in V^\perp \smallsetminus \{0\}$. Then $L := v^{-1}(V^\perp)$ is a two-dimensional subspace of C. We have $1 \in L$, so L is a subalgebra. As the restriction $N_C|_V$ of the norm form has non-degenerate polar form, the restriction $N_C|_L$ has non-degenerate polar form, as well. So L is a separable quadratic field extension of F. We recover C by repeated doubling as $\mathbb{D}^{\beta_2}(\mathbb{D}^{\beta_1}(L))$, and obtain that the restriction of N_C to $L^\perp$ has a norm splitting $N_C|_{L^\perp} = \beta_2 N_{L/F} \ominus \beta_3 N_{L/F} \ominus \beta_2\beta_3 N_{L/F}$. As the extension field involved in the norm splitting is unique (see 2.3), we have an isomorphism of F-algebras from K onto L, and $N_{L/F}$ is equivalent to $N_{K/F}$.

Pick elements $a \in L^\perp \smallsetminus \{0\}$ and $b \in (L+aL)^\perp \smallsetminus \{0\}$ with $N_C(a) = s\alpha_1$ and $N_C(b) = s\alpha_2$, respectively (for some $s \in F^\times$; this is possible because the forms q and $N_C|_{L^\perp}$ are similar). On the one hand, we have a norm splitting $N_C = N_{K/F} \ominus s\alpha_1 N_{K/F} \ominus s\alpha_2 N_{K/F} \ominus s\alpha_3 N_{K/F}$. On the other hand, doubling yields $C = \mathbb{D}^{s\alpha_2}(\mathbb{D}^{s\alpha_1}(L))$ and a norm splitting $N_C = N_{K/F} \ominus s\alpha_1 N_{K/F} \ominus s\alpha_2 N_{K/F} \ominus s^2\alpha_1\alpha_2 N_{K/F}$.

For the similitude from q onto W, we proceed analogously. We find a subfield $M \cong K$ of D, and obtain D by doubling as $\mathbb{D}^{t\alpha_2}(\mathbb{D}^{t\alpha_1}(M))$ with some $t \in F^\times$. From 3.1 (f) we know that the products $s\alpha_1 \cdot s\alpha_2 \cdot s\alpha_3$ and $t\alpha_1 \cdot t\alpha_2 \cdot t\alpha_3$ both belong to the group $N_{K/F}(K^\times)$. As that group also contains s^2 and t^2, we obtain that s and t differ by an element of $N_{K/F}(K^\times)$. Thus $C \cong \mathbb{D}^{s\alpha_2}(\mathbb{D}^{s\alpha_1}(L)) \cong \mathbb{D}^{t\alpha_2}(\mathbb{D}^{t\alpha_1}(M)) \cong D$, as claimed. □

4 Forms over ordered fields

4.1 Lemma. **(a)** *If F admits an ordering then there exists at least one octonion field over F.*

(b) *If F admits two different orderings then there exists more than one isomorphism type of octonion fields over F.*

Proof. Assume that F admits an ordering $<$. Then $\operatorname{char} F = 0$, and -1 is not a square in F. We form the quadratic extension $E := \mathbb{D}^1(F) \cong F[X]/(X^2+1)$. By repeated doubling, we construct an octonion algebra $\mathbb{O} := \mathbb{D}^1(\mathbb{D}^1(E))$ with norm form $N_E \oplus N_E \oplus N_E \oplus N_E$. That form is positive definite, and $\mathbb{O}$ is an octonion field.

In order to prove assertion (b), assume that there is a second ordering $\prec$ on F. Pick $a \in F$ such that $a < 0$ but $0 \prec a$. Then a is not a sum of squares, and $-a$ is not a square in F. The field extension $K := \mathbb{D}^a(F) \cong F[X]/(X^2+a)$ has positive definite norm form N_K, and there exists an octonion algebra $\mathbb{D}^1(\mathbb{D}^a(K))$ over F with norm form $N_K \oplus aN_K \oplus N_K \oplus aN_K$. As before, that octonion algebra has positive definite norm form (with respect to $\prec$), and is an octonion field. In that octonion field, there exists an element of norm a, but no such element exists in $\mathbb{O}$ because a is not a sum of squares in F. □

4.2 Theorem. *Over the field $\mathbb{Q}$ of rational numbers, there are precisely two isomorphism types of octonion algebras: the split one, and the one with the positive definite norm form equivalent to the sum of squares of coordinates.*

Proof. Without loss, we concentrate on the non-split case. As in 4.1 (a), we construct the "standard" non-split octonion algebra $\mathbb{O} := \mathbb{D}^1(\mathbb{D}^1(\mathbb{D}^1(\mathbb{Q})))$.

Let C be an octonion algebra over the field $\mathbb{Q}$, with norm form N_C. The local-global principle by Hasse-Minkowski [5, VI.3] asserts that the quadratic form N_C is determined, up to isometry, by the isometry types of the forms $L \otimes N_C$, where L runs over the completions of $\mathbb{Q}$. For every non-archimedean (i.e., p-adic) completion L, the form $L \otimes N_C$ is isotropic (cf. [5, 2.12]), and $L \otimes C$ is a split octonion algebra. So $L \otimes N_C$ is hyperbolic, for each p-adic completion L of $\mathbb{Q}$. The archimedean completion $\mathbb{R}$ either leads to a positive definite form $\mathbb{R} \otimes N_C$, or to an isotropic form $\mathbb{R} \otimes N_C$. In the latter case, the algebra $\mathbb{R} \otimes C$ is the split octonion algebra over $\mathbb{R}$, and the form $\mathbb{R} \otimes N_C$ is hyperbolic. We have thus shown that there are just two isomorphism types of octonion algebras over $\mathbb{Q}$. Obviously, the split octonion algebra and the non-split octonion algebra $\mathbb{O}$ over $\mathbb{Q}$ represent these two types. □

The arguments used in 4.2 can be generalized, and yield the following[3]

4.3 Theorem. *Let K be an algebraic number field (i.e., a finite extension field of $\mathbb{Q}$).*

(a) *If K admits an ordering (i.e., if at least one completion of K is isomorphic to $\mathbb{R}$) then there exists at least one isomorphism type of octonion fields over K.*

(b) *If more than one completion is isomorphic to $\mathbb{R}$ then there exist at least two isomorphism type of octonion fields over K.*

(c) *If precisely one of the archimedean completions of K is isomorphic to $\mathbb{R}$ then there exists precisely one isomorphism type of octonion fields over K (represented by $\mathbb{D}^1(\mathbb{D}^1(\mathbb{D}^1(K))) \cong K \otimes \mathbb{D}^1(\mathbb{D}^1(\mathbb{D}^1(\mathbb{Q})))$).*

[3] The results of 4.3 have already been obtained by Zorn [11, p. 400], who attributes them to Brandt (without giving an explicit reference). We could not locate any pertinent publication.

(d) *If K does not admit an ordering (i.e., if every archimedean completion of K is isomorphic to $\mathbb{C}$) then every octonion algebra over K splits, and there exists no octonion field over K.* □

5 Forms over fields with unique octonion field

Motivated by the observation made in 4.2, we study fields with the property

($\diamondsuit$) There is precisely one isomorphism type of octonion field over F.

If F is a field with this property, we will denote the (unique) octonion field over F by $\mathbb{O}$.

Recall from 4.2 that the field $\mathbb{Q}$ of rational numbers has property ($\diamondsuit$), and so do many (but not all, see 4.3) algebraic number fields. The related property for quaternion algebras has been studied by Kaplansky [4] (see [5, XI 6.23]) who proved the following. If F is a field with $\operatorname{char} F \neq 2$ admitting no ordering and there is precisely one quaternion field over F then every quadratic form in more than four variables over F is isotropic (i.e., the u-invariant of F equals 4).

5.1 Lemma. *If F is an ordered field with property ($\diamondsuit$) then the octonion field $\mathbb{O}$ over F is isomorphic to $\mathbb{D}^1(\mathbb{D}^1(\mathbb{D}^1(F)))$. In particular, the norm form is positive definite.*

Proof. Each norm in $\mathbb{D}^1(\mathbb{D}^1(\mathbb{D}^1(F)))$ is a sum of squares in F. Therefore, the norm form is positive definite, and thus anisotropic. Thus $\mathbb{D}^1(\mathbb{D}^1(\mathbb{D}^1(F)))$ is the (unique) octonion field over F. □

5.2 Lemma. *Assume that F has property ($\diamondsuit$). Then the group $N_\mathbb{O}(H^\times)$ has index two in $F^\times$, for every quaternion subalgebra H of $\mathbb{O}$.*

Proof. Let H be a quaternion subalgebra, and consider $\gamma, \lambda \in F^\times$. Then $\mathbb{D}^\gamma(H)$ is split precisely if $-\gamma \in N_\mathbb{O}(H)$, and $\mathbb{D}^\gamma(H) \cong \mathbb{D}^\lambda(H) \iff \lambda^{-1}\gamma \in N_\mathbb{O}(H^\times)$. The first one of these observations shows that $N_\mathbb{O}(H^\times)$ is a proper subgroup of $F^\times$ because there *exists* an octonion field, and index two follows from uniqueness. □

5.3 Lemma. *Assume that F has property ($\diamondsuit$). If $N_\mathbb{O}(\mathbb{O}) = F$ then $-1 \in N_\mathbb{O}(H)$ holds for each quaternion field H in $\mathbb{O}$. In particular, there exists no ordering on F, and F is not an algebraic number field.*

Proof. Let H be a quaternion algebra in $\mathbb{O}$. Pick $w \in H^\perp \smallsetminus \{0\}$. Our assumption $N_\mathbb{O}(\mathbb{O}) = F$ yields $F = N_\mathbb{O}(H) + N_\mathbb{O}(w)N_\mathbb{O}(H)$, and $N_\mathbb{O}(w) \notin N_\mathbb{O}(H)$. From 5.2 we know that $N_\mathbb{O}(H^\times)$ is a subgroup of index two in $F^\times$, and infer $F^\times = N_\mathbb{O}(H^\times) \cup N_\mathbb{O}(w)N_\mathbb{O}(H^\times)$. Now $-1 \in N_\mathbb{O}(w)N_\mathbb{O}(H^\times)$ would imply that $\mathbb{O}$ is split, so $-1 \in N_\mathbb{O}(H^\times)$.

If there were an ordering on F, property ($\diamondsuit$) would (by 5.1) imply that the norm form on $\mathbb{O}$ is positive definite, contradicting $-1 \in N_\mathbb{O}(H)$. Over an algebraic number field without ordering, every octonion algebra splits (see 4.3 (d)). □

5.4 Theorem. *Assume that F has property ($\diamondsuit$), and that $N_\mathbb{O}(\mathbb{O}) \neq F$. Then $N_\mathbb{O}(\mathbb{O}^\times)$ forms the set of positive elements for an ordering on F, and that ordering is the only one turning F into an ordered field. In particular, we have $\mathbb{O} \cong \mathbb{D}^1(\mathbb{D}^1(\mathbb{D}^1(F)))$. For every quaternion subalgebra H in $\mathbb{O}$, we have $N_H(H^\times) = N_\mathbb{O}(\mathbb{O}^\times)$. An abstract quaternion algebra H over F is isomorphic to a subalgebra of $\mathbb{O}$ if, and only if, its norm form N_H is positive definite.*

Proof. Pick a quaternion subalgebra H in $\mathbb{O}$. From 5.2 we know $N_{\mathbb{O}}(\mathbb{O}^\times) \in \{N_{\mathbb{O}}(H^\times), F^\times\}$, and our assumption yields $N_{\mathbb{O}}(\mathbb{O}^\times) = N_{\mathbb{O}}(H^\times)$. For $w \in H^\perp \smallsetminus \{0\}$ there exists $x \in H$ with $N_{\mathbb{O}}(w) = N_H(x)$. Now $v := wx^{-1} \in wH$ has norm 1 in $\mathbb{O}$, and $\mathbb{O} = \mathbb{D}^1(H)$ follows. As $\mathbb{O}$ is not split, we have $-1 \in F \smallsetminus N_{\mathbb{O}}(H)$.

For $\alpha, \beta \in N_H(H)$ we pick $x, y \in H$ with $N_H(x) = \alpha$ and $N_H(y) = \beta$, respectively. Then $N_{\mathbb{O}}(x + vy) = N_H(x) + N_{\mathbb{O}}(v)N_H(y) = N_H(x) + N_H(y)$ shows that the subgroup $P := N_{\mathbb{O}}(\mathbb{O}^\times) = N_H(H^\times)$ of $F^\times$ is also closed under addition. As $\mathbb{O}$ is not split, we have $P \cap -P = \varnothing$, and $P \cup -P = F^\times$ because P has index two in $F^\times$ (see 5.2). So P forms the set of positive elements for an ordering of the field F, cf. [6, Section 11].

The choice of a quaternion subalgebra does not affect our argument. In particular, every such subalgebra has the same group of norms (namely, P).

The last assertion follows from the observations that $N_{\mathbb{O}}$ (and thus each restriction to any subspace of $\mathbb{O}$) is positive definite, and that every quaternion algebra H with positive definite norm is non-split, with positive definite double $\mathbb{D}^1(H) \cong \mathbb{O}$. □

5.5 Remark. Lagrange's four-square theorem asserts that every positive element of $\mathbb{Q}$ is the sum of four squares. We have used this theorem (indirectly, via the local-global principle by Hasse-Minkowski) in 4.2 to show that there exists only one isomorphism type of octonion field over $\mathbb{Q}$. Conversely, we see from 5.4 that (under the assumption ($\diamondsuit$) of uniqueness of the octonion field over F) every positive element of F is a norm in every quaternion field with positive definite norm over F (a special case of the Hilbert-Siegel Theorem, see [8, Hauptsatz, p. 259]). In particular, this holds for the quaternion field $\mathbb{D}^1(\mathbb{D}^1(F))$; viz., every positive element of F is a sum of four squares.

5.6 Theorem. *Assume that F is an ordered field with property ($\diamondsuit$). Then every form of type E_6 over F is either positive or negative definite.*

The equivalence classes of positive definite forms of type E_6 over F correspond uniquely to the pairs of isomorphism classes of quadratic extension fields with positive definite norm and positive scalars modulo the group of norms. The similarity classes correspond to isomorphism classes of quadratic extension fields with positive definite norm.

More precisely: If q has a norm splitting $q = \alpha N_{K/F} \oplus \beta N_{K/F} \oplus \gamma N_{K/F}$ with $N_{K/F}$ positive definite and positive factors $\alpha, \beta, \gamma \in F$, then q is equivalent to $N_{K/F} \oplus N_{K/F} \oplus \lambda N_{K/F}$ with $\lambda := \alpha\beta\gamma$. Its equivalence class corresponds to the pair consisting of the extension K/F and the coset $\lambda N_{K/F}(K^\times)$, and its similarity class is represented by $N_{K/F} \oplus N_{K/F} \oplus N_{K/F}$.

Proof. Let $\mathbb{O}$ be the unique octonion field over F. Definiteness of q follows from the fact that q is similar to a subform of $N_{\mathbb{O}}$.

The orthogonal sum $q \oplus \lambda N_{K/F}$ is equivalent to $\alpha(N_{K/F} \oplus \alpha\beta N_{K/F} \oplus \alpha\gamma N_{K/F} \oplus \beta\gamma N_{K/F}) = \alpha N_C$, where the composition algebra $C := \mathbb{D}^{\alpha\gamma}(\mathbb{D}^{\alpha\beta}(K))$ is obtained by suitable doubling. As q is anisotropic, the algebra C is not split. Our uniqueness assumption gives $C \cong \mathbb{O}$, and $q \oplus \lambda N_{K/F}$ is equivalent to $\alpha N_{\mathbb{O}}$. From 5.4 we know $\alpha \in N_{\mathbb{O}}$, and infer that $\alpha N_{\mathbb{O}} =$ is equivalent to $N_{\mathbb{O}}$.

We abbreviate $s_\lambda := N_{K/F} \oplus N_{K/F} \oplus \lambda N_{K/F}$. The form $s_\lambda \oplus \lambda N_{K/F}$ is equivalent to $N_{K/F} \oplus N_{K/F} \oplus \lambda(N_{K/F} \oplus N_{K/F}) = N_D$, where $D := \mathbb{D}^\lambda(\mathbb{D}^1(K))$. As its norm is positive definite, the composition algebra D is not split, and $D \cong \mathbb{O}$ follows from ($\diamondsuit$). Now Witt's cancellation theorem yields the claimed equivalence of q and s_λ.

It remains to note that s_λ is similar to $\lambda s_\lambda = \lambda N_{K/F} \oplus \lambda N_{K/F} \oplus \lambda^2 N_{K/F}$,which is equivalent to s_{λ^4}, and thus equivalent to s_1. □

5.7 Theorem. *Assume that F is an ordered field with property $(\diamondsuit)$, let K/F be a separable quadratic extension, and let $d > 2$ be an integer. If $q\colon F^{2d} \to F$ is a quadratic form with norm splitting $q = \alpha_1 N_{K/F} \oplus \cdots \oplus \alpha_{d-1} N_{K/F} \oplus \alpha_d N_{K/F}$ then q is either positive or negative definite.*

If q is positive definite then q also has a norm splitting $q = N_{K/F} \oplus \cdots \oplus N_{K/F} \oplus \lambda N_{K/F}$, where $\lambda = \prod_{j=1}^{d} \alpha_j$. If q is negative definite then $-q$ has such a norm splitting.

Proof. Replacing q by $-q$ if necessary, we may (and will) assume $\alpha_1 > 0$; this will put us into the positive definite case.

We proceed by induction on d. The case $d = 3$ is treated in 5.6. Now consider the case $d > 3$. Applying our induction hypothesis to the restriction of q to $F^{2d-2} \times \{0\}^2$, we may assume $\alpha_1 = 1 = \cdots = \alpha_{d-2}$. Now we apply 5.6 to the restriction of q to $\{0\}^{2d-6} \times F^6$. $\square$

5.8 Remarks. If F is an ordered field with property $(\diamondsuit)$ then 5.6 and 5.7 yield a complete description of forms of type E_6 and of type E_7: up to a change of sign, these are forms with norm splitting $N_{K/F} \oplus N_{K/F} \oplus \lambda N_{K/F}$ with positive λ, and $N_{K/F} \oplus N_{K/F} \oplus N_{K/F} \oplus \lambda N_{K/F}$ with positive $\lambda \in F \smallsetminus N_{K/F}(K)$, respectively. Recall from 2.3 and 2.4 that K/F is uniquely determined if the form is of type E_6, but the situation is different for forms of type E_7.

Under the present assumptions on F, there do not exist any forms of type E_8 because such a form would be a (positive or negative) definite one, so the scalars $\alpha_1, \ldots, \alpha_6$ involved in the norm splitting all have the same sign, and $-\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ will be negative while $N_{K/F}(K^\times)$ is contained in the set of positive elements of F.

5.9 Remark. Theorem 5.7 excludes forms in two or four variables. While norm splittings for forms in two variables are not really interesting (obviously), the case of four variables is completely different from the case studied in 5.7: From 3.1 (b) we know that a form in four variables has a norm splitting precisely if it is similar to the norm of a quaternion field, and the norm of the quaternion field determines the isomorphism type of the quaternion field (cf. [9, 1.7]). Note that there exist quaternion fields over $\mathbb{Q}$ with indefinite norm form; e.g., take $\mathbb{D}^{-3}(\mathbb{D}^1(\mathbb{Q}))$.

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