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Andrea Blunck, Norbert Knarr,
Bernhild Stroppel,
Markus J. Stroppel

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Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
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Abstract

We define (left and right) Clifford parallelisms on a seven-dimensional projective space algebraically, using an octonion division algebra. Thus, we generalize the two well-known Clifford parallelisms on a three-dimensional projective space, obtained from a quaternion division algebra. We determine (for both the octonion and quaternion case) the automorphism groups of these parallelisms. A geometric description of the parallel classes is given with the help of a hyperbolic quadric in a Baer superspace, obtained from the split octonion algebra over a quadratic extension of the ground field, again generalizing results that are known for the quaternion case.

In contrast to the quaternion case, the orbits of the two Clifford parallelisms under the group of direct similitudes of the norm form of the algebra are non-trivial in the octonion case. The two spaces of parallelisms can be seen as the point sets of two point-line geometries, both isomorphic to the seven-dimensional projective space. Together with the original space, we thus have three versions of this projective space. We introduce a triality between them which is closely related to the triality of the polar space of split type D_4.

Mathematics Subject Classification (MSC 2000): 51A15, 17A75, 51A05, 51A10, 51J15, 11E04.
Keywords: Clifford parallelism, octonion, quaternion, composition algebra, projective space, triality.

1 Introduction

Clifford's classical parallelisms ([7], see [8] for historical information and background) in the three-dimensional projective space $\mathbb{P}(\mathbb{R}^4)$ over the field $\mathbb{R}$ of real numbers can be described in various ways. Among others, there are descriptions using Hamilton's quaternions (i.e., the four-dimensional associative division algebra over the ground field $\mathbb{R}$) or using the two reguli on a hyperbolic quadric in $\mathbb{P}(\mathbb{C}^4)$ (see [15], [3]; and [6] for a generalization to arbitrary ground fields). We consider Clifford parallelisms in $\mathbb{P}(\mathbb{O})$, defined by an octonion division algebra $\mathbb{O}$ instead of quaternions. Wherever convenient, we formulate our results and proofs in such a way that the quaternion case is treated together with the octonion case. (Section 2 collects the facts about composition algebras that we need in the present paper.) Characteristic two is explicitly allowed. Note however, that octonion division algebras in characteristic two contain both quaternion subfields and four-dimensional subalgebras that are commutative. Clifford parallelisms on the latter are treated in [12], but they are beyond the scope of our present paper.

So we consider a non-split composition algebra C of dimension at least four over an arbitrary ground field F and the associated projective space $\mathbb{P}(C)$, which is a three-dimensional or a seven-dimensional projective space over F. Two lines in this space are called right (Clifford) parallel, if they can be written as Ku, Kw (with u, w non-zero elements of C) for some two-dimensional subspace K of C containing the element $1 \in C$ (which means that K is a quadratic field extension of F contained in C). The left (Clifford) parallelism is defined analogously.

In Section 4 we determine the automorphism group $\mathrm{Aut}(\mathbin{/\!/})$ of the right parallelism. It turns out that the stabilizer of the point $\mathbf{1} = F1$ in $\mathrm{Aut}(\mathbin{/\!/})$ is the group of all collineations induced by (not necessarily F-linear) automorphisms of C. Since the group generated by collineations induced by left multiplications with pure elements of C is a subgroup of $\mathrm{Aut}(\mathbin{/\!/})$ acting transitively on the point set, this yields a description of the whole group $\mathrm{Aut}(\mathbin{/\!/})$. A similar result holds for the automorphism group $\mathrm{Aut}(\mathbin{\backslash\!\backslash})$ of the left parallelism.

Since the parallel classes of $\mathbin{/\!/}$ and $\mathbin{\backslash\!\backslash}$ are regular spreads, it is clear from [4] that they can be described via (one-dimensional or three-dimensional) indicator spaces in appropriate Baer superspaces of $\mathbb{P}(C)$. In Section 5 we show the following. Let $E : F$ be a quadratic extension of F. Consider a left or right parallel class whose representative K through the point $\mathbf{1}$ is isomorphic to E as an extension of F. Then the algebra $C_E := E \otimes C$ splits, and so the norm of C_E gives rise to a hyperbolic quadric $\mathcal{Q}_E$ in the Baer superspace $\mathbb{P}(C_E)$ of $\mathbb{P}(C)$. On $\mathcal{Q}_E$ there are two families $\mathcal{M}^+$ and $\mathcal{M}^-$ of maximal totally isotropic subspaces. The indicator spaces of the given parallel class are in $\mathcal{M}^+$ if it is a right parallel class, otherwise in $\mathcal{M}^-$. See [6] for the case that C is a quaternion algebra.

Our definition of the parallelisms $\mathbin{/\!/}$ and $\mathbin{\backslash\!\backslash}$ makes special use of the element 1. In Section 6, we take an element $a \in C \smallsetminus \{0\}$ instead, and obtain parallelisms $\mathbin{/\!/}_a$ and $\mathbin{\backslash\!\backslash}_a$. If C is a quaternion algebra, each $\mathbin{/\!/}_a$ coincides with $\mathbin{/\!/}$, and each $\mathbin{\backslash\!\backslash}_a$ coincides with $\mathbin{\backslash\!\backslash}$. In the case that C is an octonion algebra $\mathbb{O}$, however, we have that $\mathbin{/\!/}_a = \mathbin{/\!/}_b$ (and $\mathbin{\backslash\!\backslash}_a = \mathbin{\backslash\!\backslash}_b$) holds exactly if $Fa = Fb$. So we have many different Clifford parallelisms in the octonion case. We show (in 6.14) that the set Π^+ of all $\mathbin{/\!/}_a$ is the orbit of $\mathbin{/\!/}$ under the action of the group of collineations induced by direct similitudes of $\mathbb{O}$ with respect to the norm (and similarly for the set Π^- of all $\mathbin{\backslash\!\backslash}_a$). Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sets of all parallel classes of all parallelisms in Π^+ (or Π^-, respectively). In Section 7 we prove that the incidence geometries $(\Pi^+, \mathcal{C}^+, \ni)$ and $(\Pi^-, \mathcal{C}^-, \ni)$ are isomorphic to the projective space $\mathbb{P}(\mathbb{O})$ (seen as a point-line incidence geometry).

In Section 8 we study these three projective spaces and a triality linking them. There is an associated action of the autotopism group of $\mathbb{O}$. If $E : F$ is a quadratic field extension such that $\mathbb{O}_E = E \otimes \mathbb{O}$ splits, then both the triality and the group action can be extended to $\mathcal{M}^+ \times \mathcal{Q}_E \times \mathcal{M}^-$, where $\mathcal{Q}_E$ is the hyperbolic quadric in $\mathbb{P}(\mathbb{O}_E)$ mentioned above and $\mathcal{M}^+, \mathcal{M}^-$ are the two families of totally isotropic subspaces contained in $\mathcal{Q}_E$. Thus, we get a connection to the polar space of split type D_4 defined on $\mathcal{Q}_E$ and the associated classical triality (see [26, §2]).

Finally, Section 9 contains remarks on older literature. We also correct an over-enthusiastic generalization, giving a characterization of composition algebras containing only one isomorphism type of two-dimensional subalgebras (see 9.1).

2 Composition algebras

Octonion algebras are special cases of composition algebras, obtained by a doubling process (cf. 2.2.(i) below) leading from separable quadratic field extensions to quaternion algebras and then to octonion algebras. We give a precise definition, and collect some of the crucial properties:

2.1 Definition. Let F be a commutative field. A *composition algebra* over F is a vector space C over F with a bilinear multiplication (written as xy) and a quadratic form $N := N_C \colon C \to F$ which is multiplicative (i.e., $N(xy) = N(x)N(y)$ holds for all $x, y \in C$) and whose polar form is not degenerate. We also assume that the algebra contains a neutral element for its multiplication, denoted by 1.

The composition algebra is called *split* if it contains divisors of zero. We recall that composition algebras occur only with dimension $d \in \{1, 2, 4, 8\}$; see [23, 1.6.2]. If $d = 4$ we call the algebra a *quaternion algebra*, such an algebra is a skew field if it is non-split; it *is* then a *quaternion field* ([20]). Composition algebras of dimension 8 are called *octonion algebras*; a non-split octonion algebra is also called an *octonion field*.

As usual, the ground field F is embedded as $F1$ in C. The polar form will be written as $(x|y) := N(x + y) - N(x) - N(y)$.

The first chapter of [23] gives a comprehensive introduction into composition algebras over arbitrary fields, including the characteristic two case.

We collect the basic facts that we need in the present paper (for proofs, consult [23]):

2.2 Properties of composition algebras. *Let C be a composition algebra over F.*

- **(a)** *The map $\kappa \colon C \to C \colon x \mapsto \overline{x} := (x|1)1 - x$ is an involutory anti-automorphism, called the* standard involution *of C. (This is the reflection at $F1$ if $\operatorname{char} F \neq 2$, and the orthogonal transvection with center 1 if $\operatorname{char} F = 2$.)*
- **(b)** *The norm and its polar form can be recovered from the standard involution as $N_C(x) = x\overline{x} = \overline{x}x$ and $(x|y) = x\overline{y} + y\overline{x}$. In particular, we have the hyperplane $\operatorname{Pu} C := 1^\perp = \{x \in C \mid \overline{x} = -x\}$ of* pure elements.
- **(c)** *In general, the multiplication is not associative, but weak versions of associativity are still there; among them* Moufang's identities [23, 1.4.1]

$$(ax)(ya) = a((xy)a), \quad a(x(ay)) = (a(xa))y, \quad x(a(ya)) = ((xa)y)a.$$

- **(d)** *Artin's Theorem* (see [23, Prop. 1.5.2])*: For any two elements $x, y \in C$, the subalgebra generated by x and y in C is associative.*
- **(e)** *An element $a \in C$ is invertible if, and only if, its norm is not zero; we have $a^{-1} = N_C(a)^{-1}\,\overline{a}$ in that case. Thus a non-split composition algebra is a division algebra, each element of $C^* := C \smallsetminus \{0\}$ is then invertible. Note that Artin's Theorem then implies $a^{-1}(ax) = x = a(a^{-1}x) = (xa)a^{-1} = (xa^{-1})a$, for each $x \in C$.*
- **(f)** *Each element $a \in C$ is a root of a polynomial of degree 2 over F, namely, the polynomial $X^2 - (a + \overline{a})X + N_C(a) \in F[X]$. We call $T_C(a) := a + \overline{a}$ the* trace *of a in C.*

(g) *For each $a \in C$ the algebra generated by a is $F(a) = F + Fa$, and this algebra is associative and commutative. For $x, y \in F(a)$ and $v \in C$ we have $x(yv) = (xy)v$, and C is a left module over $F(a)$. In particular, if the restriction of the norm to $F(a)$ is anisotropic then $F(a)$ is a commutative field, and C is a (left) vector space over $F(a)$.*
Similarly, we may consider C as a right module over $F(a)$.

(h) [18, 1.3] *Every F-semilinear automorphism of C commutes with the standard involution. If $\dim_F C \geqslant 4$ then every $\mathbb{Z}$-linear automorphism of C is F-semilinear. Consequently, every $\mathbb{Z}$-linear automorphism of such a C is a semi-similitude of the norm form, and every F-linear automorphism is an orthogonal map. We write $\operatorname{Aut}(C) = \operatorname{Aut}_{\mathbb{Z}}(C)$ for the group of all $\mathbb{Z}$-linear automorphisms, and $\operatorname{Aut}_F(C)$ for the group of all F-linear automorphisms.*

(i) [23, 1.5.3] *If D is a subalgebra of C with $\dim_F C = 2\dim_F D$ and such that $D^\perp \cap D = \{0\}$ then $D^\perp = Dw$ holds for each $w \in D^\perp$ with $N_C(w) \neq 0$, and the multiplication in $C = D \oplus D^\perp$ is given by $(x + yw)(u + vw) = (xu - N_C(w)\overline{v}y) + (vx + y\overline{u})w$.*

2.3 Lemma. *Let $\mathbb{O}$ be an octonion division algebra over F, and let S be an F-subalgebra. Then the following hold.*

(a) *If the restriction of the polar form of the norm form to S is non-zero then it is not degenerate, and S is a composition algebra.*

(b) *If the restriction of the polar form to S is zero then S is a commutative field, in fact, it is a totally inseparable extension of degree $\dim_F S \in \{1, 2, 4\}$.*

(c) *In any case, we have $\dim_F S \in \{1, 2, 4, 8\}$.*

Proof. Assertion (a) has been proved in [17, 1.4]. Now assume that the polar form is trivial on S. Then $1 \in S \subseteq 1^\perp$ implies $\operatorname{char} F = 2$, and $1^\perp = \operatorname{Fix}(\kappa)$. Non-degeneracy of the polar form on $\mathbb{O}$ implies that its Witt index (and thus also $\dim S$) is bounded by $\frac{1}{2}\dim\mathbb{O} = 4$. For $x, y \in S$ we obtain $0 = (x|y) = x\overline{y} + y\overline{x} = xy + yx$ and then $yx = -xy = xy$. Thus S is commutative (and associative by [21, 6.1.6]). Moreover, we have $a^2 = N(a) \in F$ for each $a \in S$. Thus S is a totally inseparable extension of F, and $\dim_F S$ is a power of 2 because S is obtained by a series of quadratic extensions.

The last assertion follows from the fact that composition algebras only occur in dimensions 1, 2, 4, and 8, cf. [23, Thm. 1.6.2]. □

2.4 Remark. The results in 2.3 heavily depend on the fact that our algebra has no divisors of zero; indeed there are subalgebras of dimensions 5 and 6 in *split* octonion algebras, and some of the six-dimensional ones even occur as fixed point sets of involutory automorphisms (see [10, 4.11]).

3 The Clifford parallelisms defined by quaternions or octonions

Consider an arbitrary projective space. A set $\mathcal{S}$ of lines is called a *(line) spread* if each point lies on exactly one line of $\mathcal{S}$. An equivalence relation on the set of lines of the space is called a *parallelism* if each equivalence class (called *parallel class*) is a spread, i.e., if through each point there is exactly one line of each parallel class.

In a three-dimensional pappian projective space, any three pairwise skew lines belong to a unique *regulus* (one of the two maximal sets of pairwise skew lines on a hyperbolic quadric). A spread $\mathcal{S}$ of a three-dimensional pappian projective space is called *regular* if with any three lines also the entire regulus through these lines belongs to $\mathcal{S}$. A spread of a pappian projective space of dimension greater than three is called *regular* if its intersection with the line space of the span of any two of its elements is a regular spread. A parallelism is called *regular* if all its parallel classes are regular spreads.

Recall that a spread of a pappian projective space of dimension 3 is regular if, and only if, the translation plane defined by that spread is pappian, and that these two conditions are equivalent to the existence of quadratic extension of the ground field such that the members of the spread are the one-dimensional subspaces over the extension field. See [16, Ch. 4, Prop. 4.13, p. 57].

The $(d-1)$-dimensional projective space with homogeneous coordinates from a vector space V of dimension d over a field F will be denoted as $\mathbb{P}(V) = \mathbb{P}_F(V) \cong \mathbb{P}(F^d)$, and $\mathcal{L}$ always denotes the set of lines. We will concentrate on pappian projective spaces; i.e. the field F will be commutative throughout.

Let C be a non-split composition algebra of dimension at least 4 (i.e., a quaternion or octonion *division* algebra) over the field F. The existence of such a division algebra over F imposes serious restrictions. For instance, the field F cannot be finite, it cannot be quadratically closed, and an octonion division algebra over F does not exist if F is a local field (i.e., a non-discrete locally compact Hausdorff field, like a p-adic field, for instance) unless $F \cong \mathbb{R}$. See [23, 1.10].

Consider the (three- or seven-dimensional) projective space $\mathbb{P}(C) \cong \mathbb{P}(F^{\dim C})$ as a point-line geometry; the point set consists of the one-dimensional subspaces and the line set consists of the two-dimensional subspaces of the vector space C. Let K be any line through the point $\mathbf{1}$. Then K is a commutative subfield of C, and C is a left vector space over K, see 2.2.(g).

If $\varphi\colon V \to W$ is an injective semilinear map we write $\mathbb{P}(\varphi)\colon \mathbb{P}(V) \to \mathbb{P}(W)$ for the induced map between projective spaces. For any $u \in C^*$ the maps $\lambda_u\colon C \to C\colon x \mapsto ux$ and $\rho_u\colon C \to C\colon x \mapsto xu$ are F-linear bijections and hence induce automorphisms $\mathbb{P}(\lambda_u)$ and $\mathbb{P}(\rho_u)$, respectively, of $\mathbb{P}(C)$. In particular, they map lines to lines. We write $\Gamma\mathrm{L}_F(C)$ for the group of all semi-linear bijections of the F-vector space C. Thus $\mathrm{P}\Gamma\mathrm{L}_F(C) = \mathbb{P}(\Gamma\mathrm{L}_F(C))$ is the full group of automorphisms of the projective space.

3.1 Definition. We consider two relations $/\!\!/$ and $\backslash\!\!\backslash$ on the set of lines, defined by

$$\begin{aligned} L \mathbin{/\!\!/} M &\iff \exists K \subseteq C\colon L, M \in \{Ku \mid u \in C^*\} \wedge 1 \in K \\ \text{and}\quad L \mathbin{\backslash\!\!\backslash} M &\iff \exists K \subseteq C\colon L, M \in \{uK \mid u \in C^*\} \wedge 1 \in K\,, \end{aligned}$$

respectively. Clearly, the sets K involved in the definition have to be two-dimensional vector subspaces of C. The relations $/\!\!/$ and $\backslash\!\!\backslash$ are called the *right* and the *left (Clifford) parallelism*, respectively. The following result is already contained in [28, (17)(c) and (10)], it shows that the two Clifford parallelisms are parallelisms, indeed.

3.2 Theorem. **(a)** *For each line L there exists a unique pair $(K, K^\#)$ of subfields in C such that L is a one-dimensional subspace of the left vector space C over K (see 2.2.(g)), and also a one-dimensional subspace of the right vector space over $K^\#$.*

(b) *Both $/\!\!/$ and $\backslash\!\!\backslash$ are equivalence relations, and their equivalence classes are spreads. In fact, the class of a line K through $\mathbf{1}$ with respect to $/\!\!/$ is the set of one-dimensional subspaces* *of the left vector space C over K, and the class of K with respect to $\mathbin{\backslash\!\backslash}$ is the set of one-dimensional subspaces of the right vector space over K.*

Proof. We only discuss the relation $/\!\!/$, arguments for $\mathbin{\backslash\!\backslash}$ are completely analogous.

Let L be an arbitrary line. Let $u \in L \setminus \{0\}$. Then $K := Lu^{-1}$ is a line through $\mathbf{1}$ and $L = Ku$, cf. 2.2.(e). Assume that $Ku = K'u'$. Then $u' \in Ku$, whence $Ku = Ku'$, the unique one-dimensional left K-subspace of C containing u'. So $Ku' = Ku = K'u'$, and multiplication by $(u')^{-1}$ from the right implies $K = K'$. Thus assertion (a) is established.

From $L /\!\!/ M /\!\!/ N$ we infer that L, M, N are one-dimensional subspaces of the left vector space over the field K obtained for L in assertion (a). This description of the relation $/\!\!/$ yields that $/\!\!/$ is an equivalence relation, with classes as claimed in assertion (b). □

3.3 Remark. The parallelisms $/\!\!/$ and $\mathbin{\backslash\!\backslash}$ are also considered in [14, Remark 1, p. 486].

4 The automorphism group of a Clifford parallelism

We want to determine the automorphism groups $\operatorname{Aut}(/\!\!/)$ and $\operatorname{Aut}(\mathbin{\backslash\!\backslash})$ of the right and left Clifford parallelisms, i.e. the groups of collineations of $\mathbb{P}(C)$ that leave $/\!\!/$ (or $\mathbin{\backslash\!\backslash}$, respectively) invariant. First we collect some examples.

4.1 Lemma. *Let C be a quaternion or octonion division algebra.*

(a) *Each $\alpha \in \operatorname{Aut}(C)$ induces an element $\mathbb{P}(\alpha)$ of $\operatorname{Aut}(\mathbin{\backslash\!\backslash}) \cap \operatorname{Aut}(/\!\!/)$.*
We obtain a subgroup $\mathrm{A} := \{\mathbb{P}(\alpha) \mid \alpha \in \operatorname{Aut}(C)\}$ in $\operatorname{Aut}(\mathbin{\backslash\!\backslash}) \cap \operatorname{Aut}(/\!\!/)$.

(b) *For each $a \in \operatorname{Pu} C \setminus \{0\}$ we have $\mathbb{P}(\lambda_a) \in \operatorname{Aut}(/\!\!/)$, and $\mathbb{P}(\rho_a) \in \operatorname{Aut}(\mathbin{\backslash\!\backslash})$.*
In fact, we have $\lambda_a(Ku) = (aKa^{-1})(au)$ for each line K through $\mathbf{1}$ and each $u \in C^*$.
The group Λ generated by left multiplications with nontrivial pure elements thus induces a subgroup[1] $\mathbb{P}(\Lambda)$ of $\operatorname{Aut}(/\!\!/)$.

Proof. Assertion (a) is clear from the definition of $\mathbin{\backslash\!\backslash}$ and $/\!\!/$; recall from 2.2.(h) that each element of $\operatorname{Aut}(C)$ is F-semilinear.

In order to prove assertion (b), we show that $\mathbb{P}(\lambda_a)$ maps the right parallel class of K to the parallel class of aKa^{-1}: For $u \in C^*$ we have $\lambda_a(Ku) = a(Ku) = (aKa^{-1})(au)$, where the last equality can be shown as follows: Since $a \in \operatorname{Pu} C$, we have $a^{-1} = N_C(a)^{-1}\bar{a} = -N_C(a)^{-1}a$. Using 2.2.(c), (d), and (e) we find $(aka^{-1})(au) = -N_C(a)^{-1}(aka)(au) = -N_C(a)^{-1}(a(k(a(au)))) = a(k(a^{-1}(au))) = a(ku)$. □

4.2 Remark. We note a new phenomenon if C is an octonion division algebra rather than a quaternion field: In general, $\mathbb{P}(\rho_a)$ is not in $\operatorname{Aut}(/\!\!/)$, even if $a \in \operatorname{Pu} C$. In fact, $\rho_a \in \operatorname{Aut}(/\!\!/)$ implies that $K(ca) = (Kc)a$ holds for each $c \in C^*$. If $C = \mathbb{O}$ is an octonion algebra, that equality is not true in general. For example, pick $c \in \mathbb{O} \setminus F$ with $c + \bar{c} = 1$ and $b \in \{1, c\}^\perp \setminus \{0\}$. Then $L := F + Fc$ is a separable extension, and $H := L + Lb$ is a quaternion field. Put $K := F + Fb$, and pick $a \in H^\perp \setminus \{0\}$. Then the multiplication formula for $\mathbb{O}$ (obtained as $\mathbb{O} = H + Ha$ by doubling, see 2.2.(i)) yields $(Kc)a \cap K(ca) = F(ca)$, and $(Kc)a \neq K(ca)$ follows.

[1] It turns out that, in a roundabout way, the group $\mathbb{P}(\Lambda)$ is isomorphic to $\mathrm{SO}(\operatorname{Pu} C, N|_{\operatorname{Pu} C})$. See [5, 5.2(f), 6.1].

In fact, the image of $\mathbin{/\!\!/}$ under ρ_a is another parallelism, it turns out that there is a large orbit of such parallelism if C is an octonion algebra, and that these parallelisms form an interesting and enlightening geometry. See Sections 6 and 7 below.

If C is a quaternion field then associativity of the multiplication yields that the conjugate $\mathbb{P}(\kappa\circ\Lambda\circ\kappa) = \left\langle \mathbb{P}(\rho_a) \,\middle|\, 0 \neq a \in \operatorname{Pu} C \right\rangle$ of $\mathbb{P}(\Lambda)$ also belongs to $\operatorname{Aut}(\mathbin{/\!\!/})$.

We obtain that the two subgroups A and Λ found in 4.1 make up all of $\operatorname{Aut}(\mathbin{/\!\!/})$:

4.3 Theorem. *The stabilizer of* $\mathbf{1}$ *in* $\operatorname{Aut}(\mathbin{/\!\!/})$ *is* A*, and this is also the stabilizer of* $\mathbf{1}$ *in* $\operatorname{Aut}(\mathbin{\backslash\!\!\backslash})$*. Therefore, we have* $\operatorname{Aut}(\mathbin{/\!\!/}) = \mathbb{P}(\Lambda)\circ\mathrm{A}$ *and* $\operatorname{Aut}(\mathbin{\backslash\!\!\backslash}) = \mathbb{P}(\kappa\circ\Lambda\circ\kappa)\circ\mathrm{A}$.

Proof. The group $\mathbb{P}(\Lambda)$ is contained in $\operatorname{Aut}(\mathbin{/\!\!/})$ by 4.1.(b), and it is transitive on the points of $\mathbb{P}(C)$ by [5, 1.4]. So $\operatorname{Aut}(\mathbin{/\!\!/})$ is transitive on the set of points, and the full group is the product of the stabilizer $\operatorname{Aut}(\mathbin{/\!\!/})_{\mathbf{1}}$ with the transitive subgroup: $\operatorname{Aut}(\mathbin{/\!\!/}) = \mathbb{P}(\Lambda)\circ\operatorname{Aut}(\mathbin{/\!\!/})_{\mathbf{1}}$. In 4.1.(a) we have also seen that $\mathrm{A} = \mathbb{P}(\operatorname{Aut}(C))$ is contained in the stabilizer $\operatorname{Aut}(\mathbin{/\!\!/})_{\mathbf{1}}$ of the point $\mathbf{1}$ in $\operatorname{Aut}(\mathbin{/\!\!/})$. It remains to show $\operatorname{Aut}(\mathbin{/\!\!/})_{\mathbf{1}} \leqslant \mathrm{A}$.

Consider $\alpha \in \Gamma\mathrm{L}_F(C)$ with $\mathbb{P}(\alpha) \in \operatorname{Aut}(\mathbin{/\!\!/})$, and assume that $\mathbb{P}(\alpha)$ fixes $\mathbf{1}$. Without loss of generality, we may then assume $\alpha(1) = 1$.

Let K be a line through $\mathbf{1}$. The image $K' := \alpha(K)$ is also a line through $\mathbf{1}$, and thus another quadratic field extension. By 3.2.(b), α maps one-dimensional K-subspaces to one-dimensional K'-subspaces. As α is additive, it induces an isomorphism from C considered as an affine space over K onto C considered as an affine space over K'. By the Fundamental Theorem of Affine Geometry (e.g., see [2, 2.6.3]), we have that α is a K-K'-semilinear map, i.e., there is a field isomorphism $\varphi_K\colon K \to K'$ such that $\alpha(xy) = \varphi_K(x)\alpha(y)$ holds for all $x \in K$ and all $y \in C$. The companion φ_K coincides with the restriction of α because $\alpha(1) = 1$.

For any $x, y \in C$ we choose a line K through $\mathbf{1}$ and Fx. Then $\alpha(xy) = \varphi_K(x)\alpha(y) = \alpha(x)\alpha(y)$, and we see that α is multiplicative. Thus we have proved $\operatorname{Aut}(\mathbin{/\!\!/})_{\mathbf{1}} = \mathrm{A}$, and $\operatorname{Aut}(\mathbin{/\!\!/}) = \mathbb{P}(\Lambda)\circ\mathrm{A}$.

The anti-automorphism κ induces a collineation $\mathbb{P}(\kappa)$ centralizing A and interchanging the parallelism $\mathbin{/\!\!/}$ with $\mathbin{\backslash\!\!\backslash}$. This yields our statement about $\operatorname{Aut}(\mathbin{\backslash\!\!\backslash})$. □

If C is a quaternion field, then $\operatorname{Aut}_F(C)$ consists of inner automorphisms (by the Skolem-Noether Theorem, see [1, Cor. 7.2D] or [13, § 4.6, Cor. to Th. 4.9]). This means $\operatorname{Aut}_F(C) \leqslant \Lambda\circ\kappa\circ\Lambda\circ\kappa$. Note that $\Lambda\circ\kappa\circ\Lambda\circ\kappa = \left\{ \lambda_a\circ\rho_b \,\middle|\, a, b \in C^* \right\}$ holds if C is associative. If C is a quaternion field, we have $\mathbb{P}(\Lambda) \cong (C^*/F^*)^2 \cong \mathrm{PGO}^+(C, N)$, see [5, 5.2(e)].

For the octonion case, the group Λ is studied in detail in [5]. We prove in that paper: $\operatorname{Aut}_F(\mathbb{O}) \leqslant \Lambda$, see [5, 5.7], and $\mathbb{P}(\Lambda) \cong \mathrm{SO}(\operatorname{Pu}\mathbb{O}, N|_{\operatorname{Pu}\mathbb{O}})$, see [5, 6.1]. Therefore, we obtain:

4.4 Theorem. *Let* $\operatorname{Aut}_F(\mathbin{/\!\!/}) := \operatorname{Aut}(\mathbin{/\!\!/}) \cap \mathrm{PGL}_F(C)$ *denote the group of all automorphisms of the parallelism* $\mathbin{/\!\!/}$ *that are induced by* F*-linear maps.*

- **(a)** *If* C *is a quaternion field then* $\operatorname{Aut}_F(\mathbin{/\!\!/}) = \mathbb{P}(\Lambda\circ\kappa\circ\Lambda\circ\kappa) = \operatorname{Aut}_F(\mathbin{\backslash\!\!\backslash}) \cong (C^*/F^*)^2 \cong \mathrm{PGO}^+(C, N)$.
- **(b)** *If* C *is an octonion division algebra then* $\operatorname{Aut}_F(\mathbin{/\!\!/}) = \mathbb{P}(\Lambda) \cong \mathrm{SO}(\operatorname{Pu}\mathbb{O}, N|_{\operatorname{Pu}\mathbb{O}})$. □

4.5 Corollary. *If* $\operatorname{Aut}(F)$ *is trivial then* $\operatorname{Aut}(\mathbin{/\!\!/}) = \operatorname{Aut}_F(\mathbin{/\!\!/})$*. In particular, if* C *is the standard quaternion field over* $F = \mathbb{R}$ *then* $\operatorname{Aut}(\mathbin{/\!\!/}) = \operatorname{Aut}(\mathbin{\backslash\!\!\backslash}) \cong \mathrm{PGO}^+(\mathbb{R}^4) = \mathrm{PSO}_4(\mathbb{R})$*, and we have* $\operatorname{Aut}(\mathbin{/\!\!/}) \cong \mathrm{SO}_7(\mathbb{R})$ *if* C *is the (standard) octonion division algebra over* $\mathbb{R}$. □

See [3, Sect. 9] for an alternative approach to 4.5 for the special case where C is the quaternion field over $F = \mathbb{R}$.

5 Geometric description

The aim of this section is to find indicator spaces for all parallel classes of our Clifford parallelisms. We shall make use of a description of regular spreads found in [4], but concentrate on the only case we need.

For any field extension $E : F$ and any algebra C over F, we consider the tensor product $C_E := E \otimes C$ over F as an algebra over E, with multiplication $(e \otimes x)(e' \otimes x') = (ee') \otimes (xx')$. If C is a composition algebra with standard involution $x \mapsto \overline{x}$ then C_E is also a composition algebra, with (E-linear) standard involution extending $e \otimes x \mapsto \overline{e \otimes x} = e \otimes \overline{x}$. The norm N_C of a composition algebra C over F extends naturally to the norm of $C_E = E \otimes C$: we have $N_{C_E}(e \otimes x) = e^2 N_C(x)$.

Now let C be a quaternion or octonion division algebra, and let $m := \frac{1}{2} \dim_F C \in \{2, 4\}$. Via $U \mapsto E \otimes U$ we embed $\mathbb{P}(C) = \mathbb{P}_F(C)$ as a subspace of $\mathbb{P}(C_E) = \mathbb{P}_E(C_E)$. If $E : F$ is a quadratic field extension, then $\mathbb{P}(C)$ is a *Baer subspace* of $\mathbb{P}(C_E)$, i.e., through each point p of $\mathbb{P}(C_E)$ that does not belong to $\mathbb{P}(C)$ there is a unique line of $\mathbb{P}(C)$ passing though p (the line *indicated* by p).

Let I be an $(m-1)$-dimensional projective subspace of $\mathbb{P}(C_E)$ that does not contain points of $\mathbb{P}(C)$. We define $\mathcal{S}(I)$ to be the set of all the lines indicated by the points of I.

5.1 Lemma ([4, 1.2]). *For each $(m-1)$-dimensional projective subspace I of $\mathbb{P}(C_E)$ that does not contain points of $\mathbb{P}(C)$, the set $\mathcal{S}(I)$ is a regular spread in $\mathbb{P}(C)$.* □

Following the ideas of [6], where the quaternion case was considered, we study $\mathbb{P}(C_E) \cong \mathbb{P}_E(E^{2d})$ as a Baer superspace of $\mathbb{P}(C) \cong \mathbb{P}_F(F^{2d})$.

5.2 Theorem ([4, Thm. 1.2]). *Let $\mathcal{S}$ be a regular spread in a $(2m-1)$-dimensional pappian projective space $\mathbb{P}(F^{2m})$. Then there is a quadratic extension E of F and an $(m-1)$-dimensional projective subspace I of $\mathbb{P}(E^{2m})$ such that I contains no point of $\mathbb{P}(F^{2m})$ and $\mathcal{S} = \mathcal{S}(I)$. The space I is called an* indicator space *of $\mathcal{S}$.*

If $E : F$ is separable, then there are exactly two indicator spaces I, I' of $\mathcal{S}$, and $I' = \beta(I)$, where the Baer involution β is induced by the generator of the Galois group $\mathrm{Gal}(E : F)$ of $E : F$. If $E : F$ is inseparable, then there is exactly one indicator space of $\mathcal{S}$. □

Note that the quadratic extension in 5.2 depends on the spread. In general, there will not be a universal Baer superspace $\mathbb{P}(E^{2m})$ providing indicator spaces for all spreads of the parallelisms simultaneously. See 9.1 below, and also [6] for the three-dimensional case (where $m = 2$).

Let again $E : F$ be a quadratic field extension. We study the geometric interpretation of the norm form N_{C_E} of C_E in more detail. If C_E is split then the norm form N_{C_E} is hyperbolic (i.e., it is not degenerate, and its Witt index is $\frac{1}{2} \dim_E(C_E) = \frac{1}{2} \dim_F C = m$). We are then interested in the associated quadric $\mathcal{Q}_E$ in $\mathbb{P}(C_E) \cong \mathbb{P}(E^{2m})$; this quadric is *hyperbolic*, i.e., it is not degenerate and contains projective $(m-1)$-spaces.

On the quadric $\mathcal{Q}_E$ there are two families $\mathcal{M}^+, \mathcal{M}^-$ of maximal totally isotropic subspaces. If $m = 2$ then $C_E \cong E^{2\times 2}$, and $N_{E^{2\times 2}}(x) = \det x$ (cf. [23, p. 19 f]). The two families form the two reguli on a hyperbolic quadric in projective 3-space. If $m = 4$, however, two elements of the same family have non-trivial intersection in general; this follows readily from the algebraic description given in the next paragraph.

As C_E is a split quaternion or octonion algebra, the members of the two families admit a nice algebraic description (see[2] [26, Thm. 3]): up to a change of names for the families (or an application of the standard involution), we have

$$\mathcal{M}^+ = \left\{aC_E \,\middle|\, a \in C_E \smallsetminus \{0\}, N_{C_E}(a) = 0\right\}, \text{ and } \mathcal{M}^- = \left\{C_E a \,\middle|\, a \in C_E \smallsetminus \{0\}, N_{C_E}(a) = 0\right\},$$

and $aC_E = bC_E \iff Ea = Eb \iff C_E a = C_E b$ if $N_{C_E}(a) = 0 = N_{C_E}(b)$; cf. [26, Thm. 4].

In the sequel, we assume that some quadratic field extension $E : F$ is chosen such that C_E splits. We consider the projective space $\mathbb{P}(C_E)$ and the Baer subspace $\mathbb{P}(C)$. Since the norm form N_C is anisotropic on C, the quadric $\mathcal{Q}_E$ has empty intersection with the Baer subspace $\mathbb{P}(C)$. So according to 5.1 each element of $\mathcal{M}^+ \cup \mathcal{M}^-$ indicates a regular spread in $\mathbb{P}(C)$.

If $E : F$ is a quadratic extension and C is a non-split composition algebra then C_E splits if, and only if, the algebra C contains an F-subalgebra isomorphic to E. We put this in the geometric context studied here:

5.3 Proposition. *Let $E : F$ be a quadratic field extension. Let L be any line in $\mathbb{P}(C)$, and let K be its right parallel passing through* $\mathbf{1}$. *Then the following hold.*

- **(a)** *For each $e \in E$ and each $y \in K$ we have $N_{C_E}(e \otimes 1 - 1 \otimes y) = e^2 - T_C(y)e + N_C(y)$.*
- **(b)** *The line $E \otimes K$ of $\mathbb{P}(C_E)$ meets $\mathcal{Q}_E$ in at least one point if, and only if, the algebra $E \otimes K$ contains divisors of zero. This happens precisely if the extensions $E : F$ and $K : F$ are isomorphic, i.e., if there exists an F-linear isomorphism from E onto K.*

  Another equivalent condition is that the restriction $N_K = N_C|_K$ is similar to the norm N_E of the field extension $E : F$.
- **(c)** *The line $E \otimes L$ meets the quadric $\mathcal{Q}_E$ if, and only if, the field extensions $E : F$ and $K : F$ are isomorphic as F-algebras. This happens precisely if $N|_L$ and N_E are similar.*
- **(d)** *Assume that $N|_L$ and N_E are similar. If Eq is a point of $(E \otimes K) \cap \mathcal{Q}_E$ and S denotes the $/\!\!/$-class of L then $S = S(I) \in \mathcal{M}^+$ for $I = qC_E$, and $q := e \otimes 1 - 1 \otimes \varphi(e)$ for some F-linear homomorphism $\varphi\colon E \to C$ of algebras mapping E onto L, and any $e \in E \smallsetminus F$.*
- **(e)** *Analogously, the left parallel class of L is indicated by $J := C_E(e \otimes 1 - 1 \otimes \varphi(e)) \in \mathcal{M}^-$.*

Proof. We compute $(e \otimes 1 - 1 \otimes y)(\overline{e \otimes 1 - 1 \otimes y}) = (e \otimes 1 - 1 \otimes y)(e \otimes \bar{1} - 1 \otimes \bar{y}) = e^2 \otimes 1 - e \otimes \bar{y} - e \otimes y + 1 \otimes y\bar{y} = \big(e^2 - T_K(y)e + N_K(y)\big) \otimes 1$ and thus verify assertion (a).

Note that $E \otimes K$ splits if, and only if, the line $E \otimes K$ meets $\mathcal{Q}_E$. Hence, it remains to show that the algebra $E \otimes K$ splits precisely if the extensions $E : F$ and $K : F$ are isomorphic. Choose $y \in K \smallsetminus F$; then 1 and y form a basis for K over F. Thus $1 \otimes 1$ and $1 \otimes y$ form a basis for $E \otimes K$ over E.

If $E \otimes K$ splits then there exist $e, d \in E$ such that $w := e \otimes 1 - d \otimes y$ is not trivial, but has norm 0. If $d = 0$ then $N_{C_E}(w) = N_{C_E}(e \otimes 1) = e^2$ gives $e = 0$, contradicting our choice of w. Replacing w by $d^{-1}w$ we may therefore assume $d = 1$. Now assertions (a) and 2.2.(f) yield that e and y have the same minimal polynomial over F, and the extensions $E : F$ and $K : F$ are isomorphic.

[2] For the case of a split quaternion algebra $C_E \cong E^{2\times 2}$, it is easy to verify that each $a \in E^{2\times 2} \smallsetminus (\mathrm{GL}_2(E) \cup \{0\})$ gives one-sided ideals $aE^{2\times 2}$ and $E^{2\times 2}a$ which are two-dimensional totally isotropic subspaces, and that every such subspace is such an ideal.

Conversely, assume that the extensions are isomorphic, and let $\varphi\colon E \to K$ be an F-linear multiplicative bijection. Then φ conjugates $\mathrm{Gal}(E:F)$ onto $\mathrm{Gal}(K:F)$. For the sake of clarity, we write the (possibly trivial) generator of the Galois group $\mathrm{Gal}(E:F)$ as $x \mapsto \tilde{x}$. We pick $e \in E \smallsetminus F$, and put $q := e \otimes 1 - 1 \otimes \varphi(e) \in C_E$. Then $\overline{\varphi(e)} = \varphi(\tilde{e})$ yields $N_{C_E}(q) = q\bar{q} = (e\otimes 1 - 1\otimes\varphi(e))(e\otimes 1 - 1\otimes\varphi(\tilde{e})) = e^2\otimes 1 - e\otimes T_K(\varphi(e)) + 1\otimes N_K(\varphi(e)) = \big(e^2 - T_E(e)e + N_E(e)\big)\otimes 1 = 0$. Thus $N_{C_E}(q) = 0$, the algebra $E\otimes K$ is split, and qC_E belongs to $\mathcal{M}^+$. The point Eq of qC_E lies on $E\otimes K$, and indicates K because $E\otimes K$ is the E-linear span of $1\otimes 1$ and $1\otimes\varphi(e)$.

Any other member of the parallel class $\mathcal{S}$ is of the form Ku with $u \in C^*$. Now Ku is embedded in the line $E\otimes Ku = (E\otimes K)(1\otimes u)$ which contains the vector $q(1\otimes u) \in qC_E$. This shows $\mathcal{S} \subseteq \mathcal{S}(qC_E)$. As a spread is never properly contained in another spread, we have equality.

Assertion (e) follows by an application of the standard involution on C_E. □

5.4 Proposition. *Let $E:F$ be a quadratic field extension such that C_E splits. Then $\mathrm{Aut}_F(/\!\!/)$ acts transitively on the quadric $\mathcal{Q}_E$, and also transitively on the set of all lines L in $\mathbb{P}(C)$ such that $N|_L$ is similar to N_E.*

Proof. Choose $w \in E \smallsetminus F$, and let $\varphi\colon E \to C$ be an F-linear homomorphism. We claim that $\mathcal{Q}_E$ is the orbit of $E(w\otimes 1 - 1\otimes\varphi(w))$.

Every point $Eq \in \mathcal{Q}_E$ lies on (the extension of) a line of the Baer subspace, and every such line is of the form $E(1\otimes x) + E(1\otimes y)$ with $x, y \in C^*$. As $\mathcal{Q}_E$ contains no points of the Baer subspace, we may assume that $q = e\otimes x + 1\otimes y$, with $e \in E\smallsetminus F$. Moreover, we may choose $x \in \mathrm{Pu}\,C \smallsetminus \{0\}$ because the Baer line meets the hyperplane $\mathbb{P}(\mathrm{Pu}\,C)$.

Applying $\mathbb{P}(\lambda_x^{-1}) \in \mathbb{P}(\Lambda) \leqslant \mathrm{Aut}_F(/\!\!/)$ we obtain the point $E(e\otimes 1 - 1\otimes v)$, with $v = -x^{-1}y$. As that point lies in $\mathcal{Q}_E$, we find that e and v have the same minimal polynomial over F, cf. 5.3.(a) and 2.2.(f). This means that there exists $\alpha \in \mathrm{Aut}_F(C) \leqslant \mathrm{Aut}_F(/\!\!/)$ with $\alpha(v) = \varphi(e)$. Applying the natural E-linear extension of α, we obtain $E(e\otimes 1 - 1\otimes\varphi(e))$ in the orbit of Eq under $\mathrm{Aut}_F(/\!\!/)$.

By our choice of $w \in E\smallsetminus F$, there exist $s, d \in F$ such that $e = sw + d$ with $s \neq 0$. Now we compute $E(e\otimes 1 - 1\otimes\varphi(e)) = E(w\otimes 1 - 1\otimes\varphi(w))$, and obtain transitivity of $\mathrm{Aut}_F(/\!\!/)$ on $\mathcal{Q}_E$. Transitivity on the given set of lines follows from the fact (see 5.3) that these are just those lines of the Baer subspace that meet the quadric $\mathcal{Q}_E$, and that the point on the quadric indicates the line in question. □

5.5 Remark. Proposition 5.3 says that each right (or left) parallel class whose representative through $\mathbf{1}$ is isomorphic (as an extension of F) to E is indicated by exactly two elements of $\mathcal{M}^+$ (or $\mathcal{M}^-$, respectively) in the case that $E:F$ is separable and by exactly one such element if $E:F$ is inseparable.

Since any two such right (left) parallel classes are disjoint, the indicator sets form a partial spread[3] $\mathcal{I}^+$ (or $\mathcal{I}^-$, respectively) of $\mathcal{Q}_E$. On the other hand, each point of $\mathcal{Q}_E$ indicates a line. From 5.3 we know that these lines are exactly those whose unique parallel through $\mathbf{1}$ is isomorphic to E. This means that $\mathcal{I}^+$ (or $\mathcal{I}^-$, respectively) is a spread of $\mathcal{Q}_E$.

[3] A partial spread of a quadric is a collection of mutually disjoint maximally totally isotropic subspaces. Such a collection is called a spread if it covers the whole quadric.

We are going to introduce a Baer subspace $\mathbb{P}(V^E)$ of $\mathbb{P}(C_E)$ and an ovoid $\mathcal{Q}^E$ in that Baer subspace next. Suitable versions of triality will show that this ovoid corresponds to the two systems $\mathcal{I}^+$ and $\mathcal{I}^-$ of indicator sets for the left or right parallelism, respectively. See also 8.2 below.

5.6 Lemma. *Let $E : F$ be a separable quadratic field extension such that C_E splits. Then*

$$V^E := \left\{ e \otimes 1 - 1 \otimes y \,\middle|\, e \in E, y \in C, T_E(e) = T_C(y) \right\}$$

is an F-subspace of F-dimension $2m$ in C_E, and the following hold.

- **(a)** *The projective space $\mathbb{P}(V^E)$ is a Baer subspace in $\mathbb{P}(C_E)$.*
- **(b)** *The restriction of N_{C_E} to V^E takes its values in F, and may be considered as a quadratic form (over F) on V^E.*
- **(c)** *The quadric $\mathcal{Q}^E$ induced on $\mathbb{P}(V^E)$ is an ovoid in $\mathbb{P}(V^E)$; i.e., every line of $\mathbb{P}(V^E)$ meets $\mathcal{Q}^E$ in at most two points.*
- **(d)** *Every maximal totally isotropic subspace in $\mathcal{Q}_E$ meets the projective space $\mathbb{P}(V^E)$ in a unique point. In other words, the quadric $\mathcal{Q}^E$ is an ovoid of the quadric $\mathcal{Q}_E$.*

Proof. Pick a hyperplane W in C such that $1 \notin W \neq \operatorname{Pu} C$. Then there exists $w_2 \in W$ with $T(w_2) = 1$. We find a basis $w_1 = 1, w_2, w_3, \ldots, w_{2m}$ for C where the elements $w_3, \ldots, w_{2m}$ form a basis for $W \cap \operatorname{Pu} C$. Now pick $p, u \in E^*$ with $T(p) = 0$ and $T(u) = 1$. Then $p \otimes 1$, $u \otimes 1 - 1 \otimes w_2, -1 \otimes w_3, \ldots, -1 \otimes w_{2m}$ form an F-basis for V^E and also an E-basis for C_E. Thus $\mathbb{P}(V^E)$ is a Baer subspace of $\mathbb{P}(C_E)$, and assertion (a) is established.

Consider an arbitrary element $w = e \otimes 1 - 1 \otimes x \in V^E$. From 5.3.(a) we know $N_{C_E}(w) = e^2 - eT_C(x) + N_C(x)$. Now $T_E(e) = T_C(x)$ yields $e^2 - eT_C(x) + N_C(x) = eT_E(e) - N_E(e) - eT_C(x) + N_C(x) = -N_E(e) + N_C(x) \in F$, and $N_{C_E}(w) \in F$ follows. Thus assertion (b) is proved.

The restriction of the norm form to the hyperplane $V^E \cap (1 \otimes C) = \left\{ 1 \otimes p \,\middle|\, T_C(p) = 0 \right\}$ in V^E is anisotropic. Thus the Witt index of the form on V^E is one, and the quadric $\mathcal{Q}^E$ in $\mathbb{P}(V^E)$ is an ovoid, as claimed in assertion (c).

We consider the F-subspace $U := E \otimes 1 + 1 \otimes C$. This subspace has dimension $2m + 1$, and its intersection with any maximal totally isotropic subspace M of C_E has dimension at least one because $\dim_F(M) = 2m$ and $\dim_F(C_E) = 4m$. This shows $\dim_F(M \cap U) \geqslant 1$.

We claim that every isotropic vector in U is actually contained in V^E. Consider $e \in E$ and $y \in C$ such that $w := e \otimes 1 - 1 \otimes y$ has norm 0. From 5.3.(a) we infer that e is a root of the polynomial $X^2 - T_C(y)X + N_C(y)$.

We distinguish two cases: If $e \in F$ then $w = 1 \otimes (e - y)$ with $e - y \in C$, and $0 = N_{C_E}(w) = N_C(e - y)$ yields $e - y = 0$ because N_C is anisotropic. Then $w = 0 \in V^E$. If $e \notin F$ then the minimal polynomial of e over F has degree two, and coincides with $X^2 - T_C(y)X + N_C(y)$. By 2.2.(f), this yields that e and y have the same norm and trace, and $w \in V^E$ follows, again. So $M \cap V^E = M \cap U$ and $\dim_F(M \cap V^E) \geqslant 1$. From assertion (c) we know $\dim_F(M \cap V^E) \leqslant 1$, and assertion (d) is proved. □

6 The set of all Clifford parallelisms: orbits under similitudes

The definition of parallelisms in 3.1 appears to depend on the choice of a point of reference (namely, the point $\mathbf{1}$). If we use an *octonion* algebra $\mathbb{O}$ for C, this is indeed serious, while associativity of the multiplication in a quaternion field can be used to see that this choice does not affect the resulting parallelism. In fact (as we shall see below), the group $\mathrm{PGO}(\mathbb{O}, N_{\mathbb{O}})$ does not normalize the set $\{\mathrm{Aut}_F(/\!\!/), \mathrm{Aut}_F(\backslash\!\!\backslash)\} = \{\mathbb{P}(\Lambda), \mathbb{P}(\kappa\circ\Lambda\circ\kappa)\}$ of stabilizers of the two parallelisms $/\!\!/$ and $\backslash\!\!\backslash$ defined in 3.1, see 4.3. This is in marked contrast to the situation for the Clifford parallelisms obtained from a quaternion field $\mathbb{H}$ playing the role of C: in that case, the group $\mathrm{GO}(\mathbb{H}, N_{\mathbb{H}})$ normalizes the set $\{\Lambda_{\mathbb{H}}, \kappa\circ\Lambda_{\mathbb{H}}\circ\kappa\}$, where $\Lambda_{\mathbb{H}} := \{\lambda_a \mid a \in \mathbb{H}^*\}$ coincides with our group $\Lambda = \langle \lambda_u \mid u \in \mathrm{Pu}\, C \smallsetminus \{0\}\rangle$ from 4.1.(b). The groups $\Lambda_{\mathbb{H}}$ and $\kappa\circ\Lambda_{\mathbb{H}}\circ\kappa$ centralize each other, and it is known[4] that their product is a subgroup of index 2 in $\mathrm{GO}(\mathbb{H}, N_{\mathbb{H}})$. The proof of these observations makes essential use of associativity in $\mathbb{H}$. Now take an octonion algebra $\mathbb{O}$ for C. While it is still true that the groups $\Lambda_{\mathbb{O}} := \langle \lambda_a \mid a \in \mathbb{O}^*\rangle$ and $\kappa\circ\Lambda_{\mathbb{O}}\circ\kappa$ normalize each other (cf. [5, 3.12]) and that $\mathrm{GO}^+(\mathbb{O}, N)$ is generated by $\Lambda \cup (\kappa\circ\Lambda\circ\kappa)$ (cf. [5, 3.13(e)]), our group Λ may be a *proper* subgroup of $\Lambda_{\mathbb{O}}$, and the union $\Lambda\cup(\kappa\circ\Lambda\circ\kappa)$ will not be invariant under conjugation in $\mathrm{GO}^+(\mathbb{O}, N)$. If we apply an element of $\mathrm{PGO}(\mathbb{O}, N_{\mathbb{O}})$ to $/\!\!/$, say, we will thus in general obtain a regular parallelism different from both $/\!\!/$ and $\backslash\!\!\backslash$.

6.1 Definitions. The orbit Π of $/\!\!/$ under the group $\mathrm{GO}(\mathbb{O}, N_{\mathbb{O}})$ of similitudes of the norm form is called the set of all *Clifford parallelisms* in $\mathbb{P}(\mathbb{O})$. In order to describe the elements of Π, we generalize our definitions of $/\!\!/$ and of $\backslash\!\!\backslash$, putting $/\!\!/_a := \big\{\{Lu \mid u \in \mathbb{O}^*\} \bigm| a \in L \in \mathcal{L}\big\}$ and ${}_a\backslash\!\!\backslash := \big\{\{uL \mid u \in \mathbb{O}^*\} \bigm| a \in L \in \mathcal{L}\big\}$. In fact, we have $/\!\!/ = /\!\!/_1$ and $\backslash\!\!\backslash = {}_1\backslash\!\!\backslash$.

6.2 Remarks. Clearly, $\kappa(/\!\!/) = \backslash\!\!\backslash$. We will verify that $/\!\!/_a$ and ${}_a\backslash\!\!\backslash$ are parallelisms; in fact they lie in the orbits of $/\!\!/$ and $\backslash\!\!\backslash$ under the group of direct similitudes, see 6.14 below. We use autotopisms to understand the action of that group on lines, parallel classes, and parallelisms.

One could also consider the images of $/\!\!/$ under the group $\Gamma\mathrm{O}(\mathbb{O}, N)$ of all *semi*-similitudes. However, that orbit is not larger than the orbit under $\mathrm{GO}(\mathbb{O}, N)$ because every companion automorphism $\varphi \in \mathrm{Aut}(F)$ occurring with a semi-similitude in $\mathrm{GO}(\mathbb{O}, N)$ can already be realized as the companion of an automorphism of $\mathbb{O}$ (cf. [23, 1.7.2]), which stabilizes the parallelism $/\!\!/$ by 4.1.(a).

6.3 Definition. For each line L in $\mathbb{P}(\mathbb{O})$ and each $a \in \mathbb{O}^*$ we denote the parallel classes by $[L]_{{}_a\backslash\!\!\backslash} := \{M \mid M\, {}_a\backslash\!\!\backslash L\}$ and by $[L]_{/\!\!/_a} := \{M \mid M /\!\!/_a L\}$. We also introduce names for special choices of classes, writing $L^{\backslash\!\!\backslash} := \{xL \mid x \in \mathbb{O}^*\}$ and $L^{/\!\!/} := \{Lx \mid x \in \mathbb{O}^*\}$. For each $a \in L\smallsetminus\{0\}$ we then have $L^{\backslash\!\!\backslash} = [L]_{{}_a\backslash\!\!\backslash}$ and $L^{/\!\!/} = [L]_{/\!\!/_a}$.

Clearly, every parallel class is of the form $L^{\backslash\!\!\backslash}$ or $L^{/\!\!/}$, respectively. We study the map $L \mapsto L^{/\!\!/}$ in 7.5 below.

6.4 Definition. Let C be any algebra. An *autotopism* of C is a triplet[5] $(\alpha|\beta|\gamma)$ of additive bijections of C such that $\beta(sx) = \gamma(s)\alpha(x)$ holds for all $s, x \in C$.

[4] In fact, the group Λ is a transitive subgroup of $\mathrm{GO}(\mathbb{H}, N_{\mathbb{H}})$, the conjugacy class κ^Λ of hyperplane reflections generates $\mathrm{O}(\mathbb{H}, N_{\mathbb{H}})$ (by the Cartan-Dieudonné Theorem, see [9, Prop. 8, p. 20, Prop. 14, p. 42, Prop. 17, p. 55], cf. [11, 14.16, p. 135]), and that group contains the stabilizer of 1 in $\mathrm{GO}(\mathbb{H}, N_{\mathbb{H}})$. A Frattini argument yields $\mathrm{GO}(\mathbb{H}, N_{\mathbb{H}}) = \langle\{\kappa\}\cup\Lambda\rangle$. Now $\Lambda\circ(\kappa\circ\Lambda\circ\kappa)$ is a normal subgroup, of index 2 in $\mathrm{GO}(\mathbb{H}, N_{\mathbb{H}})$.

[5] As a reminder for the reader, triplets that are autotopisms will be written as $(\alpha|\beta|\gamma)$ rather than (α, β, γ).

6.5 Examples. Each automorphism $\alpha \in \mathrm{Aut}(\mathbb{O})$ yields an autotopism $(\alpha|\alpha|\alpha)$.

For each $u \in \mathbb{O}^*$, the triplet $(\rho_u|\lambda_u \circ \rho_u|\lambda_u)$ is an autotopism of $\mathbb{O}$. In fact, we have $(\lambda_u \circ \rho_u)(sx) = u(sx)u = (us)(xu) = \lambda_u(s)\rho_u(x)$ by one of Moufang's identities.

Using Moufang's identities in the forms $(\bar{u}(s\bar{u}))(ux) = \bar{u}(s(\bar{u}(ux)))$ and $(su)(\bar{u}(x\bar{u})) = (((su)\bar{u})x)\bar{u}$, respectively, we also see that the two triplets $(N(u)^{-2}\lambda_u|\lambda_{\bar{u}}|N(u)\lambda_{\bar{u}} \circ \rho_{\bar{u}})$ and $(\lambda_{\bar{u}} \circ \rho_{\bar{u}}|N(u)^{-2}\rho_{\bar{u}}|N(u)^{-3}\rho_u)$ are autotopisms. See 8.1.(a) for a deeper understanding how these two autotopisms arise from the first one.

6.6 Remarks. If $C = \mathbb{O}$, then each autotopism is semilinear, see [18, 1.9] and 6.11 below. Every linear autotopism of $\mathbb{O}$ has a unique E-linear extension to $E \otimes \mathbb{O}$, for each field extension $E : F$. For semilinear autotopisms (even for automorphisms) it is not true in general that the companion extends to the extension field E.

Using the defining property of autotopisms, we obtain the following.

6.7 Lemma. *For every autotopism $(\alpha|\beta|\gamma)$ of $\mathbb{O}$ we have $\beta({}_x\!\!\setminus\!\!\setminus) = {}_{\alpha(x)}\!\!\setminus\!\!\setminus$ and $\beta(/\!/_x) = /\!/_{\gamma(x)}$. The action on the set of parallel classes is given by $\beta(L^{\setminus\!\setminus}) = (\alpha(L))^{\setminus\!\setminus}$ and $\beta(L^{/\!/}) = (\gamma(L))^{/\!/}$; here we use the special representatives introduced in 6.3.* □

6.8 Definition. The subgroup generated by $\{\lambda_u \mid u \in \mathbb{O}^*\} \cup \{\rho_u \mid u \in \mathbb{O}^*\} \subset \mathrm{GO}(\mathbb{O}, N_{\mathbb{O}})$ is called the group of *direct similitudes* of the norm form, and denoted by $\mathrm{GO}^+(\mathbb{O}, N_{\mathbb{O}})$.

6.9 Remarks. From [5, 3.11] we infer $\mathrm{GO}(\mathbb{O}, N_{\mathbb{O}}) = \big\langle \{\kappa\} \cup \mathrm{GO}^+(\mathbb{O}, N_{\mathbb{O}}) \big\rangle$. As conjugation by κ interchanges multiplications from the left with multiplications from the right, the group $\mathrm{GO}^+(\mathbb{O}, N_{\mathbb{O}})$ is normalized by κ, and we obtain a semi-direct product $\mathrm{GO}(\mathbb{O}, N_{\mathbb{O}}) = \langle\kappa\rangle \ltimes \mathrm{GO}^+(\mathbb{O}, N_{\mathbb{O}})$. Similarly, we know from [5, 3.13.(d)] that the stabilizer of the vector 1 is $\mathrm{GO}(\mathbb{O}, N_{\mathbb{O}})_1 = \langle\kappa\rangle \ltimes \mathrm{GO}^+(\mathbb{O}, N_{\mathbb{O}})_1$.

The intersection $\mathrm{O}^+(\mathbb{O}, N_{\mathbb{O}}) = \mathrm{GO}^+(\mathbb{O}, N_{\mathbb{O}}) \cap \mathrm{O}(\mathbb{O}, N_{\mathbb{O}})$ is the kernel of the Dickson invariant, cf. [5, 3.13.(a)]. So $\mathrm{O}^+(\mathbb{O}, N_{\mathbb{O}}) = \mathrm{SO}(\mathbb{O}, N_{\mathbb{O}})$ if $\operatorname{char} F \neq 2$ but $\mathrm{O}^+(\mathbb{O}, N_{\mathbb{O}}) < \mathrm{SO}(\mathbb{O}, N_{\mathbb{O}}) = \mathrm{O}(\mathbb{O}, N_{\mathbb{O}})$ if $\operatorname{char} F = 2$. In any case, $\mathrm{O}^+(\mathbb{O}, N_{\mathbb{O}})$ has index 2 in $\mathrm{O}(\mathbb{O}, N_{\mathbb{O}})$, and κ represents the coset $\mathrm{O}(\mathbb{O}, N_{\mathbb{O}}) \smallsetminus \mathrm{O}^+(\mathbb{O}, N_{\mathbb{O}})$.

6.10 Lemma. *Let C be a composition algebra over F, and assume $\dim C \geqslant 4$. Then every element of norm 0 is the sum of two invertible elements of the same norm.*

Proof. Consider $x \in C$ with $N(x) = 0$. As $0 = 1 + (-1)$ we may assume $x \neq 0$. There exists a quaternion subalgebra $H \leqslant C$ which contains x. This quaternion algebra is split, and thus isomorphic to $F^{2\times 2}$. A corresponding isomorphism carries x to a conjugate either of $s\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) = s\left(\begin{smallmatrix}1&1\\-1&0\end{smallmatrix}\right) + s\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ or of $s\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) = s\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) + s\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$; here s is a suitable scalar, and the two invertible summands have norm s^2 in any case. □

6.11 Lemma. *Let C be a composition algebra with $\dim C \geqslant 4$. Then every autotopism $(\alpha|\beta|\gamma)$ of C is built from semi-similitudes α, β, γ, and these three semi-linear maps have the same companion.*

Proof. Let $(\alpha|\beta|\gamma)$ be an autotopism of C. Then both $\alpha(1)$ and $\gamma(1)$ are invertible (see [19, 2.6.2]). Multiplying the given autotopism with the autotopism $(\rho_u|\lambda_u \circ \rho_u|\lambda_u)$ for $u = \gamma(1)^{-1}$, we obtain an autotopism $(\alpha'|\beta'|\gamma')$ with $\gamma'(1) = 1$. We multiply with up to two autotopisms of the form $(\lambda_v|\lambda_v| - N(v)^{-1}\lambda_v \circ \rho_v)$ with $v \in \mathrm{Pu}\, C$ and obtain a product $(\alpha''|\beta''|\gamma'')$ with $\alpha''(1) = 1$ and $\gamma''(1) = 1$. Now $\alpha'' = \beta'' = \gamma''$ is an automorphism of C. As the center F of C

is invariant under each automorphism, α'' is semilinear. From 2.2.(h) we know that α'' is a semi-similitude, and so are the components α, β, γ. Each one of these semi-linear maps has the same companion as α''. □

6.12 Remarks. If $\mathrm{Aut}(F)$ contains a non-trivial element α then the automorphism group of the two-dimensional composition algebra $F \times F$ contains elements that are not F-semilinear. For instance $(x, y) \mapsto (x, \alpha(y))$ maps $F(1, 1)$ to a subset which is not an F-subspace.

It also happens that a two-dimensional non-split composition algebra (i.e., a separable quadratic field extension $E : F$) admits automorphisms that are not F-linear. E.g., one knows (see [22, 14.15]) that $\mathrm{Aut}(\mathbb{C})$ contains conjugates κ^α of the standard involution, with fixed fields $\alpha(\mathbb{R})$ that are different from (though isomorphic to) $\mathbb{R}$.

6.13 Lemma. *Let $(\alpha|\beta|\gamma)$ be any autotopism of a composition algebra C with $\dim C \geqslant 4$. Then $\beta(Cx) = C\alpha(x)$ and $\alpha(\overline{x}C) = \overline{\gamma(x)}C$ holds for each $x \in C$.*

Proof. Both equations are trivial if $x = 0$ or if x is invertible. So assume $x \neq 0$ and $N(x) = 0$. In particular, we only consider the split case.

The definition of autotopism gives $\beta(Cx) = \gamma(C)\alpha(x) = C\alpha(x)$. We have to be more careful with the second equality because x is not invertible. By 6.10 there are $u, v \in C^*$ with $N(u) = N(v)$ such that $x = u + v$. The semilinear maps α, β, γ involved in the autotopism have the same companion φ, see 6.11. We define $s, t \in F$ by $\varphi(s) = N(\gamma(u))^{-1}$ and $t = \varphi(N(u)s)$. Consider $c \in C$. Using $N(u) = N(v)$ we compute $\alpha(\overline{x}c) = \alpha((\overline{u} + \overline{v})c) = \varphi(s)^{-1}\left(\alpha(s\overline{u}c) + \alpha(s\overline{v}c)\right) = \overline{\gamma(u)}\gamma(u)\alpha(\overline{u}sc) + \overline{\gamma(v)}\gamma(v)\alpha(\overline{v}sc) = \overline{\gamma(u)}\beta(u\overline{u}sc) + \overline{\gamma(v)}\beta(v\overline{v}sc) = \overline{\gamma(u+v)}\beta(N(u)sc) = \overline{\gamma(x)}t\beta(c)$ and then infer $\alpha(\overline{x}C) = \overline{\gamma(x)}C$ because $t\beta(C) = C$. □

6.14 Theorem. **(a)** *For each $a \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}$ and each $u \in \mathbb{O}^*$, we have $(\lambda_a \circ \rho_a^{-1})({/\!\!/}_u) = {/\!\!/}_{au}$.*

(b) *The stabilizer $\mathrm{GO}(\mathbb{O}, N_\mathbb{O})_1$ of 1 in $\mathrm{GO}(\mathbb{O}, N_\mathbb{O})$ acts transitively on Π, and the stabilizer $\mathrm{GO}^+(\mathbb{O}, N_\mathbb{O})_1$ acts transitively both on $\Pi^+ := \left\{{/\!\!/}_y \;\middle|\; y \in \mathbb{O}^*\right\}$ and on $\Pi^- := \left\{{}_{y}{\backslash\!\!\backslash} \;\middle|\; y \in \mathbb{O}^*\right\}$. In particular, every element of Π is a parallelism.*

(c) *We have $\Pi = \Pi^- \cup \Pi^+$.*

(d) *The stabilizer of the set Π^+ in $\mathrm{GO}(\mathbb{O}, N_\mathbb{O})$ is the group $\mathrm{GO}^+(\mathbb{O}, N_\mathbb{O})$.*

(e) *Let P be any member of Π, and let C be any parallel class in P. Then the stabilizer of P and C acts transitively on the set of lines in C.*

Proof. Let $a \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}$. We use the autotopism $(\rho_a|\lambda_a \circ \rho_a|\lambda_a)$, see 6.5. Since $a \in \mathrm{Pu}\,\mathbb{O}$ yields that $a^{-1} \in Fa$, the action described in 6.7 gives $(\lambda_a \circ \rho_a^{-1})({/\!\!/}_u) = (\lambda_a \circ \rho_a)({/\!\!/}_u) = {/\!\!/}_{au}$, as claimed in assertion (a).

We abbreviate $\sigma_a := \lambda_a \circ \rho_a^{-1} \circ \kappa$. Then $\sigma_a({/\!\!/}_u) = (\lambda_a \circ \rho_a^{-1})({}_{\overline{u}}{\backslash\!\!\backslash}) = {}_{\overline{u}a}{\backslash\!\!\backslash}$, see 6.7. For each $\gamma \in \mathrm{GO}(\mathbb{O}, N_\mathbb{O})_1$ there exists (cf. [5, 3.13.(d)]) some sequence $a_1, \dots, a_m$ in $\mathrm{Pu}\,\mathbb{O}$ such that $\gamma = \sigma_{a_m} \circ \cdots \circ \sigma_{a_2} \circ \sigma_{a_1}$. If $m = 2k$ is even, we obtain $\gamma({/\!\!/}_u) = {/\!\!/}_{\nu(u)}$ with $\nu(u) := (-1)^k a_m \cdots (a_2(a_1 u) \dots)$. If $m = 2k + 1$ is odd, we use the map ν just defined for the sequence $a_1, \dots, a_{2k} = a_{m-1}$ and find $\gamma({/\!\!/}_u) = \sigma_{a_m}({/\!\!/}_{\nu(u)}) = {}_{w}{\backslash\!\!\backslash}$ for $w = \overline{\nu(u)}\,a_m$. In any case, we have $\gamma({/\!\!/}_u) \in \left\{{/\!\!/}_y \;\middle|\; y \in \mathbb{O}^*\right\} \cup \left\{{}_{y}{\backslash\!\!\backslash} \;\middle|\; y \in \mathbb{O}^*\right\}$. As every element of $\mathbb{O}^*$ is a product of two pure elements (cf. [5, 1.2]), we find that $\left\{{/\!\!/}_y \;\middle|\; y \in \mathbb{O}^*\right\} \cup \left\{{}_{y}{\backslash\!\!\backslash} \;\middle|\; y \in \mathbb{O}^*\right\}$ is the orbit of ${/\!\!/} = {/\!\!/}_1$ under the stabilizer $\mathrm{GO}(\mathbb{O}, N_\mathbb{O})_1$. Analogously, we see that $\Psi := \mathrm{GO}^+(\mathbb{O}, N_\mathbb{O})_1 = \mathrm{SO}(\mathbb{O}, N_\mathbb{O})_1$ acts transitively on $\left\{{/\!\!/}_y \;\middle|\; y \in \mathbb{O}^*\right\}$. A Frattini argument shows $\mathrm{GO}(\mathbb{O}, N_\mathbb{O}) = \mathrm{GO}(\mathbb{O}, N_\mathbb{O})_1 \circ$

Λ. As $\Lambda \leqslant \mathrm{Aut}(\mathbin{/\!/})$ leaves $\mathbin{/\!/}$ invariant, we find $\Pi := \mathrm{GO}(\mathbb{O}, N_{\mathbb{O}})(\mathbin{/\!/}) = \mathrm{GO}(\mathbb{O}, N_{\mathbb{O}})_1(\mathbin{/\!/})$, as claimed. We have proved assertions (b) and (c).

Assertion (d) follows in a similar way, using $\mathrm{GO}^+(\mathbb{O}, N_{\mathbb{O}}) = \Psi \circ \Lambda$, transitivity of Ψ on the set $\Pi^+ = \{\mathbin{/\!/}_y \mid y \in \mathbb{O}^*\}$, and the fact that Λ fixes $\mathbin{/\!/}_1$.

It remains to study the stabilizer of some parallel class C in a member of Π. Using transitivity of $\mathrm{GO}(\mathbb{O}, N_{\mathbb{O}})$, we reduce this problem to the case where the parallelism is $\mathbin{/\!/}$. Then $C = K^{/\!/} = \{Kx \mid x \in \mathbb{O}^*\}$ with a line K through $\mathbf{1}$, i.e., a commutative subfield K of $\mathbb{O}$. For $a \in K^\perp \subseteq \mathrm{Pu}\,\mathbb{O}$ with $a \neq 0$ we observe $aK = Ka$. Using 4.1.(b) we infer $\lambda_a(Kx) = (aKa^{-1})(ax) = K(ax)$. Thus C is invariant under λ_a. For any $u \in \mathbb{O}^*$, we find (cf. [5, 1.2]), $a, b \in K^\perp$ such that $u = ab$. Now $\lambda_a \circ \lambda_b$ fixes both C and $\mathbin{/\!/}$ (cf. 4.1), and $(\lambda_a \circ \lambda_b)(K) = \lambda_a(Kb) = K(ab) = Ku$ shows that the stabilizer Λ_C of C and $\mathbin{/\!/}$ is indeed transitive on C. □

7 The space of all Clifford parallelisms

In this section, we interpret the set Π of all Clifford parallelisms (cf. 6.1) in terms of incidence geometries. The application of trialities (see Sec. 8 below) will shed further light on this.

7.1 Lemma. *For any $a, b \in \mathbb{O}^*$, the equalities $\mathbin{/\!/}_a = \mathbin{/\!/}_b$ and $Fa = Fb$ are equivalent.*

Proof. Quite obviously, we have $\mathbin{/\!/}_{sa} = \mathbin{/\!/}_a$ for each $s \in F^*$. It remains to show that $\mathbin{/\!/}_a = \mathbin{/\!/}_b$ implies $Fa = Fb$. Without loss, we may assume $b = 1$; cf. 6.14.(b). Let $\mathbb{H}$ be a quaternion subalgebra containing a (cf. [23, 1.6.4]), and pick $u, v \in \mathrm{Pu}\,\mathbb{H}$ such that $a = vu$ (this is possible by [5, 1.2]. Then $(\lambda_v \circ \rho_v^{-1})\left((\lambda_u \circ \rho_u^{-1})(\mathbin{/\!/}_1)\right) = (\lambda_v \circ \rho_v^{-1})(\mathbin{/\!/}_u) = \mathbin{/\!/}_{vu} = \mathbin{/\!/}_a = \mathbin{/\!/}_1$. In other words, $\alpha := \lambda_v \circ \rho_v^{-1} \circ \lambda_u \circ \rho_u^{-1}$ belongs to the stabilizer Λ of $\mathbin{/\!/}_1$, cf. 4.4. From $\alpha(1) = 1$ we then infer $\alpha \in \mathrm{Aut}_F(\mathbb{O})$, cf 4.3.

Now $\mathrm{Fix}(\alpha)$ contains the subspace $\{1, u, v\}^\perp$ of dimension at least 5, and generates $\mathbb{O}$ as an algebra. Thus $\alpha = \mathrm{id}$. The restriction of α to $\mathbb{H}$ is conjugation by a, and we find that a lies in the center F of $\mathbb{H}$. So $Fa = F = Fb$, as claimed. □

7.2 Lemma. *Consider $a, c \in \mathbb{O}^*$. If $Fa \neq Fc$ then there is exactly one parallel class belonging to both $\mathbin{/\!/}_a$ and $\mathbin{/\!/}_c$, namely, the class $[Fa + Fc]_{/\!/_a} = [Fa + Fc]_{/\!/_c} = (Fa + Fc)^{/\!/}$.*

Proof. Without loss, we may assume $c = 1$. We abbreviate $K := F + Fa$.

Since K contains both 1 and a, the classes $[K]_{/\!/_1} = \{Ky \mid y \in \mathbb{O}^*\} = [K]_{/\!/_a}$ coincide. Aiming at a contradiction, we assume that there is another line $L \neq K$ through $\mathbf{1}$ such that $[L]_{/\!/_1}$ is also a class in $\mathbin{/\!/}_a$. We pick some $b \in L \setminus F$, then $L = F + Fb$. The union $K \cup L$ spans a subalgebra $H := F + Fa + Fb + Fab$ in $\mathbb{O}$.

For each $y \in \mathbb{O}^*$, the line $(La)y$ is the (unique) $\mathbin{/\!/}_a$-parallel to L through $F(ay)$, and $L(ay)$ is the $\mathbin{/\!/}_1$-parallel to L through $F(ay)$. Therefore, we have $(La)y = L(ay)$ for each $y \in \mathbb{O}^*$. This implies that the vector $b(ay) \in L(ay) = (La)y$ is a linear combination of the vectors $ay = (1a)y$ and $(ba)y$. Equivalently, we have $(b(ay))\overline{y} \in Fa + F(ba) \leqslant H$.

There are two cases, cf. 2.3:

Case 1: The subalgebra H is a quaternion subfield. For each $w \in H^\perp \setminus \{0\}$ we obtain our octonion algebra as the double $\mathbb{O} = H + Hw$, see 2.2.(i). We choose $y := 1 + w$. Using the multiplication formula from 2.2.(i), we compute $(b(ay))\overline{y} = (b(a(1 + w)))\,(\overline{1 + w}) = (ba + b(aw))\,(1 - w) = (ba + (ab)w)\,(1 - w) = (ba + N(w)(ab)) + (-ba + ab)w$. Our condition

$(b(ay))\bar{y} \in Fa + F(ba) \leqslant H$ now yields $-ba + ab = 0$, contradicting the fact that H is not commutative.

Case 2: The subalgebra H is commutative. Then H is totally isotropic, and a totally inseparable field extension of degree 4 over F. In particular, we have $\operatorname{char} F = 2$, and $H = H^{\perp}$. Choose $y \in \mathbb{O} \smallsetminus H$ perpendicular to 1, a, and b. Then y is not perpendicular to ab but $ay \perp 1$.

Using $(b|ay) = b(ay) + (ay)b$ and one of Moufang' identities, we compute $(b(ay))\,\bar{y} = (b(ay))\,y = ((b|ay) + (ay)b)\,y = (b|ay)y + ((ay)b)\,y = (b|ay)y + a(y^2b) = (b|ay)y + y^2(ab)$. Now $y \notin H \ni ab$ yields that this linear combination of y and ab lies in H precisely if $(b|ay) = 0$. However, using the general property $(cx|y) = (x|\bar{c}y)$ (see [23, 1.3.2]) we obtain $(b|ay) = (\bar{a}b|y) = (ab|y)$, and the latter is nonzero by our choice of y. This contradiction yields the claim also in the inseparable case. □

7.3 Theorem. *Let K be a line through $\mathbf{1}$ in $\mathbb{P}(\mathbb{O})$. If $E \otimes K$ meets $\mathcal{Q}_E$ then the parallel class $[K]_{/\!\!/ a}$ is indicated by an element of $\mathcal{M}^+$, and the parallel class $[K]_{\backslash\!\!\backslash a}$ is indicated by an element of $\mathcal{M}^-$. Conversely, each element of $\mathcal{M}^+$ indicates a right parallel class, and each element of $\mathcal{M}^-$ indicates a left parallel class. Explicitly, the spread indicated by $b\mathbb{O}_E \in \mathcal{M}^+$ is the parallel class $L^{/\!\!/}$ for the unique line L indicated by Eb.*

Proof. Recall from 5.3 that $E \otimes K$ meets $\mathcal{Q}_E$ if, and only if, K is (as an F-algebra) isomorphic to E. The parallel class $[K]_{/\!\!/ a}$ is equal to $L^{/\!\!/} = \{Lu \mid u \in \mathbb{O}^*\}$, where $a \in L$ and $K = Lx$ for some $x \in \mathbb{O}^*$. Then $L = Kx^{-1}$. We know that $E \otimes K$ meets $\mathcal{Q}_E$ in some point Ec. Thus $E \otimes L$ meets $\mathcal{Q}_E$ in Eb, where $b = c(1 \otimes x^{-1})$, and Eb indicates L.

For an arbitrary line $Lv \in L^{/\!\!/}$ we get that the line $E \otimes (Lv) = (E \otimes L)(1 \otimes v)$ meets $\mathcal{Q}_E$ in $(Eb)(1 \otimes v) = E(b(1 \otimes v)) = b(E(1 \otimes v)) \subseteq b\mathbb{O}_E \in \mathcal{M}^+$. So each line of $L^{/\!\!/}$ is indicated by some element of $b\mathbb{O}_E \in \mathcal{M}^+$. As in the proof of 5.3 we see that $L^{/\!\!/}$ coincides with the spread indicated by the subspace $b\mathbb{O}_E \in \mathcal{M}^+$.

Conversely, each element of $\mathcal{M}^+$ has the form $b\mathbb{O}_E$ for some $b \in \mathbb{O}_E \smallsetminus \{0\}$ with $N_{\mathbb{O}_E}(b) = 0$. The computation above shows that the spread indicated by $b\mathbb{O}_E \in \mathcal{M}^+$ is the parallel class $L^{/\!\!/}$ for the unique line L indicated by Eb. □

7.4 Definitions. We denote the set of all lines of $\mathbb{P}(\mathbb{O})$ by $\mathcal{L}$. The set of all parallel classes (of all Clifford parallelisms) is denoted $\mathcal{C} := \mathcal{C}^- \cup \mathcal{C}^+$, where $\mathcal{C}^- := \{[L]_{a\backslash\!\!\backslash} \mid L \in \mathcal{L}, a \in \mathbb{O}^*\}$ and $\mathcal{C}^+ := \{[L]_{/\!\!/ a} \mid L \in \mathcal{L}, a \in \mathbb{O}^*\}$, respectively.

Note that the union $\mathcal{C}^- \cup \mathcal{C}^+$ is disjoint because the indicator sets for the parallel classes belong to different parts of $\mathcal{M}^+ \cup \mathcal{M}^-$; see 7.3.

7.5 Proposition. *The projective space $\mathbb{P}_F(\mathbb{O})$ (considered as the incidence geometry $(P, \mathcal{L}, <)$ with points and lines) is isomorphic to the incidence geometry $(\Pi^+, \mathcal{C}^+, \ni)$ and to $(\Pi^-, \mathcal{C}^-, \ni)$. The corresponding point maps are $\eta^+ \colon P \to \Pi^+ \colon Fx \mapsto /\!\!/_x$ and $\eta^- \colon P \to \Pi^- \colon Fx \mapsto {}_x\backslash\!\!\backslash$, the line maps are $\pi^+ \colon \mathcal{L} \to \mathcal{C}^+ \colon L \to L^{/\!\!/}$ and $\pi^- \colon \mathcal{L} \to \mathcal{C}^- \colon L \to L^{\backslash\!\!\backslash}$, respectively (cf. 6.3). In particular, these maps are bijections.*

Proof. Define $\eta^+(Fx) := /\!\!/_x$; this gives a bijection $\eta^+ \colon P \to \Pi^+$ by 7.1. For any line $L \in \mathcal{L}$, choose $x, y \in \mathbb{O}$ such that $L = Fx + Fy$. Then $/\!\!/_x \cap /\!\!/_y = \bigcap_{w \in L \smallsetminus \{0\}} /\!\!/_w$ contains a unique parallel class (namely, $[Fx + Fy]_{/\!\!/ x} = L^{/\!\!/}$, cf. 7.2). We denote this class by $\pi^+(L)$, and obtain a bijection $\pi^+ \colon \mathcal{L} \to \mathcal{C}^+$.

We have $Fx < L \iff x \in L \smallsetminus \{0\} \iff \pi^+(L) \in \eta^+(Fx)$. This means that $\eta = (\eta^+, \pi^+)$ is an isomorphism from $(P, \mathcal{L}, <)$ onto $(\Pi^+, \mathcal{C}^+, \ni)$. The proof for $(\Pi^-, \mathcal{C}^-, \ni)$ is obtained by applying κ. □

8 Triality

The famous principle of triality occurs in various guises: as a non-inner automorphism of order three on simple algebraic groups of split type D_4 (cf. [23, 3.3.2]), or as a non-type preserving automorphism of the polar space of split type D_4 (cf. [26, § 2]). The group version also has a non-split version (for instance, on the compact real form of the simple complex Lie group of type D_4).

We extend these notions of triality to groups of linear autotopisms (in the anisotropic case, that group is related to the group of direct similitudes, cf. 6.8 and [23, 3.2.1]), and give a new geometric variant of triality in the anisotropic case, in terms of the spaces of Clifford parallelisms that we have introduced.

The various notions of triality fit together quite neatly:

8.1 Theorem (Triality for autotopisms, parallelisms, and quadrics)**.**
Let $\mathbb{O}$ be an octonion field over some field F, and let $E : F$ be a quadratic field extension such that $\mathbb{O}_E := E \otimes \mathbb{O}$ splits. For any similitude $\xi \in \mathrm{GO}(\mathbb{O}, N)$, we denote the multiplier by μ_ξ.

(a) *For each autotopism $(\alpha|\beta|\gamma)$ of $\mathbb{O}$, the triplet $\tau(\alpha|\beta|\gamma) := (\gamma|\mu_\alpha^{-1}\kappa \circ \alpha \circ \kappa|\mu_\beta^{-1}\kappa \circ \beta \circ \kappa)$ is an autotopism, as well. The map τ is an automorphism of the group Δ of all autotopisms of $\mathbb{O}$, and has order three.*

(b) *Putting $(\alpha|\beta|\gamma) \,.\, ({}_x\!\backslash\!\!\backslash, Fy, /\!\!/_z) := \left({}_{\alpha(x)}\!\backslash\!\!\backslash, F\beta(y), /\!\!/_{\gamma(z)}\right)$ and $(\alpha|\beta|\gamma) \,.\, (K^{\backslash\!\backslash}, L, M^{/\!/}) := (\alpha(K)^{\backslash\!\backslash}, \beta(L), \gamma(M)^{/\!/})$ we obtain an action of the group Δ on the set $\Pi^- \times P \times \Pi^+$ and also on $\mathcal{C}^- \times \mathcal{L} \times \mathcal{C}^+$. Note that this is an action by triplets of collineations (of the projective spaces $(\Pi^-, \mathcal{C}^-)$, $(P, \mathcal{L})$, and $(\Pi^+, \mathcal{C}^+)$, respectively).*

(c) *The map $\nabla : \Pi^- \times P \times \Pi^+ \to \Pi^- \times P \times \Pi^+ : ({}_x\!\backslash\!\!\backslash, Fy, /\!\!/_z) \mapsto ({}_z\!\backslash\!\!\backslash, F\overline{x}, /\!\!/_{\overline{y}})$ is well-defined, and has order three. For each autotopism $(\alpha|\beta|\gamma)$ we have $\nabla\left((\alpha|\beta|\gamma) \,.\, ({}_x\!\backslash\!\!\backslash, Fy, /\!\!/_z)\right) = \tau(\alpha|\beta|\gamma) \,.\, \nabla({}_x\!\backslash\!\!\backslash, Fy, /\!\!/_z)$.*

(d) *We obtain a map $\tilde{\nabla} : \mathcal{C}^- \times \mathcal{L} \times \mathcal{C}^+ \to: \mathcal{C}^- \times \mathcal{L} \times \mathcal{C}^+ : (K^{\backslash\!\backslash}, L, M^{/\!/}) \mapsto (M^{\backslash\!\backslash}, \overline{K}, \overline{L}^{/\!/})$, and $\tilde{\nabla}\left((\alpha|\beta|\gamma) \,.\, (K^{\backslash\!\backslash}, L, M^{/\!/})\right) = \tau(\alpha|\beta|\gamma) \,.\, \tilde{\nabla}(K^{\backslash\!\backslash}, L, M^{/\!/})$.*

(e) *Every linear autotopism $(\alpha|\beta|\gamma)$ has a unique E-linear extension $(\alpha|\beta|\gamma)_E$ which is an autotopism of the composition algebra $\mathbb{O}_E$, and the restriction of the triality map τ to the group of linear autotopisms extends to an automorphism τ_E of order three on the group of all linear autotopisms of $\mathbb{O}_E$.*

(f) *For each linear autotopism $(\alpha|\beta|\gamma)$ of $\mathbb{O}$, the extension $(\alpha|\beta|\gamma)_E$ acts on $\mathcal{M}^- \times \mathcal{Q}_E \times \mathcal{M}^+$ via $(\alpha|\beta|\gamma)_E \,.\, (\mathbb{O}_E\, x, Ey, z\mathbb{O}_E) := (\mathbb{O}_E\, \alpha(x), E\beta(y), \gamma(z)\mathbb{O}_E)$.*

(g) *The map $\nabla_E : \mathcal{M}^- \times \mathcal{Q}_E \times \mathcal{M}^+ \to \mathcal{M}^- \times \mathcal{Q}_E \times \mathcal{M}^+ : (\mathbb{O}_E\, x, Ey, z\mathbb{O}_E) \mapsto (\mathbb{O}_E\, z, E\overline{x}, \overline{y}\mathbb{O}_E)$ is well-defined, and has order three. (This is the classical triality of the polar space — of split type D_4 — defined by the quadratic form N_{C_E}.)*

(h) *For each linear autotopism $(\alpha|\beta|\gamma)$ of $\mathbb{O}$ we obtain $\nabla_E\left((\alpha|\beta|\gamma)_E \,.\, (\mathbb{O}_E\, x, Ey, z\mathbb{O}_E)\right) = \tau_E\left((\alpha|\beta|\gamma)_E\right) . \nabla_E\left(\mathbb{O}_E\, x, Ey, z\mathbb{O}_E\right) = \left(\tau(\alpha|\beta|\gamma)\right)_E . \nabla_E\left(\mathbb{O}_E\, x, Ey, z\mathbb{O}_E\right)$.*

(i) *For $\varepsilon \in \{+, -\}$, let $\mathrm{ind}^\varepsilon : \mathcal{M}^\varepsilon \to \mathcal{C}^\varepsilon$ be the indicating map (in the sense of 7.3), and let $\mathrm{ind}^Q : \mathcal{Q}^E \to \mathcal{L}$ map each point of the quadric to the unique Baer line containing it. Then $(\mathrm{ind}^-, \mathrm{ind}^Q, \mathrm{ind}^+) \circ \nabla_E = \tilde{\nabla} \circ (\mathrm{ind}^-, \mathrm{ind}^Q, \mathrm{ind}^+)$.*

Proof. Assertion (a) is proved in [5, 3.4] (see also [23, 3.3.2], but note that the maps given there do not have order three). In order to verify the remaining assertions, we use 6.13, 6.7, and 7.3. □

8.2 Proposition. *Assume that $E : F$ is a separable quadratic field extension such that C_E splits. Then the triality ∇_E maps the points of the ovoid $\mathcal{Q}^E$ in the Baer subspace $\mathbb{P}(V^E)$ introduced in 5.6 to the sets of indicator sets for ${}_1\!\!\backslash\!\!\backslash$ and $/\!\!/_1$, respectively (cf. 5.5). The sets of indicator sets for other parallelisms are then obtained from the images of the ovoid under similitudes.*

If $F = \mathbb{R}$ then these ovoids, and thus these sets of indicator sets, are homeomorphic to the sphere of dimension 6.

Proof. By the proof of 5.6, the points of $\mathcal{Q}^E$ have the form $E(e \otimes 1 - 1 \otimes y)$ with $T_E(e) = T_{\mathbb{O}}(y)$ and $N_E(e) = N_{\mathbb{O}}(y)$. Now ∇_E maps $E(e \otimes 1 - 1 \otimes y)$ to $(\overline{e \otimes 1 - 1 \otimes y})\mathbb{O}_E = (e \otimes 1 - 1 \otimes \overline{y})\mathbb{O}_E$ and then to $\mathbb{O}_E(e \otimes 1 - 1 \otimes \overline{y})$.

These are exactly the indicator sets of ${}_1\!\!\backslash\!\!\backslash$ and $/\!\!/_1$, respectively, as described in 5.3 because $T_E(e) = T_{\mathbb{O}}(y) = T_{\mathbb{O}}(\overline{y})$ and $N_E(e) = N_{\mathbb{O}}(y) = N_{\mathbb{O}}(\overline{y})$ holds precisely if there is an (F-linear) algebra homomorphism $\varphi\colon E \to \mathbb{O}$ with $\varphi(e) = \overline{y}$. The statement about the other parallelisms follows from the very definition 6.1, cf. 6.14. □

9 Remarks on the literature

Van Buggenhaut [25], [24] considers geometric descriptions of our parallelisms for the special case where $F = \mathbb{R}$, using the complex quadric $\mathcal{Q}_{\mathbb{C}}$ and triality. He actually claims that his arguments extend easily to arbitrary ground fields F with $\operatorname{char} F \neq 2$ (and such that an octonion field exists over F), but as he only works with a single quadratic extension, this will not work unless the octonion algebra contains only one single isomorphism type of quadratic extensions of F. This is a severe restriction, one obtains that F is a pythagorean field (i.e., every sum of squares is a square in F, cf. [22, 12.8]) and formally real (i.e., -1 is not a sum of squares in F, cf. [22, Ch. 12]):

9.1 Theorem. *Let C be a composition algebra of dimension at least four over F. If all the two-dimensional subalgebras belong to a single isomorphism type, then F is a formally real pythagorean field, and the norm form of C is equivalent to the standard euclidean form (sum of squares).*

Proof. If C is split then C contains a split quaternion algebra (isomorphic to $F^{2\times 2}$), and there exist $x, y \in C \smallsetminus F$ with $x^2 = 0$ and $y^2 = y$. Then $F + Fx \cong F[X]/(X^2) \not\cong F \times F \cong F[X]/(X^2 + X) \cong F + Fy$. Thus we concentrate on the non-split case. If $\operatorname{char} F = 2$ then C contains both separable and inseparable field extensions of F. Thus we exclude $\operatorname{char} F = 2$ in the sequel.

Assume now that the two-dimensional subalgebras are all isomorphic. Pick $a \in 1^\perp \smallsetminus \{0\}$ (i.e., $a \in C \smallsetminus F$ with $T(a) = 0$). For any $x \in 1^\perp \smallsetminus \{0\}$, the algebras $F + Fa$ and $F + Fx$ are isomorphic, whence we find $r \in F$ such that $N(a) = N(rx) = r^2 N(x)$.

Choose $x \in (F + Fa)^\perp \smallsetminus \{0\}$, and pick r as above. Then $(1|a^{-1}rx) = N(a)^{-1}(a|rx) = 0$, and $N(a) = N(rx)$ yields $N(a^{-1}rx) = N(a)^{-1}N(rx) = 1$. Thus $a^{-1}rx \in 1^\perp$ has norm 1. Consequently, the norms of arbitrary elements of $1^\perp$ are squares in F, and we infer that there exists an orthonormal basis 1, $b_2 := a^{-1}rx$, b_3, … for C. In particular, the norm form is the standard euclidean form.

For $s, t \in F$ we now note that $s^2 + t^2 = N(sb_2 + tb_3)$ is a square because $sb_2 + tb_3 \in 1^\perp$, and obtain that the field F is pythagorean. As $F + Fa$ is not split, we have that -1 is not a square, and F is formally real. □

Note also that "parallélisme de Clifford" in Van Buggenhaut's notation means a symmetric, reflexive but not transitive relation[6]; our Clifford parallelisms are called "parallélismes de Vaney–Cartan" in [24] (referring to [27]).

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Andrea Blunck
Dept. of Mathematics
Universität Hamburg
Bundesstr. 55
D-20146 Hamburg
Germany

Norbert Knarr
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

Bernhild Stroppel
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

Markus J. Stroppel
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

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