Universität Stuttgart

# Groups of similitudes generated by octonions

Andrea Blunck, Norbert Knarr,
Bernhild Stroppel,
Markus J. Stroppel

Stuttgarter Mathematische Berichte

2017-007

mathematik

Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

**E-Mail:** preprints@mathematik.uni-stuttgart.de
**WWW:** http://www.mathematik.uni-stuttgart.de/preprints

ISSN **1613-8309**

# Transitive groups of similitudes generated by octonions

Andrea Blunck, Norbert Knarr,
Bernhild Stroppel, Markus J. Stroppel

**Abstract**

We study the structure of various groups generated by (left) multiplications in alternative division algebras, and construct epimorphisms onto orthogonal and special orthogonal groups. Throughout, we include the characteristic two case in our treatment.

**Mathematics Subject Classification (MSC 2000):** 11E57, 17A75, 20G15, 20G20, 11E04, 11E39.
**Keywords:** octonion, quaternion, division algebra, composition algebra, orthogonal group, unitary group, similitude, autotopism

## Introduction

We study groups generated by certain sets of (left) multiplications by non-zero elements of an octonion field $\mathbb{O}$ of arbitrary characteristic. These groups are contained in the group $\mathrm{GO}(\mathbb{O}, N)$ of similitudes of the norm form $N$ of the octonion field. Among our results, we have the following:

**(a)** The group $\Lambda_{\mathbb{O}}$ generated by all left multiplications is normalized by all right multiplications (see 3.12), and together the left and right multiplications generate a subgroup $\mathrm{GO}^+(\mathbb{O}, N)$ of index two in $\mathrm{GO}(\mathbb{O}, N)$ (Def. 3.8, see 3.11 and 3.13). The group $\mathrm{GO}^+(\mathbb{O}, N)$ consists of all elements of $\mathrm{GO}(\mathbb{O}, N)$ that occur as entries in triplets that form autotopisms of $\mathbb{O}$ (see 3.11).

**(b)** Using an embedding into the group of autotopisms, we show (in 4.2, 4.5.(c)) that the group $\Lambda_{\mathrm{Pu}\,\mathbb{O}}$ generated by left multiplications by non-zero elements from the space $\mathrm{Pu}\,\mathbb{O}$ of pure octonions has a quotient isomorphic to $\mathrm{SO}(\mathrm{Pu}\,\mathbb{O}, N|_{\mathrm{Pu}\,\mathbb{O}})$; i.e. a special orthogonal group of a suitable form in seven variables, see 6.1.

**(c)** For each quadratic separable subfield $K$ in $\mathbb{O}$ (i.e., every two-dimensional subalgebra $K$ with $K \nleqslant 1^\perp$), the left multiplications by non-zero elements from the space $K^\perp$ generate a subgroup $\Lambda_{K^\perp}$ of the group $\Gamma\mathrm{U}_K(\mathbb{O}, g)$ of semi-similitudes of a certain hermitian form $g$ on $\mathbb{O}$ (considered as a vector space over $K$, see 2.9, 6.2, and 6.3). We show (in 6.3) that said subgroup of $\Gamma\mathrm{U}_K(\mathbb{O}, g)$ has a quotient isomorphic to $\mathrm{O}(K^\perp, N|_{K^\perp})$; i.e. an orthogonal group of a suitable form in six variables.

**(d)** For each quaternion subfield $H$ in $\mathbb{O}$, we choose a three-dimensional subspace $W \leqslant H$ and show (in 6.6) that the left multiplications by non-zero elements from the space $W^\perp$ generate a subgroup $\Lambda_{W^\perp}$ of the group $\Gamma\mathrm{U}_H(\mathbb{O}, h)$ of semi-similitudes of a certain hermitian form $h$ on $\mathbb{O}$ (considered as a vector space over $H$, see 2.12). We show (in 6.6) that said subgroup of $\Gamma\mathrm{U}_H(\mathbb{O}, h)$ has a quotient isomorphic to $\mathrm{SO}(W^\perp, N|_{W^\perp})$; i.e. a special orthogonal group of a suitable form in five variables. The group of all semi-similitudes of $h$ with $F$-linear companions is obtained as the product $\Lambda_W \circ \Lambda^+_{W^\perp}$.

The isomorphisms onto orthogonal groups in seven, six and five variables are rather well known in the case where the forms are isotropic (see [8, Ch. 4, § 8, p. 102–106]) and in the case where the ground field is the field $\mathbb{R}$ of real numbers. In the latter case, our methods lead to explicit constructions of certain coverings and exceptional isomorphisms between compact Lie groups, namely $\mathrm{Spin}_7(\mathbb{R}) \to \mathrm{SO}_7(\mathbb{R})$, $\mathrm{SU}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{R})$, $\mathrm{U}_2(\mathbb{H}) \to \mathrm{SO}_5(\mathbb{R})$.

In fact, the present investigation has been inspired by ideas of Hermann Hähl (see [13, 3.2(b) and (c)], [14, 3.1, 3.2], cp. [26, proof of 81.17, pp. 471 f]) who used (over the ground field $\mathbb{R}$) descriptions of $\mathrm{Spin}_7(\mathbb{R})$ and $\mathrm{SU}_4(\mathbb{C})$ as groups generated by (right) multiplications in $\mathbb{O}$, together with their interpretation as components of autotopisms (viz., elements of a triangle stabilizer) and the resulting homomorphisms onto $\mathrm{SO}_7(\mathbb{R})$ and $\mathrm{SO}_6(\mathbb{R})$, respectively.

Considering groups of semi-similitudes rather than orthogonal or unitary groups allows us to extend the results to arbitrary ground fields (subject only to the condition that an octonion field exists). The restriction to subgroups of $\mathrm{O}(\mathbb{O}, N)$ rather than $\mathrm{GO}(\mathbb{O}, N)$ makes it much more difficult to determine the range of the homomorphisms that we obtain.

It is a general observation that the theory of isometry groups of *anisotropic* forms is much more involved than the theory for the isotropic case. Many results depend crucially on properties of the ground field. For instance, there exist examples of anisotropic quadratic forms over non-archimedean ordered fields where the orthogonal group has a sequence $(N_j)_{j\in\mathbb{N}}$ of normal subgroups such that $\bigcap_{j\in\mathbb{N}} N_j$ is trivial and each one of the quotients $N_j/N_{j+1}$ is abelian (see [7, p. 345 ff], cf. [1, Ch. V, § 3, pp. 179–186]). In particular, these orthogonal groups do not contain any simple subgroups, and the same assertion holds for the groups of linear automorphisms of corresponding quaternion or octonion fields. See also [16, p. 67 f].

The quadratic forms (in seven, six and five variables) that play their role in the present investigation have been characterized in a separate paper [23]. The forms in six variables, in particular, are important in the theory of spherical Tits buildings of low rank, where they are known as forms of type $\mathrm{E}_6$ (see [30, Ch. 12] and [5]). .

Some results of the present paper (in particular, information about the action and the structure of $\Lambda_{\mathrm{Pu}\,\mathbb{O}}$) are applied in [4].

## 1 Composition algebras

Let $F$ be a commutative field. A *composition algebra* over $F$ is a vector space $C$ over $F$ with a bilinear multiplication (written as $xy$) and a multiplicative quadratic form $N := N_C \colon C \to F$ such that the polar form of $N$ is not degenerate. The polar form will be written as $(x|y) := N(x + y) - N(x) - N(y)$. We also assume that the algebra contains a neutral element for its multiplication, denoted by $1$. As usual, the ground field $F$ is embedded as $F1$ in $C$. The composition algebra is called *split* if it contains divisors of zero.

The first chapter of [27] gives a comprehensive introduction into composition algebras over arbitrary (commutative) fields, including the characteristic two case.

We collect the basic facts that we need in the present paper (for proofs, consult [27]):

**1.1 Properties of composition algebras.** *Let $C$ be a composition algebra over $F$.*

**(a)** *The map $\kappa\colon C \to C\colon x \mapsto \bar{x} := (x|1)1 - x$ is an involutory anti-automorphism, called the* standard involution *of $C$. Note that $-\kappa$ is the hyperplane reflection with center $F1$, see 2.3.*

**(b)** *The norm and its polar form can be recovered from the standard involution as $N_C(x) = x\bar{x} = \bar{x}x$ and $(x|y) = x\bar{y} + y\bar{x}$. In particular, we have the hyperplane $\operatorname{Pu} C := 1^\perp = \{x \in C \mid \bar{x} = -x\}$ of* pure elements.

**(c)** *For all $c, x, y \in C$, we have $(cx|y) = (x|\bar{c}y)$ and $(xc|y) = (x|y\bar{c})$, see* [27, 1.3.2].

**(d)** *In general, the multiplication is not associative, but weak versions of the associative law are still valid; among them* Moufang's identities [27, 1.4.1, 1.4.2]

$$(ax)(ya) = a((xy)a), \quad a(x(ay)) = ((ax)a)y, \quad x(a(ya)) = ((xa)y)a.$$

**(e)** *Artin's Theorem* (see [27, Prop. 1.5.2])*: For any two elements $x, y \in C$, the subalgebra generated by $x$ and $y$ in $C$ is associative.*

**(f)** *An element $a \in C$ is invertible if, and only if, its norm is not zero; we have $a^{-1} = N_C(a)^{-1}\,\bar{a}$ in that case. Thus a non-split composition algebra is a division algebra. Note that Artin's Theorem then implies $a^{-1}(ax) = x = a(a^{-1}x) = (xa)a^{-1} = (xa^{-1})a$, for each $x \in C$. If $C$ is not split then $C^* := C \smallsetminus \{0\}$ is closed under multiplication, and forms a (Moufang) loop.*

**(g)** *Each element $a \in C$ is a root of a polynomial of degree 2 over $F$, namely, the polynomial $X^2 - (a + \bar{a})X + N_C(a) \in F[X]$. We call $T_C(a) := a + \bar{a}$ the* trace *of $a$ in $C$.*

**(h)** *In particular, for each $a \in C$ the algebra generated by $a$ is $F(a) = F + Fa$, and this algebra is associative and commutative. For $x, y \in F(a)$ and $v \in C$ we have that $v, x, y$ lie in the algebra generated by $v$ and $a$. Thus $v(xy) = (vx)y$ by (e), and $C$ is a left module over $F(a)$. If the restriction of the norm to $F(a)$ is anisotropic then $F(a)$ is a commutative field, and $C$ is a (left) vector space over $F(a)$.*
*Similarly, we may consider $C$ as a right module over $F(a)$.*

**(i)** [27, 1.9] *If $\dim_F C > 2$ then $F$ is the* center *of $C$; i.e. $F = \{z \in C \mid \forall x \in C\colon zx = xz\}$. If $\dim_F C > 4$ then $F$ coincides with each one of the* nuclei *of $C$; i.e.*

$$\begin{aligned} F \;=\; \{z \in C \mid \forall x, y \in C\colon z(xy) = (zx)y\} \;&=\; \{z \in C \mid \forall x, y \in C\colon (xz)y = x(zy)\} \\ &=\; \{z \in C \mid \forall x, y \in C\colon (xy)z = x(yz)\}. \end{aligned}$$

**(j)** [21, 1.3] *Every $F$-semilinear automorphism of $C$ commutes with the standard involution. If $\dim_F C \geqslant 4$ then every $\mathbb{Z}$-linear automorphism of $C$ is $F$-semilinear. Consequently, every $\mathbb{Z}$-linear automorphism of such a $C$ is a semi-similitude of the norm form, and every $F$-linear automorphism is an orthogonal map (cf. 5.4 below).*

*We write $\operatorname{Aut}(C)$ for the group of all $\mathbb{Z}$-linear automorphisms (i.e., all additive and multiplicative bijections), and $\operatorname{Aut}_F(C)$ for the group of all $F$-linear automorphisms.*

**(k)** [27, 1.5.3] *If $D$ is a subalgebra of $C$ with $\dim_F C = 2\dim_F D$ and such that $D^\perp \cap D = \{0\}$ then $D^\perp = Dw$ holds for each $w \in D^\perp$ with $N_C(w) \neq 0$, and the multiplication in $C = D \oplus D^\perp$ is given by $(x + yw)(u + vw) = (xu - N_C(w)\bar{v}y) + (vx + y\bar{u})w$.*

For a reader versed in associative algebras, the construction of the module structures in 1.1.(h) will appear natural, and one might conjecture a straightforward extension to any subalgebra of $C$. However, the lack of associativity in octonion algebras prevents this. A pertinent construction will be given in 2.10 below.

We need a variant of the result [21, 1.6], as follows.

**1.2 Lemma.** *Let $C$ be a non-split composition algebra, and let $X, Y$ be vector subspaces such that $\dim X + \dim Y > \dim C$. Then $C = XY := \{xy \mid x \in X, y \in Y\}$.*

*Proof.* For $c \in C \setminus \{0\}$, the sets $X$ and $c\overline{Y}$ are vector subspaces; their intersection has dimension at least 1. For $x \in (X \cap c\overline{Y}) \setminus \{0\}$ there exists $v \in \overline{Y}$ such that $x = cv$. Now $y := v^{-1} = N(v)^{-1}\bar{v}$ belongs to $Y$, and $c = xv^{-1} = xy \in XY$ yields the claim. □

**1.3 Definition.** Let $V$ be any vector subspace of a non-split composition algebra $C$. By $\Lambda_V := \langle \lambda_v \mid v \in V \setminus \{0\} \rangle$ we denote the group generated by all left multiplications with non-trivial elements of $V$. The subgroup $\Lambda_V^+ := \langle \lambda_v \circ \lambda_w \mid v, w \in V \setminus \{0\} \rangle$ is generated by all products of an even number of left multiplications by elements of $V \setminus \{0\}$.

**1.4 Lemma.** *Let $V \leqslant C$ be a vector subspace with $\dim V > \frac{1}{2} \dim C$. Then $\Lambda_V^+$ acts transitively on $C^*$.*

*Proof.* From 1.2 we know $VV = C$, and $C^* \subseteq \Lambda_V^+(1)$ gives transitivity, as claimed. □

## 2 Similitudes of the norm and related forms

Notation and terminology for quadratic and hermitian forms is fairly standard. However, the literature contains different notions of degeneracy for quadratic forms in characteristic two. We call a quadratic form $q$ on a vector space $V$ *degenerate* if there exists a non-zero vector $v \in V^\perp$ with $q(v) = 0$. The orthogonal space $X^\perp := \{w \in V \mid \forall x \in X \colon f_q(w, x) = 0\}$ is meant with respect to the polar form given by $f_q(x, y) := q(x + y) - q(x) - q(y)$, as usual.

In this section, we only consider vector spaces of finite dimension.

**2.1 Definitions.** Let $V$ be a *left*[1] vector space over some (not necessarily commutative) field $K$, and let $\sigma \colon K \to K$ be an anti-automorphism with $\sigma^2 = \mathrm{id}$.

**(a)** Let $h \colon V \times V \to K$ be a $\sigma$-hermitian form, i.e., a bi-additive map such that $h(w, v) = \sigma(h(v, w))$ and $h(sv, w) = sh(v, w)$ hold[2] for all $v, w \in V$ and each $s \in K$. A *semi-similitude* of $h$ with multiplier $\mu_\psi$ and companion $\varphi_\psi$ is a semi-linear bijection $\psi \colon V \to V$ such that $h(\psi(v), \psi(w)) = \varphi_\psi(h(v, w))\, \mu_\psi$ holds for all $v, w \in V$.

**(b)** Let $q \colon V \to K$ be a quadratic form. Then a *semi-similitude* of $q$ with multiplier $\mu_\psi$ and companion $\varphi_\psi$ is a semi-linear bijection $\psi \colon V \to V$ such that $q(\psi(v)) = \varphi_\psi(q(v))\, \mu_\psi$ holds for each $v \in V$.

**(c)** In any case, a *similitude* is a semi-similitude with trivial companion.

[1] Right vector spaces (coordinatized by spaces of columns) generally fit better with mappings applied from the left. However, it turns out in 2.10 below that left vector spaces over quaternions are what we need here.

[2] Then $h(v, sw) = h(v, w)\sigma(s)$.

If $\psi$ is a semi-similitude of a non-zero hermitian form then its companion $\varphi_\psi$ is a field automorphism, and coincides with the companion of the semi-linear bijection. In particular, it is uniquely determined.

Note that each semi-similitude of a quadratic form is a semi-similitude of the corresponding polar form, with the same multiplier and companion. If the quadratic form is non-zero then the companion of the semi-similitude is the companion of the semi-linear bijection.

**2.2 Definition.** If $h\colon V \times V \to K$ is a non-zero hermitian form then the set of all semi-similitudes of $h$ forms a group which we denote by $\Gamma\mathrm{U}_K(V,h)$. The subgroups $\mathrm{GU}_K(V,h)$ of all similitudes and $\mathrm{U}_K(V,h)$ of all isometries (i.e., similitudes with multiplier $1$) clearly form normal subgroups in $\Gamma\mathrm{U}_K(V,h)$; these are the kernels of the homomorphisms $\psi \mapsto \varphi_\psi$ and $\psi \mapsto (\varphi_\psi, \mu_\psi)$, respectively.

Analogously, for a quadratic form $q\colon V \to K$ with non-zero polar form, we have the group $\Gamma\mathrm{O}(V,q)$ of all semi-similitudes, with normal subgroups $\mathrm{GO}(V,q)$ and $\mathrm{O}(V,q)$ of similitudes and isometries, respectively.

As we want to include the characteristic two case, some caution is required when dealing with involutions in orthogonal groups:

**2.3 Definition.** Let $q\colon V \to F$ be a quadratic form on a vector space $V$ with corresponding polar form $f_q\colon V \times V \to F$ (we allow the case where $f_q$ is degenerate). For each $v \in V$ with $q(v) \neq 0$ we call

$$\sigma_v\colon V \to V\colon x \mapsto x - \frac{f_q(x,v)}{q(v)} v$$

the *hyperplane reflection with center* $Fv$ (and axis $v^\perp$).

It is easy to verify that $\sigma_v \in \mathrm{O}(V,q)$ with $\sigma_v^2 = \mathrm{id}$ and $\mathrm{Fix}(\sigma_v) = v^\perp$. If $\operatorname{char} F \neq 2$ then the hyperplane reflection's axis and center are already determined by its space of fixed points (which is the axis, and forms the orthogonal complement of the center). The situation is quite different if $\operatorname{char} F = 2$, see 2.7 below.

If $\operatorname{char} F \neq 2$ then a product of a sequence of hyperplane reflections on $V$ has determinant $1$ if, and only if, the number of factors is even. If $\operatorname{char} F = 2$ then the determinant does not impose any restrictions on members of the orthogonal group.

**2.4 Remarks on special orthogonal groups, and the Dickson invariant.** Let $q\colon V \to F$ be a non-degenerate quadratic form, and let $h\colon V \times V \to F$ denote a hermitian form. As usual[3] we write $\mathrm{SO}(V,q) := \left\{\psi \in \mathrm{O}(V,q) \,\middle|\, \det\psi = 1\right\}$ and $\mathrm{SU}(V,h) := \left\{\psi \in \mathrm{U}(V,h) \,\middle|\, \det\psi = 1\right\}$.

If $\operatorname{char} F \neq 2$ and $\dim V \neq 0$ then $\mathrm{SO}(V,q)$ is a (normal) subgroup of index $2$ in $\mathrm{O}(V,q)$. However, we have $\mathrm{SO}(V,q) = \mathrm{O}(V,q)$ if $\operatorname{char} F = 2$.

We recall the definition of Dickson's invariant ([9], cf. [29, 11.43]): for a quadratic form $q$ on a vector space $V$ of arbitrary characteristic and with non-degenerate polar form, this is the (multiplicative!) map $\mathrm{D}\colon \mathrm{O}(V,q) \to \{1,-1\}\colon \gamma \mapsto (-1)^d$, where $d := \dim(V/\mathrm{Fix}(\gamma)) = \dim\{\gamma(v) - v \mid v \in V\}$. If the characteristic is different from two then $\mathrm{D}(\gamma) = \det\gamma$.

[3] The mathematical community is unanimous in its interpretation of "$\mathrm{SO}(V,q)$" as long $\operatorname{char} F \neq 2$, and confusion starts if that restriction is dropped. The reader should be warned against the fact that the definitions of $\mathrm{SO}(V,q)$ for the characteristic two case wildly vary in the existing literature; some sources use this name for the kernel of the Dickson invariant.

In any case, it is obvious that the kernel of the Dickson invariant $\mathrm{D}$ contains the subgroup consisting of products of an even number of hyperplane reflections. See 2.5 for a more precise statement.

If the characteristic is two then non-degeneracy of the polar form implies that $\dim V$ is even. In this case, the kernel $\ker \mathrm{D}$ of the Dickson invariant is a (normal) subgroup of index 2 in $\mathrm{O}(V,q)$. If the polar form is degenerate (but not zero) then every product of hyperplane reflections can be written as a product of an even number of reflections, see 2.5 below.

The following result is based on the Cartan-Dieudonné Theorem. There is an exceptional case in that theorem, namely the case where $V$ has dimension 4 over $F = \mathbb{F}_2$ and the Witt index is 2. This is why we make the extra assumption on $V$ in 2.5 below. As we are interested mainly in anisotropic forms (over infinite fields), that special case is not really relevant for us.

**2.5 Lemma.** *Let $q\colon V \to F$ be a non-degenerate quadratic form, let $A := \{v \in V \mid q(v) \neq 0\}$, and put $\mathrm{O}^+(V,q) := \langle \sigma_u \circ \sigma_v \mid u,v \in A\rangle$. If $|F| = 2$ and $\dim V = 4$, assume in addition that the Witt index of $q$ is not 2.*

**(a)** $\langle \sigma_u \mid u \in A\rangle = \mathrm{O}(V,q)$.

**(b)** *If the polar form $f_q$ is not degenerate then $\mathrm{O}^+(V,q)$ is a normal subgroup of index 2 in $\mathrm{O}(V,q)$. Indeed $\mathrm{O}^+(V,q)$ is the kernel of the Dickson invariant, and $\mathrm{O}^+(V,q) = \mathrm{SO}(V,q)$ if $\operatorname{char} F \neq 2$.*

**(c)** *If $f_q \equiv 0$ then $\mathrm{O}(V,q)$ is trivial, and so is $\mathrm{O}^+(V,q)$.*

**(d)** *If $f_q$ is degenerate (in particular, if $\operatorname{char} F = 2$ and $\dim V$ is odd) then $\mathrm{O}^+(V,q) = \mathrm{O}(V,q)$ unless $|F| = 2$ and $\dim V \in \{3,5\}$.*

*Proof.* Assertion (a) is the Cartan-Dieudonné Theorem (see [7, Prop. 8, p. 20, Prop. 14, p. 42, Prop. 17, p. 55], cf. [29, 11.39, 11.41] or [11, 14.16, p. 135]).

If the polar form is not degenerate then the Dickson invariant is defined, we clearly have $\mathrm{D}(\sigma_u) = -1$, and the Dickson invariant of any product of hyperplane reflections is $(-1)^d$ where $d$ is the number of factors. This yields assertion (b).

If $f_q$ is zero then $\operatorname{char} F = 2$ (because $q$ is not degenerate but $2q(v) = f_q(v,v) = 0$ holds for each $v \in V$). Now $q$ is a semilinear map from $V$ to $F$; the companion is the Frobenius endomorphism. That map is injective because its kernel is the radical $\{v \in V^\perp \mid q(v) = 0\}$ of $q$. That radical is trivial by our hypothesis. As the group $\mathrm{O}(V,q)$ preserves the values under $q$, that group is trivial, and so is $\mathrm{O}^+(V,q)$ (cf. [8, Ch. I, § 16, p. 35]).

Finally, assume that $f_q$ is degenerate (then $\operatorname{char} F = 2$) but $f_q$ is not zero. Choose $z \in V^\perp \smallsetminus \{0\}$; then $q(z) \neq 0$ because $q$ is not degenerate. For $x \in V$ the map $d_x\colon F \to F\colon s \mapsto q(x+sz) = q(x) + s^2 q(z)$ is injective.

If $|F| > 2$ then there exists $y \in V \smallsetminus V^\perp$ with $q(y) \notin \{0, q(z)\}$. We put $w := q(z)y + q(y)z$ and note $q(y+z) = q(y) + q(z) \neq 0 \neq q(y)q(z)q(y+z) = q(w)$. For $x \in V$, we use $z \in V^\perp$ and $f_q|_{Fy+Fz} = 0$ to compute

$$\begin{aligned}
\sigma_y\left(\sigma_{y+z}(x)\right) &= \sigma_y\left(x - \tfrac{f_q(x,y+z)}{q(y+z)}(y+z)\right) \\
&= x - \tfrac{f_q(x,y)}{q(y+z)}(y+z) - \tfrac{f_q(x,y)}{q(y)}y \;=\; x - \tfrac{f_q(x,y)}{q(y)q(y+z)}\left((q(y)+q(y+z))y + q(y)z\right) \\
&= x - \tfrac{f_q(x,y)}{q(y)q(y+z)}\left(q(z)y + q(y)z\right) \;=\; x - \tfrac{f_q(x,q(z)y+q(y)z)}{q(y)q(z)q(y+z)}w \\
&= \sigma_w(x)\,.
\end{aligned}$$

Thus $\sigma_w = \sigma_y \circ \sigma_{y+z}$, and $\mathrm{O}^+(V,q) = \mathrm{O}(V,q)$ follows in this case.

If $|F| = 2$ and $\dim V = 2n+1$ then $\mathrm{O}(V,q) \cong \mathrm{Sp}_{2n}(\mathbb{F}_2)$ (see [11, 14.2]). That group is simple if $n \geqslant 3$ (see [11, 3.11]), and coincides with its normal subgroup $\mathrm{O}^+(V,q)$. This completes the proof of assertion (d). □

Up to similitude, there is just one non-degenerate quadratic form $q_d$ on $\mathbb{F}_2^d$ if $d$ is odd. The cases $\mathrm{O}(\mathbb{F}_2^3, q_3) \cong \mathrm{S}_3$ and $\mathrm{O}(\mathbb{F}_2^5, q_5) \cong \mathrm{S}_6$ are true exceptions in 2.5.(d); we have $\mathrm{O}^+(\mathbb{F}_2^3, q_3) \cong \mathrm{A}_3$ and $\mathrm{O}^+(\mathbb{F}_2^5, q_5) \cong \mathrm{A}_6$, respectively. In fact, there are $2^5 - 1 - 1 - 15 = 15$ non-trivial reflections in $\mathrm{O}(\mathbb{F}_2^5, q_5) \cong \mathrm{S}_6$. The set of these reflections is a union of conjugacy classes of involutions, and $\mathrm{S}_6$ contains three conjugacy classes of invoutions, with $15$, $45$, and $15$ elements, respectively. Up to (possibly outer) automorphisms, the set of reflections is thus the class of transpositions.

**2.6 Lemma.** *Let $C$ be any non-split composition algebra of dimension at least two over $F$. For each $u \in C$, we consider the left and right multiplications $\lambda_u$ and $\rho_u$, respectively, and the map $\delta_u := N(u)^{-1}\lambda_u \circ \rho_u$.*

- **(a)** *For each $u \in C \smallsetminus F$ the map $\delta_u$ is the product of two hyperplane reflections, namely $\delta_u = \sigma_u \circ \sigma_1$, where $\sigma_1 = -\kappa$ and $\sigma_u = -\lambda_u \circ \kappa \circ \lambda_u^{-1}$.*
- **(b)** *Let $V$ be a vector subspace of $C$, and assume $\overline{V} = V$. Then $\left\langle \delta_u \circ \delta_v \,\middle|\, u,v \in V \smallsetminus \{0\} \right\rangle = \left\langle \sigma_x \circ \sigma_y \,\middle|\, x,y \in V \smallsetminus \{0\} \right\rangle$. This group induces $\mathrm{O}^+(V, N|_V)$ on $V$, and acts trivially on $V^\perp$.*
- **(c)** *For each vector subspace $V \leqslant C$ with $1 \in V$ the group $\mathrm{O}^+(V, N|_V)$ is generated by $\left\{ \delta_u|_V \,\middle|\, u \in V \smallsetminus \{0\} \right\}$.*

*Proof.* We know that $\sigma_1 = -\kappa$ is a hyperplane reflection, see 2.3. Therefore, its conjugate $\lambda_u \circ (-\kappa) \circ \lambda_u^{-1} = \lambda_u \circ \sigma_1 \circ \lambda_u^{-1} = \sigma_u$ is another reflection, and $\sigma_u \circ \sigma_1 = (-\lambda_u \circ \kappa \circ \lambda_u^{-1}) \circ (-\kappa) = \lambda_u \circ \kappa \circ \lambda_u^{-1} \circ \kappa = \delta_u$, as claimed in assertion (a).

Let $V \leqslant C$ be any vector subspace. For each $x \in V$ and each $u \in V \smallsetminus \{0\}$ we have $\sigma_u(x) \in x + Fu \subseteq V$. Thus $V$ is invariant under $\sigma_u$, and $\sigma_u|_V \in \mathrm{O}(V, N|_V)$ is either a hyperplane reflection on $V$, or trivial on $V$.

We observe $\delta_u = -\sigma_u \circ \kappa = -\kappa \circ \sigma_{\overline{u}}$. Thus $\delta_u \circ \delta_v = \sigma_u \circ \sigma_{\overline{v}}$, and $\overline{V} = V$ implies $\left\langle \delta_u \circ \delta_v \,\middle|\, u,v \in V \smallsetminus \{0\} \right\rangle = \left\langle \sigma_x \circ \sigma_y \,\middle|\, x,y \in V \smallsetminus \{0\} \right\rangle$. Each product $\sigma_x \circ \sigma_y$ with $x,y \in V$ leaves $V$ invariant and acts trivially on $V^\perp$. So assertion (b) follows.

Now assume $1 \in V$; then $V$ is invariant also under $\sigma_1$, and $\delta_u|_V$ belongs to $\mathrm{O}^+(V, N|_V)$. In order to show that the set $\left\{ \delta_u|_V \,\middle|\, u \in \mathbb{O}^* \right\}$ generates $\mathrm{O}^+(V, N|_V)$, we recall $\delta_u = \sigma_u \circ \sigma_1$ and obtain $\delta_u \circ \delta_v^{-1} = \sigma_u \circ \sigma_1 \circ \sigma_1 \circ \sigma_v = \sigma_u \circ \sigma_v$. □

**2.7 Remark.** If $q\colon V \to F$ is a non-degenerate quadratic form then $\left\{ \sigma_u \,\middle|\, u \in S^\perp, q(u) \neq 0 \right\}$ is the set of all hyperplane reflections with given axis $S$. Together with the identity, this set forms an elementary abelian 2-group $\Sigma_S$. The order of $\Sigma_S$ is the cardinality of $V^\perp$. If $\Sigma_S$ is infinite then the dimension of $\Sigma_S$ over $\mathbb{F}_2$ equals the cardinality of $S$, see [3, II.2, Lem 3, p. 20]. As abstract groups,[4] we thus have $(\Sigma_S, \circ) \cong (V^\perp, +)$.

If the polar form $f_q$ is not degenerate (in particular, if $\operatorname{char} F \neq 2$) then this group has order at most two. If $\operatorname{char} F = 2$ then every hyperplane reflection is a transvection.

[4] Giving an isomorphism explicitly appears to be difficult, in general.

## Construction of hermitian forms

We prove a variant of an old result by Jacobson [15], cf.[5] [28, 4.3]. The difference to Jacobson's version is that we construct the hermitian form (and do not only *re*construct it). First of all, we turn our composition algebras into vector spaces over larger fields:

**2.8 Lemma.** *Let $K$ be a two-dimensional subalgebra of a composition algebra $C$, and consider $a \in K^\perp$ with $N(a) \neq 0$. Then the left multiplication $\lambda_a \colon x \mapsto ax$ is a semilinear bijection of the $K$-module $C$ onto itself, with companion $\kappa|_K$.*

*Proof.* Consider $s \in K$ and $x \in C$. Then $\lambda_a(sx) = a(sx) = a(s(a(a^{-1}x))) = (a(sa))(a^{-1}x)$ by one of Moufang's identities (see 1.1.(d)). As $a^{-1} = N(a)^{-1}\bar{a} = -N(a)^{-1}a$ is a scalar multiple of $a$, this shows $\lambda_a(sx) = (asa^{-1})(ax)$; parentheses in $asa^{-1}$ are not needed by Artin's Theorem 1.1.(e). Now $a \in K^\perp$ yields $as = \bar{s}a$, and $\lambda_a(sx) = \bar{s}\lambda_a(x)$, as claimed. □

**2.9 Lemma.** *Let $C$ be a composition algebra over $F$, and let $K$ be a two-dimensional subalgebra of $C$. Moreover, assume that the restriction $N|_K$ of the norm is anisotropic, and that the restriction $\sigma := \kappa|_K$ is not trivial (in other words, assume that $K/F$ is a separable quadratic field extension, with $\mathrm{Gal}(K/F) = \langle\sigma\rangle$). Pick $c \in K \smallsetminus F$ with $T(c) = 1$. Then $j := (c - \bar{c})^{-1}$ is a pure element, and*

$$g(x, y) := j\left((cx|y) - \bar{c}(x|y)\right)$$

*defines a non-degenerate $\sigma$-hermitian form on $C$ (considered as a left vector space over $K$, see 1.1.(h)) with $g(x, x) = N(x)$ for each $x \in C$.*

*Every $K$-linear similitude of $N$ is a similitude of the form $g$, with the same multiplier.*

*Proof.* Clearly, the map $g \colon C \times C \to K$ is bilinear over $F$.

Using $\bar{c} = 1 - c$ and $c^2 = c\,T(c) - N(c) = c - N(c)$, we find $j^{-1}\,g(cx, y) = (c^2x|y) - \bar{c}(cx|y) = (cx|y) - Nc(x|y) - (cx|y) + c(cx|y) = c\left((cx|y) - \bar{c}(x|y)\right) = cj^{-1}g(x, y)$. Now $K = F + Fc$ yields $g(sx, y) = s\,g(x, y)$ for each $s \in K$.

Using $\bar{j} = -j$ and $(x|cy) = (\bar{c}x|y)$ (see 1.1.(c)), we compute $g(y, x) = \overline{g(x, y)} = \sigma(g(x, y))$. We have proved that $g$ is a $\sigma$-hermitian form.

Computing in the (associative) subalgebra generated by $c$ and $x$ we obtain $g(x, x) = j\left((cx|x) - \bar{c}(x|x)\right) = j\left(cx\bar{x} + x\overline{cx} - 2\bar{c}x\bar{x}\right) = j(c - \bar{c})x\bar{x} = x\bar{x} = N(x)$, as claimed. □

## Quaternion structures and corresponding hermitian forms

For suitable vector subspaces $W < \mathbb{O}$ of dimension three, we find that $\Lambda_W$ is the multiplicative group of an associative subalgebra both of $\mathbb{O}$ and of $\mathrm{End}_F(\mathbb{O})$, and $\mathbb{O}$ is, in a natural way, a left vector space over that subalgebra of $\mathrm{End}_F(\mathbb{O})$.

**2.10 Lemma.** *Let $W$ be a three-dimensional vector subspace of $\mathbb{O}$, with $1 \in W$. Let $M_W$ be the subalgebra generated by $W$ in $\mathbb{O}$.*

- **(a)** *The group $\Lambda_{W^\perp}$ normalizes $\Lambda_W$, and $\Lambda^+_{W^\perp}$ centralizes $\Lambda_W$; for each $u \in W^\perp \smallsetminus \{0\}$ conjugation by $\lambda_u$ induces the same algebra automorphism $\iota_W$ of $M_W$, with $\iota_W|_W = \kappa|_W$.*
- **(b)** *Mapping $a \in W$ to $\lambda_a$ has a unique extension to an algebra isomorphism $\zeta_W$ from $M_W$ onto a subalgebra $L$ of $\mathrm{End}_F(\mathbb{O})$. In particular, the group $\Lambda_W$ is the multiplicative group of $L$, and thus isomorphic to $M_W^*$.*

[5] There is a typo in the formulation of the lemma [28, 4.3]; it should read "$\bar{c} = 1 - c$" instead of "$\bar{c} = 1 + c$".

**(c)** *The additive group of $\mathbb{O}$ becomes a left vector space of dimension 2 over $L$ (and thus over $M_W$, via $\zeta_W$). We have $\Lambda_{W^\perp} \leqslant \Gamma\mathrm{L}_{M_W}(\mathbb{O})$ and $\Lambda^+_{W^\perp} \leqslant \mathrm{GL}_{M_W}(\mathbb{O})$.*

*Proof.* Consider $a \in W$ and $u \in W^\perp$. Then $u\bar{a} = -a\bar{u} = au$, and $-N(u)\bar{a}x = (uau)x = u(a(ux)) = -N(u)u^{-1}(a(ux))$ holds for all $x \in \mathbb{O}$ by one of Moufang's identities (see 1.1.(d)). If $u \neq 0$ then this means $\lambda_{\bar{a}} = \lambda_u^{-1} \circ \lambda_a \circ \lambda_u$, and $\Lambda^+_{W^\perp}$ centralizes $\Lambda_W$.

From 1.4 we know that $\Lambda^+_{W^\perp}$ acts transitively on $\mathbb{O}^*$, and thus irreducibly on $\mathbb{O}$. By Schur's lemma (cf. [17, 3.5, p. 118] or [24, Ch. XVII, Prop. 1.1]) the centralizer $B$ of $\Lambda^+_{W^\perp}$ in $\mathrm{End}_F(\mathbb{O})$ is a (not necessarily commutative) field which forms an algebra of finite dimension over $F$. In the previous paragraph we have obtained $\Lambda_W \leqslant B$. The subalgebra $M_W$ generated by $W$ in $\mathbb{O}$ forms a vector subspace of dimension 4 in $\mathbb{O}$ by Artin's Theorem (see 1.1.(e)), and is clearly invariant under $\Lambda_W$. Thus $M_W$ is also invariant under the subalgebra $L$ generated by $\Lambda_W$ in $\mathrm{End}_F(\mathbb{O})$. Now $L$ forms a subfield of $B$, and acts regularly on $M_W$. This yields $\dim L = 4$. As $M_W$ is associative, the linear map $\zeta_W \colon W \to L|_{M_W} : a \mapsto \lambda_a|_{M_W}$ extends to an algebra homomorphism from $M_W$ to $L$, as claimed, and that homomorphism is bijective by dimension reasons. The multiplicative group $M_W^*$ equals $WW$, cf. 1.2.

If $1, a, b$ is any basis for $W$ then $M_W = W \oplus Fab$ and $\iota_W(ab) = \bar{a}\bar{b} = \overline{ba}$; clearly $\iota_W^2 = \mathrm{id}$. The automorphism $\iota_W$ is the unique automorphism of $M_W$ with $\iota_W|_W = \kappa|_W$. The rest is clear. □

If $W \not\leqslant 1^\perp$ then $M_W$ is a quaternion field, and $\iota_W \neq \mathrm{id}$ is an inner automorphism (namely, conjugation by any non-trivial element of $W^\perp \cap M_W$). A hermitian form on the 2-dimensional left vector space $\mathbb{O}$ over $M_W$ will be constructed in 2.11 below (cf. [28, 4.2]).

If $W \leqslant 1^\perp$ (this case can occur only if $\mathrm{char}\, F = 2$) but $W \not\leqslant W^\perp$ then $\iota_W = \mathrm{id}$, but $M_W$ is still a quaternion field. We construct a hermitian form for that case in 2.13 below. However, if $W \leqslant W^\perp$ then the restriction of the polar form to $W$ is trivial, and $M_W$ is a commutative field (a totally inseparable extension of degree four over $F$, and every $F$-linear automorphism of $M_W$ is trivial). The construction of the form in 2.13 still works for the commutative case but yields an alternating form.

**2.11 Definition.** Let $W$ be a three-dimensional vector subspace of $\mathbb{O}$ such that $1 \in W \not\leqslant 1^\perp$. Choose $a, b \in W \setminus F$ with $a + \bar{a} = 1$ and $b \in \{1, a\}^\perp$, so $b\bar{a} = ab$ and $ba = \bar{a}b$.

Put $j := (a - \bar{a})^{-1} = (2a - 1)^{-1}$, then $j$ lies in $1^\perp \cap F(a) \subseteq b^\perp$, so $\bar{j} = -j$, $ja = aj$ and $bj = -jb$. Note that $j, a, b$ lie in the associative algebra generated by $a$ and $b$. We define

$$h_W(v, w) := j\left(a(v|w) - (v|aw)\right) + b^{-1}j\left(a(bv|w) - (bv|aw)\right) .$$

**2.12 Lemma.** *Let $W$ be a three-dimensional vector subspace of $\mathbb{O}$, assume $1 \in W \not\leqslant 1^\perp$, and let $M_W$ be the subalgebra of $\mathbb{O}$ generated by $W$. Endow $\mathbb{O}$ with the structure of a left vector space over $M_W$ as in 2.10. Then the map $h_W \colon \mathbb{O} \times \mathbb{O} \to M_W$ is a non-degenerate (in fact, anisotropic) hermitian form, with respect to the standard involution on $M_W$. For each $v \in \mathbb{O}$, we have $h_W(v, v) = N(v)$. Every $M_W$-linear similitude of $N$ is a similitude of the form $h_W$.*

*Proof.* In the following computations, we use $(bx|ay) = -(ax|by)$. By 1.1.(c), that equation is equivalent to the equation $(x|\bar{b}(ay)) + (x|\bar{a}(by)) = 0$ which can be verified as follows. Using 1.1.(d) and 1.1.(e) we see $a\left(\bar{b}(ay) + \bar{a}(by)\right) = \left((a\bar{b})a\right)y + (a\bar{a})(by)$; our assumption $a\bar{b} = -b\bar{a}$ then yields $(x|\bar{b}(ay) + \bar{a}(by)) = 0$.

We first note

$$
\begin{aligned}
\overline{h_W(w,v)} &= \big(\bar{a}(w|v) - (w|av)\big)\,\bar{j} + \big(\bar{a}(bw|v) - (bw|av)\big)\,\overline{b^{-1}j} \\
&= \big(\bar{a}(v|w) - (av|w)\big)\,\bar{j} + \big(\bar{a}(v|bw) - (av|bw)\big)\,(-j)(-b)^{-1} \\
&= j\,\big(-\bar{a}(v|w) + (v|\bar{a}w)\big) + \big(-\bar{a}(bv|w) + (bv|aw)\big)\,jb^{-1} \\
&= j\,\big(-\bar{a}(v|w) + (v|w) - (v|aw)\big) + \big(\bar{a}(bv|w) - (bv|aw)\big)\,b^{-1}j \\
&= j\,\big((-\bar{a}+1)(v|w) - (v|aw)\big) + b^{-1}j\,\big(a(bv|w) - (bv|aw)\big) \quad = \quad h_W(v,w)\,.
\end{aligned}
$$

From $a+\bar{a}=1$ we infer $a^2 = a - \bar{a}a$. Using this we obtain

$$
\begin{aligned}
h_W(av,w) &= j\,\big(a(av|w) - (av|aw)\big) + b^{-1}j\,\big(a(b(av)|w) - (b(av)|aw)\big) \\
&= j\,\big(a(v|\bar{a}w) - (v|\bar{a}(aw)\big) + b^{-1}j\,\big(-a(av|bw) + (a(av)|bw)\big) \\
&= j\,\big(a(v|w) - a(v|aw) - \bar{a}a(v|w)\big) + b^{-1}j\,\big(-a(av|bw) + (av|bw) - \bar{a}a(v|bw)\big) \\
&= j\,\big(a^2(v|w) - a(v|aw)\big) + b^{-1}j\,\big(\bar{a}(av|bw) - \bar{a}a(v|bw)\big) \\
&= aj\,\big(a(v|w) - (v|aw)\big) + ab^{-1}j\,\big((av|bw) - a(v|bw)\big) \\
&= aj\,\big(a(v|w) - (v|aw)\big) + ab^{-1}j\,\big(-(bv|aw) + a(bv|w)\big) \quad = \quad a\;h_W(v,w)\,.
\end{aligned}
$$

Now we use $b^2 = -N(b) \in F$ to obtain

$$
\begin{aligned}
h_W(bv,w) &= j\,\big(a(bv|w) - (bv|aw)\big) + b^{-1}j\,\big(a(b^2v|w) - (b^2v|aw)\big) \\
&= b\,b^{-1}j\,\big(a(bv|w) - (bv|aw)\big) + b\,j\,\big(a(v|w) - (v|aw)\big) \quad = \quad b\;h_W(v,w)\,.
\end{aligned}
$$

We have established that the $F$-linear map $\psi_w\colon \mathbb{O} \to M_W\colon v \mapsto h_W(v|w)$ centralizes both $\lambda_a$ and $\lambda_b$. Therefore, the centralizer of $\psi_w$ contains the subalgebra $L = \zeta_W(M_W)$ of $\mathrm{End}_F(\mathbb{O})$ generated by these left multiplications (cf. 2.10), and $\psi_w$ is $M_W$-linear. Together with the relation $\overline{h_W(w,v)} = h_W(v,w)$ from above, this completes the proof that $h_W$ is a hermitian form.

For $v \in \mathbb{O}$, we compute

$$
\begin{aligned}
h_W(v,v) &= j\,\big(a(v|v) - (v|av)\big) + b^{-1}j\,\big(a(bv|v) - (bv|av)\big) \\
&= N(v)\,\big(j\,\big(2a - (1|a)\big) + b^{-1}j\,\big(a(b|1) - (b|a)\big)\big) \\
&= N(v)\,\big(j(2a - a - \bar{a}) + b^{-1}j(ab + a\bar{b} - b\bar{a} - a\bar{b})\big) \quad = \quad N(v)(1+0) = N(v)\,,
\end{aligned}
$$

by our choice of $j = (a-\bar{a})^{-1}$ and $b\bar{a} = ab$. In particular, the form $h_W$ is anisotropic, and thus not degenerate.

Let $\varphi$ be an $M_W$-linear similitude of $N$, with multiplier $s \in F$. Then $(\varphi(v)|\varphi(w)) = s(v|w)$ and $\varphi\left(\zeta_W(x)(v)\right) = \zeta_W(x)\left(\varphi(v)\right)$ hold for all $v, w \in \mathbb{O}$ and each $x \in M_W$. Note that $\zeta_W(x)(v) = xv$ if $x \in \{a,b\} \subset W$. We compute

$$
\begin{aligned}
h_W(\varphi(v),\varphi(w)) &= j\,\big(a(\varphi(v)|\varphi(w)) - (\varphi(v)|a\varphi(w))\big) + b^{-1}j\,\big(a(b\varphi(v)|\varphi(w)) - (b\varphi(v)|a\varphi(w))\big) \\
&= j\,\big(a(\varphi(v)|\varphi(w)) - (\varphi(v)|\varphi(aw))\big) + b^{-1}j\,\big(a(\varphi(bv)|\varphi(w)) - (\varphi(bv)|\varphi(aw))\big) \\
&= j\,\big(as(v|w) - s(v|aw)\big) + b^{-1}j\,\big(as(bv|w) - s(bv|aw)\big) \\
&= s\,h_W(v,w)\,.
\end{aligned}
$$

Thus we see that $\varphi$ is a similitude of $h_W$, with the same multiplier $s$. □

**2.13 Lemma.** *Assume that $W \leqslant \mathbb{O}$ is a three-dimensional vector space such that $1 \in W \leqslant 1^\perp$. Choose $a, b \in W$ such that $1, a, b$ is a basis for $W$, and such that $t := (a|b)$ equals $1$ if $W \nleqslant W^\perp$.*

**(a)** *The map $h_W^\perp \colon \mathbb{O} \times \mathbb{O} \to M_W$ defined by*

$$h_W^\perp(v, w) := (ab)(v|w) + b(v|aw) + a(v|bw) + (v|a(bw))$$

*is a non-degenerate sesquilinear form.*

**(b)** *If $W \nleqslant W^\perp$ then $h_W^\perp$ is hermitian, with respect to the standard involution on the algebra $M_W$ generated by $W$. For each $v \in \mathbb{O}$, we then have $h_W^\perp(v, v) = N(v)$.*

**(c)** *If $W \leqslant W^\perp$ then $h_W^\perp$ is alternating; in particular, we have $h_W^\perp(v, v) = 0$ for each $v \in \mathbb{O}$.*

**(d)** *In any case, every $M_W$-linear similitude of $N$ is a similitude of the form $h_W^\perp$.*

*Proof.* It is easy to see that every $M_W$-linear similitude of $N$ is a similitude of the form $h_W^\perp$, see the arguments given in the proof of 2.12 above.

We note that $1 \in W \leqslant 1^\perp$ implies $\operatorname{char} F = 2$ and then that the polar form of the norm $N$ is alternating. In the following arguments, we use $N(c) = c^2$ and $(1|c) = 0 = (c|c)$ for $c \in W$ without further mention. Using one of Moufang's identities (cf. 1.1.(d)) and $t = (a|b)$, we compute $b(aw) = b(a(b(b^{-1}w) = N(b)^{-1}(bab)(bw) = N(b)^{-1}\left((t - ab)\,b\right)(bw) = N(b)^{-1}\left((t + ab)b\right)(bw) = N(b)^{-1}\left(b^2tw + ab^2(bw)\right) = tw + a(bw)$.

In order to see that $v \mapsto h_W^\perp(v, w)$ is $M_W$-linear, it suffices to check the pertinent relation for the generators $a, b$. We obtain

$$\begin{aligned} h_W^\perp(av, w) &= (ab)(av|w) + b(av|aw) + a(av|bw) + (av|a(bw)) \\ &= ba^2(v|w) + (ab)(v|aw) + a^2(v|bw) + a(v|a(bw)) \quad = \quad a\,h_W^\perp(v, w)\,. \end{aligned}$$

Fix $w \in \mathbb{O}$. With the relation $b(aw) = tw + a(bw)$ from above we compute

$$\begin{aligned} h_W^\perp(bv, w) &= (ab)(bv|w) + b(bv|aw) + a(bv|bw) + (bv|a(bw)) \\ &= (ab)(v|bw) + b(v|b(aw)) + ab^2(v|w) + (bv|b(aw) - tw) \\ &= ab^2(v|w) + b^2(v|aw) + (ab)(v|bw) + (bv|tw) + b(v|a(bw)) - (bv|tw) \\ &= b\,h_W^\perp(v, w)\,. \end{aligned}$$

Using $b(aw) = tw + a(bw)$ again, we find

$$\begin{aligned} h_W^\perp(w, v) &= (ab)(w|v) + b(w|av) + a(w|bv) + (w|a(bv)) \\ &= (ab)(v|w) + b(av|w) + a(bv|w) + (a(bv)|w) \\ &= (ab)(v|w) + b(v|aw) + a(v|bw) + (v|b(aw)) \\ &= (ab + t)(v|w) + b(v|aw) + a(v|bw) + (v|a(bw)) \quad = \quad \overline{h_W^\perp(v, w)}\,, \end{aligned}$$

so the form $h_W^\perp$ is indeed hermitian or symmetric, depending on the restriction $\kappa|_{M_W}$.

In any case, we have

$$\begin{aligned} h_W^\perp(v, v) &= (ab)(v|v) + b(v|av) + a(v|bv) + (v|a(bv)) \\ &= 0 + bN(v)(1|a) + aN(v)(1|b) + (av|bv) \quad = \quad N(v)(a|b)\,. \end{aligned}$$

If $W \nleqslant W^\perp$ then our assumption $(a|b) = 1$ yields $h_W^\perp(v, v) = N(v)$ for each $v \in \mathbb{O}$, and the form $h_W^\perp$ is anisotropic (like $N$) and thus not degenerate. If $W \leqslant W^\perp$ then $h_W^\perp(v, v) = 0$ for each $v \in \mathbb{O}$, and the form $h_W^\perp$ is alternating. In order to see non-degeneracy in that case, pick $u \in \{a, b, ab\}^\perp$ such that $(1|u) = 1$; then $h_W^\perp(u, 1) = ab \neq 0$. □

## 3 Autotopisms and anti-autotopisms

**3.1 Definition.** Let $C$ be any algebra. An *autotopism* of $C$ is a triplet[6] $(\alpha|\beta|\gamma)$ of additive bijections of $C$ such that $\beta(sx) = \gamma(s)\alpha(x)$ holds for all $s, x \in C$. An *anti-autotopism* of $C$ is a triplet $(\delta|\varepsilon|\varphi)$ of additive bijections of $C$ such that $\varepsilon(sx) = \delta(x)\varphi(s)$ holds for all $s, x \in C$.

If $(\alpha|\beta|\gamma)$ is an autotopism then it is already determined by any one of the maps $\alpha$, $\beta$, or $\gamma$ together with a single non-zero value of any one of the remaining two maps. For instance, we have $\gamma(1)\alpha(x) = \beta(x) = \gamma(x)\alpha(1)$ for each $x \in C$. Being a component of an autotopism imposes severe restrictions on the additive bijection, see 3.3.(e) and 3.11 below. However, each automorphism $\alpha$ of $C$ yields an autotopism $(\alpha|\alpha|\alpha)$, and each anti-automorphism $\beta$ of $C$ yields an anti-autotopism $(\beta|\beta|\beta)$ of $C$. In particular, the standard involution $\kappa$ yields the anti-autotopism $(\kappa|\kappa|\kappa)$.

The set of all autotopisms forms a subgroup $\Delta$ of the direct product $\mathrm{Aut}(C,+)^3$. The set $\Delta_F := \Delta \cap (\mathrm{GL}_F(\mathbb{O}))^3$ of all linear autotopisms (cf. 3.3.(d)) is a subgroup of $\Delta$. The multiplication is more involved if anti-autotopisms enter the stage: for anti-autotopisms $(\delta|\varepsilon|\varphi)$, $(\delta'|\varepsilon'|\varphi')$ and autotopisms $(\alpha|\beta|\gamma)$, $(\alpha'|\beta'|\gamma')$ we have $(\delta|\varepsilon|\varphi)(\delta'|\varepsilon'|\varphi') = (\varphi \circ \delta'|\varepsilon \circ \varepsilon'|\delta \circ \varphi')$, $(\alpha|\beta|\gamma)(\delta'|\varepsilon'|\varphi') = (\gamma \circ \delta'|\beta \circ \varepsilon'|\alpha \circ \varphi')$ and $(\delta|\varepsilon|\varphi)(\alpha'|\beta'|\gamma') = (\delta \circ \alpha'|\varepsilon \circ \beta'|\varphi \circ \gamma')$. The motivation for both the definition and the multiplication formulas comes from the theory of projective planes; autotopisms are used to describe elements of a triangle stabilizer, while anti-autopisms describe dualities fixing a triangle.

If $C$ is a division algebra, we consider the affine plane over $C$, with point set $C^2$, vertical lines $[c] := \{c\} \times C$ with $c \in C$, and lines $[s,t] := \{(x, sx+t) \mid x \in C\}$ of slope $s$ and intercept $t$. The lines of given slope $s$ form a parallel class, the corresponding point at infinity is denoted by $(s)$. Lines of slope $0$ are called horizontal.

The group $\Delta$ of autotopisms describes the stabilizer of a triangle[7] in the projective completion of that affine plane; the vertices of that triangle are the origin $(0,0)$ and the points at infinity for the coordinate axes $[0,0]$ and $[0]$. Indeed, the action of $(\alpha|\beta|\gamma)$ on the sets of points and lines are given by $(\alpha|\beta|\gamma).(x,y) = (\alpha(x), \beta(y))$, $(\alpha|\beta|\gamma).[s,t] = [\gamma(s), \beta(t)]$, and $(\alpha|\beta|\gamma).[c] = [\alpha(c)]$. Thus $\alpha$ and $\beta$ give the actions on the horizontal and vertical axis, respectively, while $\gamma$ gives the action on slopes (and thus on the line at infinity).

The global stabilizer of the triangle is the semi-direct product of $\Delta$ with a dihedral group of order $6$ which we introduce in the next lemma.

**3.2 Lemma.** *Let $\mathbb{O}$ be an octonion field.*

- **(a)** *Mapping $(x,y) \in \mathbb{O} \times \mathbb{O}$ to $(y,x)$ and any non-horizontal line $[s,t]$ to $[s^{-1}, -s^{-1}t]$ extends to an involutory automorphism $\vartheta_1$ of $\mathbb{P}_2(\mathbb{O})$ with axis $[1,0]$ and center $(-1)$.*
- **(b)** *Mapping $(x,y) \in \mathbb{O} \times (\mathbb{O} \smallsetminus \{0\})$ to $(xy^{-1}, y^{-1})$ and $[s,t]$ to $[-t^{-1}s, t^{-1}]$ extends to an involutory automorphism $\vartheta_2$ of $\mathbb{P}_2(\mathbb{O})$ with axis $[0,1]$ and center $(0,-1)$.*
- **(c)** *The conjugate $\vartheta_3 := \vartheta_1 \circ \vartheta_2 \circ \vartheta_1$ coincides with $\vartheta_2 \circ \vartheta_1 \circ \vartheta_2$. Consequently, the product $\vartheta_1 \circ \vartheta_2$ has order $3$, and the group generated by these automorphisms is dihedral of order $6$.*

*Proof.* Using 1.1.(f) one easily verifies that $\vartheta_1$ extends to an automorphism of the affine plane, and then to an automorphism of $\mathbb{P}_2(\mathbb{O})$. The axial involution $\vartheta_2$ is taken from [25, 3.5 (22),

[6] As a reminder for the reader, triplets that are (anti-)autotopisms will be written as $(\alpha|\beta|\gamma)$ rather than $(\alpha, \beta, \gamma)$.
[7] See [6, 3.1.32] for that interpretation of autotopisms as collineations; cf. also [21, Sect. 1].

p. 107], cf. [19, 2.3]. The effect of the products $\vartheta_1 \circ \vartheta_2 \circ \vartheta_1$ and $\vartheta_2 \circ \vartheta_1 \circ \vartheta_2$ on $(x, y)$ is easily computed (using associativity of the subalgebra generated by $x$ and $y$, cf. 1.1.(e)). □

**3.3 Lemma.** *Let $\mathbb{O}$ be an octonion field over $F$.*

**(a)** *If $(\alpha|\beta|\gamma)$ is an (anti-)autotopism of $\mathbb{O}$ and $r, s, t \in F^*$ are scalars then $(r\alpha|s\beta|t\gamma)$ is an (anti-)autotopism precisely if $s = tr$.*

**(b)** *Any one of the triplets $(\alpha|\beta|\mathrm{id})$, $(\mathrm{id}|\beta|\alpha)$ is an autotopism if, and only if, there exists $z \in F^*$ such that $\alpha = \lambda_z = \beta$, and $(\alpha|\mathrm{id}|\gamma)$ is an autotopism precisely if there exists $z \in F^*$ such that $\alpha = \lambda_z = \gamma^{-1}$.*

**(c)** *If $(\alpha|\beta|\gamma)$ and $(\alpha'|\beta'|\gamma)$ are both autotopisms then there exists $r \in F^*$ such that $(\alpha'|\beta'|\gamma) = (r\alpha|r\beta|\gamma)$. Conversely, if $(\alpha|\beta|\gamma)$ is an autotopism then $(r\alpha|r\beta|\gamma)$ is one, for each $r \in F^*$.*

**(d)** *Let $(\alpha|\beta|\gamma)$ be an (anti-)autotopism and let $\varphi_\xi$ denote the companion of the semi-linear map $\xi$. Then $\varphi_\alpha = \varphi_\beta = \varphi_\gamma$.*

*In particular, if one component of an (anti-)autotopism is $F$-linear then all three components are $F$-linear. We briefly call the (anti-)autotopism linear in this case.*

**(e)** *Every linear autotopism consists of similitudes of the norm form of $\mathbb{O}$, and every autotopism consists of semi-similitudes (in particular, semi-linear maps).*

**(f)** *An automorphism $\varphi$ of $F$ occurs as the companion of a semi-similitude of the norm form if, and only if, it occurs as the companion of an automorphism of $\mathbb{O}$.*

*Proof.* A straightforward computation yields assertion (a).

In order to prove assertions (b) and (c), we interpret autotopisms as elements of the triangle stabilizer in the projective completion of the affine plane over $\mathbb{O}$. If one of the entries of an autotopism is trivial then the corresponding element of the triangle stabilizer acts trivially on one side $L$ of the triangle, and acts trivially on the line pencil in the vertex $p$ opposite that side. Thus it is a homology with center $p$ and axis $L$. It is well known that, in every affine plane over an octonion field, these homologies are precisely the maps of the form $(x, y) \mapsto (zx, zy)$ if $p = (0, 0)$, of the form $(x, y) \mapsto (x, zy)$ if $L = [0, 0]$, of the form $(x, y) \mapsto (zx, y)$ if $L = [0]$, respectively, where $z \in F^*$ (see [18, 1.22], together with the fact that the center of an octonion algebra coincides with each one of its nuclei, cf. 1.1.(i)). Thus we have proved assertion (b). The quotient $(\alpha'|\beta'|\gamma)^{-1}(\alpha|\beta|\gamma)$ then acts trivially on the line at infinity, and assertion (c) follows from assertion (b).

In order to verify assertion (d), we consider $s, t \in F$ and compare $\varphi_\beta(st)\beta(1) = \beta((st)1) = \beta((s1)(t1))$ with the product of $\varphi_\alpha(s)\alpha(1) = \alpha(s1)$ and $\varphi_\gamma(t)\gamma(1) = \gamma(t1)$ in the suitable order. This yields that $(\varphi_\alpha|\varphi_\beta|\varphi_\gamma)$ is an autotopism of $F$. As the components of that autotopism are automorphisms, they coincide.

Assertion (e) is known, see [21, Cor 1.9]. Assertion (f) is also known, see [27, 1.7.2]. □

Assertion 3.3.(c) treats the third component in a special way. It can easily be transferred to the other components (mutatis mutandis), either by a direct argument or by an application of the following famous principle of triality.

**3.4 Lemma** (Triality for autotopisms[8]). *If $(\alpha|\beta|\gamma)$ is an autotopism of an octonion field $\mathbb{O}$ then $\tau(\alpha|\beta|\gamma) := (\gamma|\mu_\alpha^{-1}\kappa\circ\alpha\circ\kappa|\mu_\beta^{-1}\kappa\circ\beta\circ\kappa)$ is an autotopism, as well. The map $\tau$ is an automorphism of $\Delta$, and has order three.*

*Proof.* We note $\tau^2(\alpha|\beta|\gamma) := (\mu_\beta^{-1}\kappa\circ\beta\circ\kappa|\mu_\gamma^{-1}\kappa\circ\gamma\circ\kappa|\alpha)$, and conclude that $\tau$ has order three. Checking that $\tau(\alpha|\beta|\gamma)$ is an autotopism amounts to verification of the equation $\mu_\beta^{-1}\overline{\beta(\overline{s})}\,\gamma(x) = \mu_\alpha^{-1}\overline{\alpha(\overline{sx})}$. Multiplying with $\overline{\gamma(x)}$ from the right we obtain $\mu_\beta^{-1}\overline{\beta(\overline{s})}\,N(\gamma(x)) = \mu_\alpha^{-1}\overline{\alpha(\overline{sx})}\,\overline{\gamma(x)}$. The latter equation is equivalent to $\mu_\beta^{-1}\mu_\gamma\beta(\overline{s})\varphi(N(x)) = \mu_\alpha^{-1}\gamma(x)\,\alpha(\overline{sx})$. As $(\alpha|\beta|\gamma)$ is an autotopism, the right hand side equals $\mu_\alpha^{-1}\beta(x\overline{x}\,\overline{s}) = \varphi(N(x))\beta(\overline{s})$, and $\mu_\beta = \mu_\gamma\mu_\alpha$ yields the claim.

It is straightforward to verify that $\tau$ is a group homomorphism on $\Delta$. □

**3.5 Remark.** In fact, we obtain $\tau$ as conjugation by $\vartheta_2\circ\vartheta_1$ from 3.2; this observation could replace the arguments in the proof of 3.4. In order to verify that claim, we note first that $\left(\delta(x^{-1})\right)^{-1} = \mu_\delta^{-1}\overline{\delta(\overline{x})}$ holds for each semi-similitude $\delta$ and each $x\in\mathbb{O}^*$. Conjugation by $\vartheta_1$ induces the involutory automorphism $\pi_3\colon (\alpha|\beta|\gamma)\mapsto(\beta|\alpha|\mu_\gamma^{-1}\kappa\circ\gamma\circ\kappa)$ on $\Delta$, and conjugation by $\vartheta_2$ gives the involution $\pi_2\colon (\alpha|\beta|\gamma)\mapsto(\mu_\gamma^{-1}\kappa\circ\gamma\circ\kappa|\mu_\beta^{-1}\kappa\circ\beta\circ\kappa|\mu_\alpha^{-1}\kappa\circ\alpha\circ\kappa)$. Thus the dihedral group generated by $\vartheta_1$ and $\vartheta_2$ acts faithfully as a group of automorphisms on $\Delta$.

**3.6 Examples.** For each $u\in\mathbb{O}^*$ the triplet $(\rho_u|\lambda_u\circ\rho_u|\lambda_u)$ is an autotopism; we use Moufang's identity $u((sx)u) = (us)(xu)$ here, see 1.1.(d). The triplet $(\frac{-1}{N(u)}\rho_u|\frac{-1}{N(u)}\lambda_u\circ\rho_u|\lambda_u)$ is an autotopism by 3.3.(c). Note that $\frac{-1}{N(u)}\rho_u = \rho_u^{-1}$ holds if $u\in\operatorname{Pu}\mathbb{O}$, so the latter autotopism is $(\rho_u^{-1}|\lambda_u\circ\rho_u^{-1}|\lambda_u)$ in that case.

Our triality $\tau$ from 3.4 maps the autotopism $(\rho_u|\lambda_u\circ\rho_u|\lambda_u)$ to $(\lambda_u|N(u)^{-1}\lambda_{\overline{u}}|N(u)^{-2}\lambda_{\overline{u}}\circ\rho_{\overline{u}})$ and then on to $(N(u)^{-2}\lambda_{\overline{u}}\circ\rho_{\overline{u}}|N(u)^{-1}\rho_{\overline{u}}|\rho_u)$. Using 3.3.(c) again, we infer that the triplet $(N(u)^{-1}\lambda_{\overline{u}}\circ\rho_{\overline{u}}|\rho_{\overline{u}}|\rho_u) = (\lambda_u^{-1}\circ\rho_{\overline{u}}|\rho_{\overline{u}}|\rho_u)$ is an autotopism. Application of $\pi_3$ from 3.5 to $(\rho_u|\lambda_u\circ\rho_u|\lambda_u)$ yields that $(\lambda_u\circ\rho_u|\rho_u|N(u)^{-1}\kappa\circ\lambda_u\circ\kappa) = (\lambda_u\circ\rho_u|\rho_u|N(u)^{-1}\rho_{\overline{u}})$ is an autotopism, as well.

The standard involution $\kappa$ is an anti-automorphism, and yields an anti-autotopism $(\kappa|\kappa|\kappa)$. Multiplication (cf. 3.1) gives $(\alpha|\beta|\gamma)(\kappa|\kappa|\kappa) = (\gamma\circ\kappa|\beta\circ\kappa|\alpha\circ\kappa)$. So $(\rho_u|\lambda_u\circ\rho_u|\lambda_u)(\kappa|\kappa|\kappa) = (\lambda_u\circ\kappa|\lambda_u\circ\rho_u\circ\kappa|\rho_u\circ\kappa) = (\lambda_u\circ\kappa|\lambda_u\circ\kappa\circ\lambda_{\overline{u}}|\kappa\circ\lambda_{\overline{u}}) = (\lambda_u\circ\kappa|-N(u)\sigma_u|N(u)\kappa\circ\lambda_u^{-1})$ is an anti-autotopism, and $(\lambda_u\circ\kappa|\sigma_u|-\kappa\circ\lambda_u^{-1})$ is an anti-autotopism, as well (see 3.3.(a)).

**3.7 Definition.** For $j\in\{1,2,3\}$, the map $\operatorname{pr}^j\colon \Delta\to\Gamma\mathrm{O}(\mathbb{O},N)\colon (\alpha_1|\alpha_2|\alpha_3)\mapsto\alpha_j$ is a group homomorphism.

Note that triality (see 3.4) cyclically interchanges the kernels and images of $\operatorname{pr}^1$, $\operatorname{pr}^2$, and $\operatorname{pr}^3$, as defined in in 3.7. The homomorphism $\operatorname{pr}^j\colon\Delta\to\Gamma\mathrm{O}(\mathbb{O},N)$ is never surjective, and its restriction to the group $\Delta_F$ of linear autotopisms is also not surjective onto $\mathrm{GO}(\mathbb{O},N)$. We clarify the situation for the linear autotopisms; it turns out (in 3.11 below) that the following definition gives a direct description of $\operatorname{pr}^j(\Delta_F) = \operatorname{pr}^j(\Delta)\cap\mathrm{GO}(\mathbb{O},N)$.

**3.8 Definition.** The subgroup $\mathrm{GO}^+(\mathbb{O},N) := \langle\{\lambda_u \mid u\in\mathbb{O}^*\}\cup\{\rho_u \mid u\in\mathbb{O}^*\}\rangle \leqslant \mathrm{GO}(\mathbb{O},N)$ is called the group of *direct similitudes* of the norm form.

**3.9 Lemma.** *Let $V\leqslant\mathbb{O}$ be any vector subspace of dimension at least $5$. Then $\mathrm{GO}(\mathbb{O},N)$ is generated by $\{\kappa\}\cup\left\{\lambda_v \;\middle|\; v\in V\right\}$.*

[8] See also [27, 3.3.2], but note that the maps given there do not have order three.

*Proof.* Let $\Psi := \left\langle \{\kappa\} \cup \left\{ \lambda_v \,\middle|\, v \in V \right\} \right\rangle \leqslant \mathrm{GO}(\mathbb{O}, N)$. The subgroup $\Lambda_V = \left\langle \lambda_v \,\middle|\, v \in V \right\rangle$ is transitive on $\mathbb{O}^*$ by 1.4. For $u \in \mathbb{O}^*$, pick $\psi_u \in \Lambda_V$ with $\psi_u(1) = u$. Then $\Psi$ contains the involution $\kappa = -\sigma_1$ and its conjugate $-\sigma_u = \psi_u \circ \kappa \circ \psi_u^{-1}$. The group $\left\langle -\sigma_u \,\middle|\, u \in \mathbb{O}^* \right\rangle$ contains $\left\langle (-\sigma_u) \circ (-\sigma_v) \,\middle|\, u, v \in \mathbb{O}^* \right\rangle = \mathrm{O}^+(\mathbb{O}, N)$. As $-\mathrm{id}$ lies in $\mathrm{O}^+(\mathbb{O}, N)$, the Cartan-Dieudonné Theorem (cf. 2.5.(a)) also implies $\left\langle -\sigma_u \,\middle|\, u \in \mathbb{O}^* \right\rangle = \mathrm{O}(\mathbb{O}, N)$. The stabilizer $\mathrm{GO}(\mathbb{O}, N)_1$ is contained in $\mathrm{O}(\mathbb{O}, N) \leqslant \Psi$, and transitivity of $\Lambda_V \leqslant \Psi$ yields $\mathrm{GO}(\mathbb{O}, N) \leqslant \Psi$. □

**3.10 Lemma.** *Let $C$ be a division algebra, and let $(\alpha|\beta|\gamma)$ be an anti-autotopism of $C$. If $\gamma = \mathrm{id}$ then $C$ is isotopic to a commutative algebra.*

*Proof.* Being an anti-autotopism means $\beta(sx) = \alpha(x)\gamma(s)$ for all $s, x \in C$. Specializing $s = 1$ we find $\beta = \alpha$. Specializing $x = 1$ we then find $\alpha(s) = \beta(s) = as$ for each $s \in C$, where $a := \alpha(1)$. The existence of $a \in C \smallsetminus \{0\}$ with $a(sx) = (ax)s$ for all $s, x \in C$ implies that $C$ is isotopic to a commutative algebra; see [20, p. 592], cf. [10]. □

**3.11 Theorem.** *For each $\gamma \in \mathrm{GO}(\mathbb{O}, N)$ there exists either an autotopism $(\alpha|\beta|\gamma)$ or an anti-autotopism $(\hat\alpha|\hat\beta|\gamma)$, but not both. The set $\mathrm{pr}^3(\Delta_F)$ of those $\gamma \in \mathrm{GO}(\mathbb{O}, N)$ that occur with autotopisms of $\mathbb{O}$ is the subgroup $\mathrm{GO}^+(\mathbb{O}, N)$. In particular, the subgroup $\mathrm{GO}^+(\mathbb{O}, N)$ has index 2 in $\mathrm{GO}(\mathbb{O}, N)$, and we have a semidirect product $\mathrm{GO}(\mathbb{O}, N) = \mathrm{GO}^+(\mathbb{O}, N) \rtimes \langle\kappa\rangle$.*

*Proof.* Assume that there is an autotopism $(\alpha|\beta|\gamma)$ and an anti-autotopism $(\hat\alpha|\hat\beta|\gamma)$. Then $(\hat\alpha \circ \alpha^{-1}|\hat\beta \circ \beta^{-1}|\gamma \circ \gamma^{-1})$ is an anti-autotopism. From 3.10 we then infer that $\mathbb{O}$ is isotopic to a commutative algebra. Every algebra isotopic to $\mathbb{O}$ is in fact isomorphic to $\mathbb{O}$ (see [22, 2.6.3]), and we reach a contradiction.

In order to show existence of suitable (anti-)autotopisms, we recall from 3.9 that the group $\mathrm{GO}(\mathbb{O}, N)$ is generated by $\{\kappa\} \cup \left\{ \lambda_u \,\middle|\, u \in \mathbb{O}^* \right\}$, that $(\kappa|\kappa|\kappa)$ is an anti-autotopism, and that $(\rho_u|\lambda_u \circ \rho_u|\lambda_u)$ and $(\lambda_u^{-1} \circ \rho_{\bar u}|\rho_{\bar u}|\rho_u)$ are autotopisms (for each $u \in \mathbb{O}^*$, see 3.6). Thus every element of $\mathrm{GO}^+(\mathbb{O}, N) = \langle \{\lambda_u \mid u \in \mathbb{O}^*\} \cup \{\rho_u \mid u \in \mathbb{O}^*\} \rangle \leqslant \mathrm{pr}^3(\Delta_F)$. On the other hand, $(\kappa|\kappa|\kappa)$ is an anti-autotopism, and $\kappa \notin \mathrm{pr}^3(\Delta_F)$.

Now $\kappa \circ \lambda_u \circ \kappa = \rho_{\bar u}$ yields that $\kappa$ normalizes $\mathrm{GO}^+(\mathbb{O}, N)$, and that $\mathrm{GO}^+(\mathbb{O}, N)$ is a subgroup of index 2 in $\mathrm{GO}(\mathbb{O}, N)$. We find $\mathrm{GO}^+(\mathbb{O}, N) \leqslant \mathrm{pr}^3(\Delta_F) < \mathrm{GO}(\mathbb{O}, N)$, and $\mathrm{GO}^+(\mathbb{O}, N) = \mathrm{pr}^3(\Delta_F)$ follows. □

**3.12 Remarks.** The group $\Lambda_{\mathbb{O}} = \left\langle \lambda_u \,\middle|\, u \in \mathbb{O}^* \right\rangle$ is normalized[9] by $\kappa \circ \Lambda_{\mathbb{O}} \circ \kappa = \left\langle \rho_u \,\middle|\, u \in \mathbb{O}^* \right\rangle$, and $\Lambda_{\mathbb{O}}$ is thus a normal subgroup of $\mathrm{GO}^+(\mathbb{O}, N)$ (but not normal in $\mathrm{GO}(\mathbb{O}, N)$, in general). In order to see normality, we use one of Moufang's identities (see 1.1.(d)): For all $a, b \in \mathbb{O}^*$ and $x \in \mathbb{O}$ we have $(ab)x = (ab)((xa^{-1})a) = a((b(xa^{-1}))a)$. This means $\lambda_a^{-1} \circ \lambda_{ab} = \rho_a \circ \lambda_b \circ \rho_a^{-1}$. The product $\Lambda_{\mathbb{O}} \circ (\kappa \circ \Lambda_{\mathbb{O}} \circ \kappa)$ of two normal subgroups is a subgroup of $\mathrm{GO}^+(\mathbb{O}, N)$ and contains a set of generators; thus $\mathrm{GO}^+(\mathbb{O}, N) = \Lambda_{\mathbb{O}} \circ (\kappa \circ \Lambda_{\mathbb{O}} \circ \kappa)$.

Using a generalization of the Spinor norm, one sees that the group $\Lambda_{\mathrm{Pu}\,\mathbb{O}}$ is a proper subgroup of $\Lambda_{\mathbb{O}}$, in general. As each element of $\mathbb{O}$ is of the form $ab$ with $a, b \in \mathrm{Pu}\,\mathbb{O}$ (see 1.2), our formula $\lambda_{ab} = \lambda_a \circ \rho_a \circ \lambda_b \circ \rho_a^{-1}$ also shows that $\Lambda_{\mathrm{Pu}\,\mathbb{O}}$ is not normalized by $\left\langle \rho_a \,\middle|\, a \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\} \right\rangle$ if $\Lambda_{\mathrm{Pu}\,\mathbb{O}} \neq \Lambda_{\mathbb{O}}$.

[9] We took this idea from Theo Grundhöfer's paper [12, p. 448].

**3.13 Theorem.** **(a)** *We have* $\mathrm{GO}^+(\mathbb{O},N) \cap \mathrm{O}(\mathbb{O},N) = \mathrm{O}^+(\mathbb{O},N)$*; cf. 2.5.*

**(b)** *The stabilizer* $\mathrm{GO}(\mathbb{O},N)_{\{1,-1\}}$ *of the set* $\{1,-1\}$ *equals the direct product* $\mathrm{O}(\mathbb{O},N)_{\{1,-1\}} = \langle -\kappa\rangle \circ \langle \delta_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle$. *(Recall that* $\delta_u = \frac{1}{N(u)}\lambda_u \circ \rho_u = -\kappa\circ\sigma_u = -\sigma_u\circ\kappa$*, cf. 2.6.)*

**(c)** *The stabilizer* $\mathrm{GO}^+(\mathbb{O},N)_{\{1,-1\}} = \mathrm{O}^+(\mathbb{O},N)_{\{1,-1\}}$ *is generated by* $\{\delta_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\}$.

**(d)** *If* $\mathrm{char}\,F \neq 2$ *then* $\mathrm{GO}^+(\mathbb{O},N)_1 = \mathrm{O}^+(\mathbb{O},N)_1 = \langle \delta_u \circ \delta_v \mid u,v \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle$, *and* $\mathrm{GO}(\mathbb{O},N)_1 = \langle -\kappa\circ\delta_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle = \langle \sigma_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle = \langle\kappa\rangle \circ \mathrm{O}^+(\mathbb{O},N)_1$.

*Applying 2.6.(b), we see that* $\mathrm{GO}^+(\mathbb{O},N)_1$ *induces* $\mathrm{O}^+(\mathrm{Pu}\,\mathbb{O}, N|_{\mathrm{Pu}\,\mathbb{O}})$ *on* $\mathrm{Pu}\,\mathbb{O}$*; the kernel of the restriction map is trivial because* $-\kappa$ *does not belong to* $\mathrm{GO}^+(\mathbb{O},N)$.

*If* $\mathrm{char}\,F = 2$ *then* $\{1,-1\} = \{1\}$. *We find* $\mathrm{GO}^+(\mathbb{O},N)_{\{1,-1\}} = \mathrm{GO}^+(\mathbb{O},N)_1$, *and* $\mathrm{GO}(\mathbb{O},N)_{\{1,-1\}} = \mathrm{GO}(\mathbb{O},N)_1 = \langle\kappa\rangle \circ \mathrm{GO}^+(\mathbb{O},N)_1$ *in that case. Thus* $\mathrm{GO}^+(\mathbb{O},N)_1$ *induces* $\langle \delta_u|_{\mathrm{Pu}\,\mathbb{O}} \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle = \langle \sigma_u|_{\mathrm{Pu}\,\mathbb{O}} \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle = \mathrm{O}(\mathrm{Pu}\,\mathbb{O}, N|_{\mathrm{Pu}\,\mathbb{O}}) = \mathrm{O}^+(\mathrm{Pu}\,\mathbb{O}, N|_{\mathrm{Pu}\,\mathbb{O}})$ *on* $\mathrm{Pu}\,\mathbb{O}$ *(cf. 2.5.(d)); the kernel of the restriction map is trivial because* $-\kappa$ *does not belong to* $\mathrm{GO}^+(\mathbb{O},N)$.

**(e)** $\mathrm{GO}^+(\mathbb{O},N) = \langle \{\lambda_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\} \cup \{\rho_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\}\rangle$.

**(f)** *If* $\mathrm{char}\,F \neq 2$ *then* $\mathrm{GO}^+(\mathbb{O},N) = \{\varphi \in \mathrm{GO}(\mathbb{O},N) \mid \det\varphi = \mu_\varphi^4\}$.

*If* $\mathrm{char}\,F = 2$ *then the latter group coincides with* $\mathrm{GO}(\mathbb{O},N)$.

*Proof.* From 2.6.(c) we infer that $\mathrm{O}^+(\mathbb{O},N) = \langle \delta_u \mid u \in \mathbb{O}^*\rangle$ is contained in $\mathrm{GO}^+(\mathbb{O},N)$. The reflection $\kappa$ is contained in $\mathrm{O}(\mathbb{O},N)$ but neither in $\mathrm{O}^+(\mathbb{O},N)$ nor in $\mathrm{GO}^+(\mathbb{O},N)$. Now the inclusions $\mathrm{O}^+(\mathbb{O},N) \leqslant \mathrm{GO}^+(\mathbb{O},N) \cap \mathrm{O}(\mathbb{O},N) < \mathrm{O}(\mathbb{O},N)$ yield $\mathrm{GO}^+(\mathbb{O},N) \cap \mathrm{O}(\mathbb{O},N) = \mathrm{O}^+(\mathbb{O},N)$, as claimed in assertion (a).

We study the stabilizer $\mathrm{GO}(\mathbb{O},N)_{\{1,-1\}}$ of the set $\{1,-1\}$ next. Elements of that stabilizer clearly have multiplier 1, and $\mathrm{GO}(\mathbb{O},N)_{\{1,-1\}} = \mathrm{O}(\mathbb{O},N)_{\{1,-1\}}$. For $u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}$, we have $\delta_u \in \mathrm{GO}^+(\mathbb{O},N)_{\{1,-1\}}$. This element induces on $\mathrm{Pu}\,\mathbb{O}$ the hyperplane reflection $\sigma_u|_{\mathrm{Pu}\,\mathbb{O}}$, see 2.6, and induces $-\mathrm{id}|_F$ on $F$. The set $\{\sigma_u|_{\mathrm{Pu}\,\mathbb{O}} \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\}$ of hyperplane reflections generates the full orthogonal group $\mathrm{O}(\mathrm{Pu}\,\mathbb{O}, N|_{\mathrm{Pu}\,\mathbb{O}})$ by the Cartan-Dieudonné Theorem, see 2.5.(a).

In order to prove assertion (b), it thus remains to determine the set of all $\gamma \in \mathrm{O}(\mathbb{O},N)_{\{1,-1\}}$ that act trivially on $\mathrm{Pu}\,\mathbb{O}$. If $\mathrm{D}(\gamma) = 1$ then the codimension of $\mathrm{Fix}(\gamma)$ is even and bounded above by $\dim(\mathbb{O}/\mathrm{Pu}\,\mathbb{O}) = 1$. So $\gamma = \mathrm{id}$ in that case, and we obtain that $\mathrm{O}^+(\mathbb{O},N)$ intersects the kernel trivially. Thus $-\kappa$ is the only element of Dickson invariant $-1$ in the kernel, and the kernel is generated by $-\kappa$. We have thus proved that $\mathrm{O}(\mathbb{O},N)_{\{1,-1\}} = \langle \{-\kappa\} \cup \{\delta_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\}\rangle$.

We have $\mathrm{D}(\delta_u) = 1$, $\mathrm{D}(-\kappa) = -1$, and $-\kappa$ commutes with $\delta_u$ because $u \in \mathrm{Pu}\,\mathbb{O}$. Thus $\mathrm{O}(\mathbb{O},N)_{\{1,-1\}}$ is a direct product of $\langle -\kappa\rangle$ and $\langle \delta_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle$. This completes the proof of assertion (b), and we also obtain $\mathrm{O}^+(\mathbb{O},N)_{\{1,-1\}} = \langle \delta_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\rangle$. The last remaining claim in assertion (c) is $\mathrm{GO}^+(\mathbb{O},N)_{\{1,-1\}} = \mathrm{O}^+(\mathbb{O},N)_{\{1,-1\}}$; that equality follows from assertion (a). Assertion (d) follows from the observation that $-\kappa$ and each $\delta_u$ map 1 to $-1$.

The group generated by $\{\lambda_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\} \cup \{\rho_u \mid u \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}\}$ is a subgroup of $\mathrm{GO}^+(\mathbb{O},N)$, acts transitively on $\mathbb{O}^*$ by 1.4, and it contains the stabilizer $\mathrm{GO}^+(\mathbb{O},N)_1$ by assertion (d). Thus it coincides with $\mathrm{GO}^+(\mathbb{O},N)$, and assertion (e) is proved.

It remains to prove assertion (f). For each $a \in \mathbb{O}^*$, we have $\mu_{\lambda_a} = N(a)$. If $a \in F^*$ then clearly $\det \lambda_a = a^8$ and $\mu_{\lambda_a} = a^2$, whence $\det \lambda_a = \mu_{\lambda_a}^4$. If $a \in \mathbb{O} \smallsetminus F$ then $K := F + Fa$ is a quadratic extension of $F$. We pick $b \in \mathbb{O} \smallsetminus K$ and $c \in \mathbb{O} \smallsetminus (K + Kb)$. Then $1, a, b, ab, c, ac, bc, a(bc)$ forms a basis for $\mathbb{O}$. With respect to that basis, the matrix for $\lambda_a$ is a block diagonal matrix of the form

$$\begin{pmatrix} 0 & -N(a) & & & & & & \\ 1 & T(a) & & & & & & \\ & & 0 & -N(a) & & & & \\ & & 1 & T(a) & & & & \\ & & & & 0 & -N(a) & & \\ & & & & 1 & T(a) & & \\ & & & & & & 0 & -N(a) \\ & & & & & & 1 & T(a) \end{pmatrix},$$

with determinant $N(a)^4 = \mu_{\lambda_a}^4$. Together with $\rho_a = \kappa \circ \lambda_{\overline{a}} \circ \kappa$ this shows that $\mathrm{GO}^+(\mathbb{O}, N)$ is contained in the (normal) subgroup $\mathrm{M} := \left\{\varphi \in \mathrm{GO}(\mathbb{O}, N) \,\middle|\, \det \varphi = \mu_\varphi^4\right\}$ of $\mathrm{GO}(\mathbb{O}, N)$. If $\operatorname{char} F \neq 2$ then M does not contain $\kappa$ because $\det \kappa = -1 \neq 1^4 = \mu_\kappa^4$. The assertion follows because $\mathrm{GO}^+(\mathbb{O}, N)$ has index 2 in $\mathrm{GO}(\mathbb{O}, N)$. If $\operatorname{char} F = 2$ then $\kappa \in \mathrm{M}$ yields $\mathrm{GO}(\mathbb{O}, N) = \langle\kappa\rangle \circ \mathrm{GO}^+(\mathbb{O}, N) \leqslant \langle\kappa\rangle \circ \mathrm{M} = \mathrm{M}$. □

## 4 Some transitive groups of similitudes of the norm form

**4.1 Definition.** Let $V$ be any subspace of $\mathbb{O}$. Recall from 1.3 that $\Lambda_V := \left\langle \lambda_v \,\middle|\, v \in V \smallsetminus \{0\}\right\rangle$ denotes the group generated by all left multiplications with non-trivial elements of $V$, and $\Lambda_V^+ := \left\langle \lambda_v \circ \lambda_w \,\middle|\, v, w \in V \smallsetminus \{0\}\right\rangle$ is generated by all products of an even number of left multiplications by elements of $V \smallsetminus \{0\}$. (Note that $\Lambda_V^+ = \Lambda_V$ if $1 \in V$ because $\lambda_1 = \mathrm{id}$ is then one of the generators.) By 1.4, the group $\Lambda_V^+$ acts transitively on $\mathbb{O}^*$ if $\dim V \geqslant 5$.

In the group $\Delta$ we define the subgroups $\Gamma_V := \langle (\frac{-1}{N(v)}\rho_v | \frac{-1}{N(v)}\lambda_v \circ \rho_v | \lambda_v) \mid v \in V \smallsetminus \{0\}\rangle$ and $\Gamma_V^+ := \langle (\frac{1}{N(vw)}\rho_v \circ \rho_w | \frac{1}{N(vw)}\lambda_v \circ \rho_v \circ \lambda_w \circ \rho_w | \lambda_v \circ \lambda_w) \mid v, w \in V \smallsetminus \{0\}\rangle$; these generators belong to $\Delta$ by 3.6. Note that $(\frac{-1}{N(v)}\rho_v | \frac{-1}{N(v)}\lambda_v \circ \rho_v | \lambda_v) = (\rho_v^{-1} | \lambda_v \circ \rho_v^{-1} | \lambda_v)$ holds if $v \in \mathrm{Pu}\,\mathbb{O} \smallsetminus \{0\}$. We write $\eta(\lambda_v) := (\rho_v^{-1} | \lambda_v \circ \rho_v^{-1} | \lambda_v)$ for such $v$.

**4.2 Lemma.** *Restriction of* $\mathrm{pr}^3 \colon \Delta \to \mathrm{O}(\mathbb{O}, N) \colon (\alpha|\beta|\gamma) \mapsto \gamma$ *yields an isomorphism from* $\Gamma_{\mathrm{Pu}\,\mathbb{O}}$ *onto* $\Lambda_{\mathrm{Pu}\,\mathbb{O}}$*; the inverse will be denoted by* $\eta$*; in fact, it extends* $\eta(\lambda_v) = (\rho_v^{-1} | \lambda_v \circ \rho_v^{-1} | \lambda_v)$*, as defined above.*

*Proof.* We restrict the homomorphism $\mathrm{pr}^3 \colon \Delta \to \Gamma\mathrm{O}(\mathbb{O}, N) \colon (\alpha|\beta|\gamma) \mapsto \gamma$ from 3.7 to the subgroup $\Gamma_{\mathrm{Pu}\,\mathbb{O}} \leqslant \Delta$. The kernel of $\mathrm{pr}^3$ is just $\left\{(\lambda_z|\lambda_z|\mathrm{id}) \,\middle|\, z \in F^*\right\}$, see 3.3.(b). Therefore, the kernel of the restriction is $\Gamma_{\mathrm{Pu}\,\mathbb{O}} \cap \left\{(\lambda_z|\lambda_z|\mathrm{id}) \,\middle|\, z \in F^*\right\}$.

Every element of $\mathrm{pr}^2(\Gamma_{\mathrm{Pu}\,\mathbb{O}})$ fixes 1, and $(\lambda_z|\lambda_z|\mathrm{id}) \in \Gamma_{\mathrm{Pu}\,\mathbb{O}}$ yields $z = 1$. Therefore, the restriction $\mathrm{pr}^3|_{\Gamma_{\mathrm{Pu}\,\mathbb{O}}}$ is an injective homomorphism, its range is $\Lambda_{\mathrm{Pu}\,\mathbb{O}}$, and we obtain an isomorphism from $\Gamma_{\mathrm{Pu}\,\mathbb{O}}$ onto $\Lambda_{\mathrm{Pu}\,\mathbb{O}}$. Clearly $\eta$ is the inverse of this isomorphism. □

**4.3 Lemma.** *Let* $V \leqslant \mathbb{O}$ *be any non-trivial vector subspace. Then* $\ker \mathrm{pr}^2 = \Gamma_F^+ \leqslant \Gamma_V^+$*, and* $\Lambda_F = \Lambda_F^+ \leqslant \Lambda_V^+$*. If* $V \leqslant \mathrm{Pu}\,\mathbb{O}$ *then* $\Lambda_F = \ker(\mathrm{pr}^2 \circ \eta)$.

*Proof.* From 3.3.(b) we know $\ker \mathrm{pr}^2 = \left\{(\rho_z^{-1}|\mathrm{id}|\lambda_z) \,\middle|\, z \in F \smallsetminus \{0\}\right\}$. As $\mathbb{O}$ is an algebra over $F$, we obtain $\Gamma_F^+ = \left\{(-\rho_z^{-1}| -\mathrm{id}|\lambda_z)(-\rho_1| -\mathrm{id}|\lambda_1) \,\middle|\, z \in F^*\right\} = \ker \mathrm{pr}^2$.

Using 1.1.(f) we note $\lambda_z = \lambda_{zv} \circ \lambda_v^{-1} \in \Lambda_V^+$ if $z \in F^*$ and $v \in V \smallsetminus \{0\}$, and compute $(\rho_z^{-1}|\mathrm{id}|\lambda_z) = (-N(zv)^{-1}\rho_{zv} \,|\, -N(zv)^{-1}\lambda_{zv} \circ \rho_{zv} \,|\, \lambda_{zv})\,(-N(v)^{-1}\rho_v \,|\, -N(v)^{-1}\lambda_v \circ \rho_v \,|\, \lambda_v)^{-1}$. Thus $\Lambda_F = \Lambda_F^+ \leqslant \Lambda_V^+$, and $\Gamma_F^+ \leqslant \Gamma_V^+$. The rest is clear. □

**4.4 Definitions.** For every vector subspace $V \leqslant \mathrm{Pu}\,\mathbb{O}$, let $f_{V^\perp}$ be the restriction of the polar form to $V^\perp$, and consider the group homomorphism $\mathrm{pr}_V^2 \colon \Gamma_V \to \mathrm{O}(V, N|_V) \colon (\alpha|\beta|\gamma) \mapsto \beta|_V$.

**4.5 Lemma.** *Let $V \leqslant \mathrm{Pu}\,\mathbb{O}$ be a vector subspace.*

**(a)** *For each $a \in V \smallsetminus \{0\}$, we have $\mathrm{pr}^2(\eta(\lambda_a)) = \kappa \circ \sigma_a = \sigma_a \circ \kappa$.*

**(b)** *The group $\mathrm{pr}^2(\Gamma_V^+)$ acts trivially on $V^\perp$ and induces $\mathrm{pr}_V^2(\Gamma_V^+) = \mathrm{O}^+(V, N|_V)$ on $V$.*

**(c)** *If $\mathrm{char}\, F \neq 2$ and $\dim V$ is odd then $\mathrm{pr}_V^2(\Gamma_V) = \mathrm{O}^+(V, N|_V) = \mathrm{SO}(V, N|_V)$.*
*If $\mathrm{char}\, F = 2$ or $\dim V$ is even then $\mathrm{pr}_V^2(\Gamma_V) = \mathrm{O}(V, N|_V)$.*

**(d)** *If $f_{V^\perp}$ is not degenerate then either $\dim V$ is even and $\ker \mathrm{pr}^2 = \ker \mathrm{pr}_V^2$, or $V = \mathrm{Pu}\,\mathbb{O}$ and $\ker \mathrm{pr}^2 = \ker \mathrm{pr}_{\mathrm{Pu}\,\mathbb{O}}^2$, or $\dim V \in \{1, 3, 5\}$ and $\ker \mathrm{pr}^2$ has index $2$ in $\ker \mathrm{pr}_V^2$.*

**(e)** *If $f_{V^\perp}$ is degenerate then $\mathrm{char}\, F = 2$. We study two important special cases explicitly:*

- **(i)** *If $f_{V^\perp}$ is not zero then $\ker \mathrm{pr}^2$ has index $2$ in $\ker \mathrm{pr}_V^2$.*
- **(ii)** *If $V = \mathrm{Pu}\,\mathbb{O}$ then $f_{V^\perp} = f_F = 0$. We have $\ker \mathrm{pr}_{\mathrm{Pu}\,\mathbb{O}}^2 = \ker \mathrm{pr}^2$.*

*Proof.* Recall from 2.6 that $\sigma_a = -\lambda_a \circ \kappa \circ \lambda_a^{-1}$. Using $a \in V \leqslant \mathrm{Pu}\,\mathbb{O}$ we obtain $\sigma_a(x) = -a(\overline{a^{-1}x}) = a(\bar{x}a^{-1}) = \overline{axa^{-1}}$ for each $x \in \mathbb{O}$. This shows $\lambda_a \circ \rho_a^{-1} \circ \kappa = \sigma_a = \kappa \circ \lambda_a \circ \rho_a^{-1}$ and then $(\lambda_a \circ \rho_a^{-1}) \circ (\lambda_c \circ \rho_c^{-1}) = \sigma_a \circ \sigma_c$. These products act trivially on $\{a, c\}^\perp \geqslant V^\perp$, while their restrictions to $V$ generate $\mathrm{O}^+(V, N|_V)$, see 2.5. Thus (a) and (b) are established.

In order to prove assertion (c), we distinguish cases: If $\mathrm{char}\, F \neq 2$ and $\dim V$ is odd then $\mathrm{pr}_V^2(\lambda_a) = \kappa \circ \sigma_a|_V = -\sigma_a|_V$ has determinant $1$. Thus $\mathrm{SO}(V, N|_V) \geqslant \mathrm{pr}_V^2(\Gamma_V) \geqslant \mathrm{pr}_V^2(\Gamma_V^+) = \mathrm{SO}(V, N|_V)$ yields $\mathrm{SO}(V, N|_V) = \mathrm{pr}_V^2(\Gamma_V)$. If $\mathrm{char}\, F \neq 2$ and $\dim V$ is even then $\mathrm{pr}_V^2(\lambda_a) = -\sigma_a|_V$ has determinant $-1$, and we obtain $\mathrm{O}(V, N|_V) \geqslant \mathrm{pr}_V^2(\Gamma_V) > \mathrm{pr}_V^2(\Gamma_V) = \mathrm{SO}(V, N|_V)$ which means $\mathrm{O}(V, N|_V) = \mathrm{pr}_V^2(\Gamma_V)$. If $\mathrm{char}\, F = 2$ then $\mathrm{pr}_V^2(\lambda_a) = \kappa \circ \sigma_a|_V = \sigma_a|_V$, and $\mathrm{O}(V, N|_V) = \mathrm{pr}_V^2(\Gamma_V)$ follows.

We now turn to the investigation of the kernel of $\mathrm{pr}_V^2$. For any sequence $a_1, \dots, a_m$ in $V \smallsetminus \{0\}$, we have $\mathrm{pr}^2(\eta(\lambda_{a_m} \circ \cdots \circ \lambda_{a_1})) = \kappa^m(\sigma_{a_m} \circ \cdots \circ \sigma_{a_1})$ by assertion (a). This product acts trivially on $V$ precisely if $\sigma_{a_m} \circ \cdots \circ \sigma_{a_1}|_V = (-\mathrm{id}|_V)^m$ because $V \leqslant \mathrm{Pu}\,\mathbb{O}$. In any case, that product induces $(-\kappa|_{V^\perp})^m$ on $V^\perp$.

If $f_{V^\perp}$ is not degenerate then $f_V$ is not degenerate, and $\mathbb{O} = V^\perp \oplus V$. If $\dim V$ is even then $\sigma_{a_m} \circ \cdots \circ \sigma_{a_1}|_V \in \{\mathrm{id}|_V, -\mathrm{id}|_V\}$ implies $\sigma_{a_m} \circ \cdots \circ \sigma_{a_1}|_V \in \mathrm{O}^+(V, N|_V)$. Then $m$ is even, and $(-\kappa|_{V^\perp})^m$ is trivial. This yields $\ker \mathrm{pr}^2 = \ker \mathrm{pr}_V^2$. If $V = \mathrm{Pu}\,\mathbb{O}$ then $\delta \in \ker \mathrm{pr}_V^2 \smallsetminus \ker \mathrm{pr}^2$ would satisfy $\mathrm{pr}^2(\delta) = -\kappa$, but $-\kappa \in \mathrm{O}(\mathbb{O}, N) \smallsetminus \mathrm{O}^+(\mathbb{O}, N)$ is never a component of an autotopism. If $\dim V$ is odd and $f_{V^\perp}$ is not degenerate then $\mathrm{char}\, F \neq 2$, and $-\mathrm{id}|_V$ is the product of an odd number of reflections in $\mathrm{O}(V, N|_V)$, say $-\mathrm{id}|_V = \sigma_{a_{2k+1}} \circ \cdots \circ \sigma_{a_1}$ with $a_1, \dots, a_{2k+1} \in V \smallsetminus \{0\}$. Now $\mathrm{pr}^2(\eta(\lambda_{a_{2k+1}} \circ \cdots \circ \lambda_{a_1}))|_{V^\perp} = \kappa|_{V^\perp}$, and $V \neq \mathrm{Pu}\,\mathbb{O}$ implies that $\eta(\lambda_{a_{2k+1}} \circ \cdots \circ \lambda_{a_1})$ lies in $\ker \mathrm{pr}_V^2 \smallsetminus \ker \mathrm{pr}^2$. This proves assertion (d).

Now assume that $f_{V^\perp}$ is degenerate, then $\mathrm{char}\, F = 2$, the form $f|_V$ is also degenerate, and $\mathrm{O}^+(V, N|_V) = \mathrm{O}(V, N|_V)$; see 2.5.(d). Thus it is possible to write $\mathrm{id}|_V = \sigma_{a_{2k+1}} \circ \cdots \circ \sigma_{a_1}$ with an odd number of elements $a_1, \dots, a_{2k+1} \in V \smallsetminus \{0\}$. If $f_{V^\perp} \neq 0$ then there exists $u \in V^\perp$

with $\bar{u} \neq u$, and $\eta(\lambda_{a_{2k+1}} \circ \cdots \circ \lambda_{a_1})$ lies in $\ker \mathrm{pr}^2_V \smallsetminus \ker \mathrm{pr}^2$ because $\eta(\lambda_{a_{2k+1}} \circ \cdots \circ \lambda_{a_1})|_{V^\perp} = \kappa|_{V^\perp} \neq \mathrm{id}|_{V^\perp}$. This settles the first of the subcases in (e).

Consider $\delta \in \ker \mathrm{pr}^2_{\mathrm{Pu}\,\mathbb{O}}$ next; then $\gamma := \mathrm{pr}^2(\delta) \in \mathrm{O}^+(\mathbb{O}, N)_1$. Pick $b \in \mathbb{O}$ with $(1|b) = 1$. Then $\{1, b\}^\perp \leqslant \mathrm{Pu}\,\mathbb{O}$ is invariant under $\gamma$, and so is $(\{1, b\}^\perp)^\perp = F + Fb$. Thus there exist $u_1, y_1 \in F$ with $\gamma(b) = u_1 + y_1 b$. Then $1 = (1|b) = (\gamma(1)|\gamma(b)) = (1|u_1 + y_1 b) = y_1(1|b) = y_1$ gives $y_1 = 1$, and $N(b) = N(\gamma(b)) = N(u_1 + b) = N(u_1) + (u_1|b) + N(b) = u_1^2 + u_1 + N(b)$ yields $u_1^2 + u_1 = 0$ and thus $u_1 \in \{0, 1\}$. We obtain $\gamma \in \{\mathrm{id}, \kappa\}$, but $\kappa = \gamma \in \mathrm{GO}^+(\mathbb{O}, N)$ is impossible. This settles the second subcase in (e). □

## 5 Automorphisms of octonion algebras

The results of this section will be used to determine $\Lambda^+_V$ for suitable $V \leqslant \mathrm{Pu}\,\mathbb{O}$, see 6.3 below.

**5.1 Lemma.** *Let $H$ be a quaternion field, and consider $u, x \in H$. Then $u$ and $x$ are conjugates if, and only if, they satisfy $N_H(u) = N_H(x)$ and $T_H(u) = T_H(x)$.*

*Proof.* Since conjugation preserves the norm on $H$ it remains to prove the non-trivial implication, starting from the assumptions $N_H(u) = N_H(x)$ and $T_H(u) = T_H(x)$.

If $x \neq \bar{u}$ then $a := x - \bar{u}$ lies in $\mathrm{Pu}\,H = \ker T_H$ and satisfies $a^{-1}ua = a^{-1}u(x - \bar{u}) = a^{-1}(ux - N_H(u)) = a^{-1}(ux - N_H(x)) = a^{-1}(u - \bar{x})x = a^{-1}(-\bar{a})x = x$.

If $x = \bar{u}$ we pick $a \in \{1, \bar{u}\}^\perp \smallsetminus \{0\}$. Then $\bar{a} \in \{1, u\}^\perp$ yields $0 = (u|\bar{a}) = ua + \bar{a}\bar{u} = ua - a\bar{u}$. This gives $a^{-1}ua = \bar{u}$ and proves the lemma. □

The assertion of 5.1 cannot be generalized to the case of a split quaternion algebra. In fact, such an algebra is isomorphic to a matrix algebra $F^{2\times 2}$, and it is easy to construct examples of non-central matrices with the same norm (i.e., determinant) and trace as some central element. For instance, take $x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$; these elements cannot be conjugates, of course.

**5.2 Proposition.** *Let $H$ be a quaternion field, with norm $N$ and standard involution $\kappa$. We define $\mathrm{GO}^+(H, N) := \{\lambda_a \circ \rho_c^{-1} \mid a, c \in H^*\}$. Then the following hold.*

**(a)** $\mathrm{GO}^+(H, N) = \left\langle \lambda_u \circ \rho_v \,\middle|\, u, v \in \mathrm{Pu}\,H \smallsetminus \{0\} \right\rangle$.

**(b)** $\mathrm{O}^+(H, N) = \{\lambda_a \circ \rho_c^{-1} \mid a, c \in H^*, N(a) = N(c)\} = \mathrm{GO}^+(H, N) \cap \mathrm{O}(H, N)$.

**(c)** $\mathrm{O}(H, N) = \langle \kappa \rangle \ltimes \mathrm{O}^+(H, N)$.

**(d)** $\mathrm{GO}(H, N) = \langle \kappa \rangle \ltimes \mathrm{GO}^+(H, N)$.

**(e)** $\mathrm{PGO}^+(H, N) \cong (H^*/F^*)^2$.

**(f)** $H^*/F^* \cong \mathrm{SO}(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H})$.

*Proof.* Assertion (a) follows from 1.2.

Clearly $\mathrm{GO}^+(H, N) \leqslant \mathrm{GO}(H, N)$, and $\Xi := \{\lambda_a \circ \rho_c^{-1} \mid a, c \in H^*, N(a) = N(c)\}$ equals $\mathrm{GO}^+(H, N) \cap \mathrm{O}(H, N)$. We use the Dickson invariant D, see 2.4 and recall that $\mathrm{O}^+(H, N)$ is defined as the kernel of D. For $a \in H \smallsetminus \{0, 1\}$ we have $\mathrm{Fix}(\rho_a) = \{0\}$, and $\mathrm{Fix}(\lambda_a \circ \rho_a^{-1})$ is the centralizer of $a$ in $H$. That centralizer is the subalgebra $F + Fa$ unless $a \in F$ (then it is $H$). In any case we have $\mathrm{D}(\lambda_a \circ \rho_a^{-1}) = 1$, and $\mathrm{D}(\rho_u) = 1$ if $N(u) = 1$. Now $\lambda_a \circ \rho_c^{-1} = (\lambda_a \circ \rho_a^{-1}) \circ \rho_u$ with $u = ac^{-1}$ yields $\Xi \leqslant \mathrm{O}^+(H,N)$.

In 2.6.(c) we have proved that $\mathrm{O}^+(H,N)$ is generated by the set $\{\delta_a \mid a \in H^*\}$, where $\delta_a = N(a)^{-1}\lambda_a \circ \rho_a = \lambda_a \circ \rho_{\overline{a}}^{-1} \in \Xi$. This yields $\Xi = \mathrm{O}^+(H,N)$. Assertion (c) now follows from the facts that $\mathrm{O}^+(H,N)$ has index 2 in $\mathrm{O}(H,N)$ (see 2.5.(b)) and $\kappa \notin \mathrm{O}^+(H,N)$. Thus we have proved assertions (b) and (c).

Assertion (d) is obtained by a Frattini argument: clearly, the group $\mathrm{GO}^+(H,N)$ is transitive on $H^*$, so it remains to note that the stabilizer $\mathrm{GO}(H,N)_1 = \mathrm{O}(H,N)_1 = \langle\kappa\rangle \ltimes \mathrm{O}^+(H,N)_1$ is contained in $\langle\kappa\rangle \ltimes \mathrm{GO}^+(H,N)$.

In order to prove assertion (e), we consider the surjective homomorphism $\psi$ from $(H^*)^2$ onto $\mathrm{PGO}^+(H,N)$ mapping $(a,c)$ to $\mathbb{P}(\lambda_a \circ \rho_c^{-1})$, where $\mathbb{P}(\beta)\colon \mathbb{P}(H) \to \mathbb{P}(H)\colon Fx \mapsto F(\beta(x))$ for each $\beta \in \mathrm{GL}_F(H)$. The kernel of $\psi$ consists of pairs $(a,c)$ such that there exists $s \in F^*$ with $sx = axc^{-1}$ for each $x \in H$. Specializing $x = 1$, we see $sc = a$. This gives $sxc = scx$ for each $x$, and $c \in F^*$ follows. So $\ker\psi = (F^*)^2$, and $\mathrm{PGO}^+(H,N) \cong (H^*)^2/\ker\psi = (H^*/F^*)^2$, as claimed in assertion (e).

Under the action of $H^*$ on $H$ by conjugation, the vector $1$ is invariant, and so is $1^\perp = \mathrm{Pu}\,H$. We obtain a homomorphism $\gamma\colon H^* \to \mathrm{O}(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H})\colon a \mapsto \lambda_a \circ \rho_a^{-1}|_{\mathrm{Pu}\,H}$. If $\operatorname{char} F \neq 2$ then $\mathrm{Fix}(\gamma(a)) \cap \mathrm{Pu}\,H$ has even co-dimension, and we find $\gamma(H^*) \leqslant \mathrm{O}^+(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H}) = \mathrm{SO}(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H})$. If $\operatorname{char} F = 2$ then the equalities $\mathrm{O}(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H}) = \mathrm{SO}(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H}) = \mathrm{O}^+(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H})$ hold anyway (see 2.5.(d)).

The restriction map from $\mathrm{O}^+(H,N)_1$ to $\mathrm{O}^+(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H})$ is surjective because every hyperplane reflection $\sigma_p|_{\mathrm{Pu}\,H}$ on $\mathrm{Pu}\,H$ is the restriction of a hyperplane reflection on $H$. So $\mathrm{O}^+(H,N)_1 = \{\lambda_a \circ \rho_a^{-1} \mid a \in H^*\}$ yields that $\gamma$ describes a surjective map from $H^*$ onto $\mathrm{O}^+(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H})$. The kernel of $\gamma$ is $F^*$, and $H^*/F^* \cong \mathrm{SO}(\mathrm{Pu}\,H, N|_{\mathrm{Pu}\,H})$ follows. □

**5.3 Proposition** ([27, Sect. 2.1]). *Let $H$ be a quaternion subalgebra of an octonion algebra $C$, and pick $a \in H^\perp \smallsetminus \{0\}$ with $N(a) \neq 0$. Then the elements of the global stabilizer $\mathrm{Aut}_F(C)_H$ of $H$ in $\mathrm{Aut}_F(C)$ are precisely the maps of the form*

$$\alpha_{c,t}\colon C = H \oplus Ha \to C\colon x + ya \mapsto (cxc^{-1}) + (tcyc^{-1})a\,,$$

*with $c, t \in H \smallsetminus \{0\}$ such that $N(c) \neq 0$ and $N(t) = 1$.* □

**5.4 Theorem.** *Let $\mathbb{O}$ be an octonion division algebra over an arbitrary commutative field $F$. Then $\mathrm{Aut}_F(\mathbb{O})$ is contained in the kernel $\mathrm{O}^+(\mathbb{O},N) \leqslant \mathrm{SO}(\mathbb{O},N)$ of the Dickson invariant.*

*Proof.* We know $\mathrm{Aut}_F(\mathbb{O}) \leqslant \mathrm{O}(\mathbb{O},N)$, see 1.1.(j); it remains to show $\mathrm{Aut}_F(\mathbb{O}) \leqslant \ker \mathrm{D}$. (See 2.4 for the definition and properties of the Dickson invariant D.)

If $\mu$ is an automorphism of $\mathbb{O}$ fixing a two-dimensional subalgebra $K$ elementwise then $\mu$ is a $K$-linear map, and $\mathrm{Fix}(\mu)$ is a vector space over $K$. Now $\dim_F \mathrm{Fix}(\mu) = 2\dim_K \mathrm{Fix}(\mu)$ is even, and so is the co-dimension $d = \dim(\mathbb{O}/\mathrm{Fix}(\mu))$. This yields $\mathrm{D}(\mu) = 1$.

In particular, for each quaternion subalgebra $H$ in $\mathbb{O}$, each $a \in H^\perp \smallsetminus \{0\}$ and each automorphism of the form $\alpha_{c,1}$ as in 5.3 we obtain $\mathrm{D}(\alpha_{c,1}) = 1$ because $F + Fc + (F + Fc)a$ is fixed elementwise.

Now consider $\alpha \in \mathrm{Aut}_F(\mathbb{O})$. Since $\alpha$ commutes with the standard involution (see 1.1.(j)), we have $T_{\mathbb{O}}(w) = T_{\mathbb{O}}(\alpha(w))$ and $N(w) = N(\alpha(w))$ for each $w \in \mathbb{O}$. As the affine hyperplane $\{u \in \mathbb{O} \mid u + \overline{u} = 1\}$ generates the vector space $\mathbb{O}$, we may assume that there exists $w \in \mathbb{O}$ with $w + \overline{w} = 1$ and $\alpha(w) \neq w$. Then $w \in \mathbb{O} \smallsetminus F$, and $\{w, \alpha(w)\}$ generates an associative subalgebra $A$ of dimension at least $2$. If $A$ is a quaternion algebra, we use the fact that elements of the same norm and trace are conjugates in $A$ (see 5.1) to find $c \in A$ such that $\alpha_{c,1}(w) = \alpha(w)$, where $\alpha_{c,1}$ is defined as in 5.3, with $A$ playing the role of $H$. Then $\beta := \alpha^{-1} \circ \alpha_{c,1}$ acts trivially on the subfield $F + Fw$. Thus $\mathrm{D}(\beta) = 1$, and $\mathrm{D}(\alpha_{c,1}) = 1$ yields $\mathrm{D}(\alpha) = 1$.

There remains the case where $\dim A = 2$. There exists a quaternion field $H$ in $C$ containing $w$ (see [27, 1.6.4]). Now $H$ contains $A = F + Fw$, by 5.1 and 5.3 we find $c \in H$ such that $\alpha^{-1} \circ \alpha_{c,1}$ fixes $F + Fw$ elementwise, and $\mathrm{D}(\alpha) = 1$ follows as above. □

**5.5 Corollary.** *For each $F$-linear automorphism $\alpha \in \mathrm{Aut}_F(\mathbb{O})$, the subalgebra $\mathrm{Fix}(\alpha)$ of fixed points has even dimension. (In particular, there is no automorphism fixing each element of $F$ and no others.)* □

**5.6 Lemma.** *Let $V \leqslant \mathrm{Pu}\,\mathbb{O}$ be a non-trivial vector space, and let $\Psi \leqslant \mathrm{Aut}(\mathbb{O})$ be a subgroup. If $\mathrm{pr}^2(\Gamma_V^+)$ contains $\Psi$ then $\Gamma_V^+$ contains $\{(\alpha|\alpha|\alpha) \mid \alpha \in \Psi\}$, and $\Lambda_V^+$ contains $\Psi$.*

*Proof.* The claim follows from the fact that $\{(\alpha|\alpha|\alpha) \mid \alpha \in \Psi\}$ is contained in the group $\Delta$ of all autotopisms, and the observation (made in 4.3) that $\Gamma_V^+$ contains the full kernel of the homomorphism $\mathrm{pr}^2 \colon \Delta \to \Gamma\mathrm{O}(\mathbb{O}, N)$. □

**5.7 Theorem.** *If $C$ is a subalgebra of $\mathbb{O}$ then both $\mathrm{pr}^2(\Gamma_{C^\perp}^+)$ and $\Lambda_{C^\perp}^+$ contain $\mathrm{Aut}_C(\mathbb{O}) := \{\alpha \in \mathrm{Aut}(\mathbb{O}) \mid \alpha|_C = \mathrm{id}\}$. In fact, we have $\{(\alpha|\alpha|\alpha) \mid \alpha \in \mathrm{Aut}_C(\mathbb{O})\} \leqslant \Gamma_{C^\perp}^+$.*

*Proof.* As $1 \in C$ implies $C^\perp \leqslant \mathrm{Pu}\,\mathbb{O}$, we know from 4.5.(b) that $\mathrm{pr}^2(\Gamma_{C^\perp}^+)$ acts trivially on $C^{\perp\perp} = C$ and induces $\mathrm{pr}^2_{C^\perp}(\Gamma_{C^\perp}^+) = \mathrm{O}^+(C^\perp, N|_{C^\perp})$ on $C^\perp$. Every hyperplane reflection on $C^\perp$ is induced by a hyperplane reflection on $\mathbb{O}$, and the latter reflection acts trivially on $C$. Therefore, the group $\mathrm{pr}^2(\Gamma_{C^\perp}^+)$ equals the pointwise stabilizer $\mathrm{O}^+(\mathbb{O}, N)_{[C]}$ of $C$ in $\mathrm{O}^+(\mathbb{O}, N)$, see 2.5.(b). From 5.4 we know $\mathrm{Aut}_F(\mathbb{O}) \leqslant \mathrm{O}^+(\mathbb{O}, N)$. Thus $\mathrm{Aut}_C(\mathbb{O}) \leqslant \mathrm{Aut}_F(\mathbb{O})$ is contained in $\mathrm{O}^+(\mathbb{O}, N)_{[C]} = \mathrm{pr}^2(\Gamma_{C^\perp}^+)$. From 5.6 we know that $\Gamma_{C^\perp}^+$ contains $\{(\alpha|\alpha|\alpha) \mid \alpha \in \mathrm{Aut}_C(\mathbb{O})\}$, and $\Lambda_{C^\perp}^+$ contains $\mathrm{Aut}_C(\mathbb{O})$. □

**5.8 Lemma.** *Let $K$ be a two-dimensional subalgebra of a composition algebra $C$, and assume that the restriction $N|_K$ of the norm is anisotropic (so $K/F$ is a quadratic field extension). If $\alpha$ is an automorphism of $C$ and fixes each element of $K$ then $\alpha$ is $K$-linear (with respect to the natural structure of $C$ as a left vector space over $K$, see 1.1.(h)).* □

**5.9 Lemma.** *Let $K$ be a two-dimensional subalgebra of a non-split octonion algebra $\mathbb{O}$ with $K \not\leqslant \mathrm{Pu}\,C$, and define a hermitian form $g$ as in 2.9. Then the group $\mathrm{Aut}_K(\mathbb{O})$ of $K$-linear automorphisms induces the group $\mathrm{SU}_K(K^\perp, g|_{K^\perp \times K^\perp})$ on $K^\perp$.*

*Proof.* The space $K^\perp$ is invariant under $\mathrm{Aut}_K(\mathbb{O})$ because $\mathrm{Aut}(\mathbb{O})$ acts by semi-similitudes of the norm. Each element of $\mathrm{Aut}_K(\mathbb{O})$ fixes $c$, induces a $K$-linear map, and thus acts by a similitude of the form $g$ by 2.9. The existence of non-trivial fixed points implies that the multiplier is $1$. So $\mathrm{Aut}_K(\mathbb{O})$ induces a subgroup of $\mathrm{U}_K(K^\perp, g|_{K^\perp \times K^\perp})$ on $K^\perp$.

Fix a non-trivial element $p \in K^\perp \smallsetminus \{0\}$, and consider $q \in \mathbb{S} := \{x \in K^\perp \mid N(x) = N(p)\}$. We construct an automorphism $\alpha_q \in \mathrm{Aut}_K(\mathbb{O})$ with $\alpha_q(p) = q$, as follows.

The subspaces $H_p := K + Kp$ and $H_q := K + Kq$ are quaternion subalgebras of $\mathbb{O}$, and they are both obtained by doubling $K$. The multiplication formula 1.1.(k) shows that the $K$-linear bijection fixing $1$ and mapping $p$ to $q$ is an isomorphism from $H_p$ onto $H_q$. Pick a non-trivial element $w \in H_p^\perp \cap H_q^\perp$; then the octonion algebra is recovered in two ways as double $\mathbb{O} = H_p \oplus H_p w$ and $\mathbb{O} = H_q \oplus H_q w$, cf. 1.1.(k) again. Thus the $K$-linear isomorphism from $H_p$ onto $H_q$ extends to a $K$-linear automorphism $\alpha_q$ of $\mathbb{O}$, with $\alpha_q(w) = w$.

As $\alpha_q$ acts trivially on the quaternion subfield $W := K + Kw$, we know from 5.3 that there exists $t \in W$ with $N(t) = 1$ and $\alpha_q(up) = (tu)p$ holds for each $u \in W$ (recall that $W^\perp = Wp$). Now $tp = \alpha_q(p) = q$ yields $(tw)p = \alpha_q(wp) = w\alpha_q(p) = wq = -qw$. Writing $t = r + sw$ with $r, s \in K$, we find $tp = rp + \bar{s}(wp)$ and $(tw)p = -sN(w)p + \bar{r}(wp)$. With respect to the $K$-basis $p, wp$ for $Wp$, the restriction $\alpha_q|_{Wp}$ is thus described by the matrix $A_q := \begin{pmatrix} r & -sN(w) \\ \bar{s} & \bar{r} \end{pmatrix}$ with $\det_K(A_q) = r\bar{r} + N(w)s\bar{s} = N(t) = 1$. So $\det_K(\alpha_q) = 1$, and $\alpha_q|_{K^\perp} \in \mathrm{SU}_K(K^\perp, g|_{K^\perp \times K^\perp})$ because $\alpha_q$ fixes $1$.

The set $\Omega := \{\alpha_q \mid q \in \mathbb{S}\}$ induces a subset of $\mathrm{SU}_K(K^\perp, g|_{K^\perp\times K^\perp})$ which is transitive on $\mathbb{S}$. A Frattini argument tells us that $\mathrm{Aut}_K(\mathbb{O}) = \Omega \circ \mathrm{Aut}_K(\mathbb{O})_p$. Pick $c \in K \smallsetminus F$, then 5.3 asserts that $\mathrm{Aut}_K(\mathbb{O})_p = \mathrm{Aut}_F(\mathbb{O})_{c,p}$ equals $\{(u + va) \mapsto (u + (Bv)a) \mid B \in \mathrm{SU}_K(Ha, g|_{Ha\times Ha})\}$, where $H = H_p = K + Kp$ is the quaternion field from above, $a \in H^\perp \smallsetminus \{0\}$, and $u + va \in H \oplus Ha = \mathbb{O}$. Thus we obtain $\mathrm{Aut}_K(\mathbb{O}) = \{(u + y) \mapsto (u + Ay) \mid A \in \mathrm{SU}_K(K^\perp, g|_{K^\perp\times K^\perp})\}$, where $u + y \in K \oplus K^\perp$. □

**5.10 Lemma.** *Let $H$ be a quaternion subalgebra in $\mathbb{O}$, and pick any $w \in H^\perp \smallsetminus \{0\}$. Then $H^\perp = Hw$, and $\mathbb{O} = H \oplus Hw$. We abbreviate $\psi_{a,c}\colon \mathbb{O} \to \mathbb{O}\colon x + yw \mapsto ax + (y\bar{c})w$ and $\tilde{\psi}_{a,c}\colon \mathbb{O} \to \mathbb{O}\colon x + yw \mapsto x\bar{a} + (cy)w$.*

- **(a)** $\Lambda_H = \Lambda_H^+ = \{\psi_{a,c} \mid a, c \in H^*, N(a) = N(c)\} \cong \{(a, c) \in H^* \times H^* \mid N(a) = N(c)\}$.
- **(b)** $\Lambda_H/\Lambda_F \cong \mathrm{O}^+(H, N|_H) = \{(x \mapsto axc^{-1}) \mid (a, c) \in H^* \times H^*, N(a) = N(c)\}$.
- **(c)** $\Lambda_{H^\perp}^+ = \{\tilde{\psi}_{a,c} \mid a, c \in H^*, N(a) = N(c)\}$ *is a subgroup of index* $2$ *in* $\Lambda_{H^\perp} = \langle\{\lambda_w\} \cup \Lambda_{H^\perp}^+\rangle$, *and* $\Lambda_{H^\perp}^+ \cong \Lambda_H$; *in fact, those groups are conjugates in* $\mathrm{O}^+(\mathbb{O}, N)$.
- **(d)** $\Gamma_H \cap \ker \mathrm{pr}^3 = \langle(-\mathrm{id}| - \mathrm{id}|\mathrm{id})\rangle$, *and* $\Gamma_H^+ \cap \ker \mathrm{pr}^3$ *is trivial.*

*Proof.* We note first that $\Lambda_H = \Lambda_H^+$ follows from the fact that $1 \in H$.

For a deeper understanding of the group $\Lambda_H$ we recall that $\mathbb{S}_H := \{s \in H \mid N(s) = 1\}$ coincides with the commutator group of $H^*$ (e.g., see [30, 20.26] for a proof of this folklore result). Consider $a, c \in H^*$ and $x, y \in H$. Using 1.1.(k) again, we see $\lambda_a(x + yw) = ax + (ya)w$ and $(\lambda_a \circ \lambda_c)(x + yw) = acx + (yca)w$. This yields $(\lambda_{ac}^{-1} \circ \lambda_a \circ \lambda_c)(x + yw) = x + (y\, cac^{-1}a^{-1})w$, and $\{\psi_{1,s} \mid s \in \mathbb{S}_H\} \subseteq \Lambda_H$ follows.

We further obtain $\lambda_a = \psi_{a,\bar{a}}$ and $\lambda_{a_1} \circ \cdots \circ \lambda_{a_k} = \psi_{u,v}$ with $u = a_1 \cdots a_k$ and $v = \overline{a_k \cdots a_1}$, respectively. Then $N(\bar{u}) = N(v)$, and there exists $s \in \mathbb{S}_H$ such that $v = s\bar{u}$ and $\lambda_{a_1} \circ \cdots \circ \lambda_{a_k} = \psi_{u,s\bar{u}} = \psi_{1,s} \circ \psi_{u,\bar{u}} = \psi_{1,s} \circ \lambda_u$. We obtain $\Lambda_H \leqslant \{\psi_{u,s\bar{u}} \mid u \in H^*, s \in \mathbb{S}_H\} = \{\psi_{u,v} \mid u, v \in H^* \times H^*, N(u) = N(v)\} \subseteq \{\psi_{1,s} \mid s \in \mathbb{S}_H\} \circ \{\lambda_u \mid u \in H^*\} \subseteq \Lambda_H$, and equality follows.

Mapping $(u, v)$ to $\psi_{u,v}$ is an isomorphism from $\{(u, v) \in H^* \times H^* \mid N(u) = N(v)\}$ onto $\Lambda_H$. This completes the proof of assertion (a). We postpone the proof of assertion (b).

In order to understand $\Lambda_{H^\perp}$, we consider $a, c, x, y \in H$ with $a \neq 0 \neq c$ and compute $\lambda_{aw}(x + yw) = -N(w)\bar{y}a + (a\bar{x})w$. This yields that $\lambda_{aw}$ interchanges $H$ with $H^\perp = Hw$, and $(\lambda_{aw} \circ \lambda_{cw})(x + yw) = -N(w)\,(x\bar{c}a + (a\bar{c}y)w)$. We compute $\tilde{\psi}_{\bar{a},a} = \lambda_{aw} \circ \lambda_{-N(w)^{-1}w} \in \Lambda_{H^\perp}^+$. From $\tilde{\psi}_{\bar{a},a} \circ \tilde{\psi}_{\bar{c},c} \circ \tilde{\psi}_{\overline{ca}^{-1},(ca)^{-1}} = \tilde{\psi}_{1,aca^{-1}c^{-1}}$ we then see $\{\tilde{\psi}_{1,s} \mid s \in \mathbb{S}_H\} \subseteq \Lambda_{H^\perp}^+$. By arguments like those used for $\Lambda_H$ above, we now obtain $\Lambda_{H^\perp}^+ = \{\tilde{\psi}_{a,c} \mid a, c \in H^*, N(a) = N(c)\}$.

Mapping $x+yw$ to $\overline{x}+\overline{y}w = -N(w)^{-1}w(x+yw)w$ is an element of $\mathrm{O}^+(\mathbb{O},N)$. It is obvious from assertions (a) and (d) that conjugation by that element interchanges the groups $\Lambda_H$ and $\Lambda^+_{H^\perp}$. Conjugation by $\lambda_w$ normalizes both $\Lambda_H$ and $\Lambda^+_{H^\perp}$; in fact, we have $\lambda_w\circ\psi_{a,c}\circ\lambda_w^{-1} = \psi_{c,a}$ and $\lambda_w\circ\tilde\psi_{a,c}\circ\lambda_w^{-1} = \tilde\psi_{c,a}$. We also note $\psi_{a,c}\circ\tilde\psi_{b,d} = \tilde\psi_{b,d}\circ\psi_{a,c}$ (by the associative law in $H$), and assertion (c) is proved.

In order to prove (d), consider a sequence $u_1,\dots,u_k \in H^*$ with $\lambda_{u_k}\circ\cdots\circ\lambda_{u_1} = \mathrm{id}$. Evaluating $\lambda_{u_k}\circ\cdots\circ\lambda_{u_1}$ at $1$ and at $w\in H^\perp$ (using 1.1.(k)) we obtain $u_k\cdots u_1 = 1 = u_1\cdots u_k$; note that these products are taken in the associative subalgebra $H$. In particular, we have $N(u_k)\cdots N(u_1) = 1$. Now $\alpha := (-1)^k\rho_{u_k}\circ\cdots\circ\rho_{u_1}$ is the first component of some autotopism $(\alpha|\beta|\mathrm{id}) \in \Gamma_H < \Delta$, see 3.6. From 3.3.(b) we infer $\alpha = t\,\mathrm{id}$ for some $t\in F^*$. Evaluating $t = \alpha(1) = (-1)^k u_1\cdots u_k = (-1)^k$ we find $t = (-1)^k \in \{1,-1\}$.

For elements in $\Gamma^+_H\cap\ker\mathrm{pr}^3$ we have even $k$, and obtain that $\Gamma^+_H\cap\ker\mathrm{pr}^3$ is trivial. Noting that $(-\mathrm{id}|-\mathrm{id}|\mathrm{id}) = (-\mathrm{id}|-\mathrm{id}|\lambda_1)\in\Gamma_H\cap\ker\mathrm{pr}^3$ completes the proof of assertion (d).

It remains to prove assertion (b). The inverse of the restriction $\mathrm{pr}^3|_{\Gamma^+_H}$ is an isomorphism $\xi\colon \Lambda^+_H\to\Gamma^+_H$ extending $\xi(\lambda_u\circ\lambda_v) = (N(uv)^{-1}\rho_u\circ\rho_v|\delta_u\circ\delta_v|\lambda_u\circ\lambda_v)$, cf. 4.1 and 2.6. The homomorphism $\mathrm{pr}^2_H\circ\xi\colon\Lambda^+_H\to\mathrm{O}^+(H,N|_H)$ is surjective (see 2.6.(b)). As $\delta_u$ acts trivially on $H^\perp$ and $\mathbb{O} = H\oplus H^\perp$, we have $\ker(\mathrm{pr}^2_H\circ\xi) = \ker(\mathrm{pr}^2\circ\xi)$. Each element of that kernel is of the form $\gamma = \lambda_{u_k}\circ\cdots\circ\lambda_{u_1}$ with even $k$ and $(\delta_{u_k}\circ\cdots\circ\delta_{u_1})(x) = x$ for each $x\in H$. That means $(u_k\cdots u_1)x = x(\overline{u_k}\cdots\overline{u_1})$ for each $x\in H$. Evaluating at $x=1$, we obtain $s := u_k\cdots u_1 = \overline{u_k}\cdots\overline{u_1}$. The general condition now is $sx = xs$, and yields $s\in F^*$. In particular, we have $u_1\cdots u_k = \overline{\overline{u_k}\cdots\overline{u_1}} = \overline{s} = s = u_k\cdots u_1$.

Now $\gamma$ maps $x+yw\in\mathbb{O} = H\oplus Hw$ to $sx+(ys)w = s(x+yw)$, and $\gamma = \lambda_s\in\Lambda_F$ follows. Conversely, every $s\in F^*$ yields an element $\lambda_s = \lambda_s\circ\lambda_1\in\Lambda_F = \Lambda^+_F < \Lambda^+_H$ in the kernel under consideration. So $\mathrm{O}^+(H,N|_H) = \mathrm{pr}^2_H(\Gamma^+_H)\cong\Lambda^+_H/\ker(\mathrm{pr}^2_H\circ\xi) = \Lambda^+_H/\Lambda_F = \Lambda_H/\Lambda_F$.

The equality $\mathrm{O}^+(H,N|_H) = \left\{\left(x\mapsto axc^{-1}\right)\,\middle|\,(a,c)\in H^*\times H^*, N(a) = N(c)\right\}$ follows from our result $\mathrm{pr}^2_H(\xi(\Lambda^+_H)) = \mathrm{O}^+(H,N|_H)$, the observations $\mathrm{pr}^2_H(\xi(\lambda_a\circ\lambda_1)) = (x\mapsto ax\overline{a}^{-1})$ and $\mathrm{pr}^2_H(\xi(\lambda_{ac}^{-1}\circ\lambda_a\circ\lambda_c)) = (x\mapsto x(\overline{c}^{-1}\overline{a}^{-1}\overline{ca}))$, and arguments as in the proof of assertion (a) above. □

## 6 Transitive groups of similitudes of hermitian forms

We start with a closer look at the group $\mathrm{O}^+(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}})$ studied in 3.13.(d):

**6.1 Theorem.** *The group $\Lambda^+_{\mathrm{Pu}\,\mathbb{O}}$ contains $\Lambda_F$ as a central subgroup. We have $\Lambda^+_{\mathrm{Pu}\,\mathbb{O}}/\Lambda_F \cong \mathrm{O}^+(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}})$ and $\Lambda_{\mathrm{Pu}\,\mathbb{O}}/\Lambda_F\cong\mathrm{SO}(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}})$; the latter is a special orthogonal group in 7 variables.*

*Proof.* In 4.5.(b) we have seen $\mathrm{pr}^2_{\mathrm{Pu}\,\mathbb{O}}(\Gamma^+_{\mathrm{Pu}\,\mathbb{O}}) = \mathrm{O}^+(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}})$. If $\mathrm{char}\,F\neq 2$ then 4.5.(c) gives $\mathrm{pr}^2_{\mathrm{Pu}\,\mathbb{O}}(\Gamma_{\mathrm{Pu}\,\mathbb{O}}) = \mathrm{O}^+(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}}) = \mathrm{SO}(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}})$ because $\dim\mathrm{Pu}\,\mathbb{O}$ is odd. In this case, the kernels of $\mathrm{pr}^2_{\mathrm{Pu}\,\mathbb{O}}$ and $\mathrm{pr}^2$ coincide, see 4.5.(d). If $\mathrm{char}\,F = 2$ then 4.5.(c) gives $\mathrm{pr}^2_{\mathrm{Pu}\,\mathbb{O}}(\Gamma_{\mathrm{Pu}\,\mathbb{O}}) = \mathrm{O}(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}}) = \mathrm{SO}(\mathrm{Pu}\,\mathbb{O},N|_{\mathrm{Pu}\,\mathbb{O}})$. The kernels of $\mathrm{pr}^2_{\mathrm{Pu}\,\mathbb{O}}$ and $\mathrm{pr}^2$ coincide also in this case, see 4.5.(e). From 4.3 we know $\ker(\mathrm{pr}^2\circ\eta) = \Lambda_F < \Lambda^+_{\mathrm{Pu}\,\mathbb{O}}$. □

The result 6.1 has generalizations that shed light on certain exceptional (iso)morphisms between classical groups, see 6.3 and 6.6 below. Instead of $\mathrm{Pu}\,\mathbb{O} = F^\perp$, we study the orthogonal space $K^\perp$ of a two-dimensional separable subalgebra $K$ of $\mathbb{O}$ next. As in 2.9, we consider $\mathbb{O}$ as a vector space over $K$, and construct the hermitian form $g$.

**6.2 Lemma.** *Let $a, b$ be non-zero elements in $K^\perp$, and let $u, v$ be non-zero elements in $K$.*

**(a)** *The map $\lambda_a$ is a semi-similitude of the form $g$; the companion field automorphism is $\kappa|_K$, and the multiplier is $N(a)$.*

**(b)** *The product $\lambda_b \circ \lambda_a$ is a similitude of the form $g$, with multiplier $N(ba)$, and $\det_K(\lambda_b \circ \lambda_a) = N(ba)^2$.*

**(c)** *The map $\rho_u$ is a similitude of the form $g$; the multiplier is $N(u)$, and $\det_K(\rho_u) = \bar{u}^3 u$. The set $\mathrm{P}_K := \left\{\rho_c \,\middle|\, c \in K^*\right\}$ forms a group; we have $\rho_u \circ \rho_v = \rho_{uv}$.*

*Proof.* From 2.8 we know that $\lambda_a$ is semilinear with companion $\kappa|_K$. Pick $c \in K \setminus F$ with $Tc = 1$, as required for the construction of the form $g$ in 2.9. Using 1.1.(c) and $\bar{a}(c(ax)) = -a(c(ax)) = -((ac)a)x = -(\bar{c}\bar{a}a)x = -N(a)\bar{c}x$ we see $g(ax, ay) = N(a)\overline{g(x, y)}$.

In order to prove assertion (b), we describe the $K$-linear map $\gamma := \lambda_b \circ \lambda_a$ by a matrix with respect to a suitable $K$-basis for $\mathbb{O}$, as follows. As $K \not\leq K^\perp$ and $a \in K^\perp \setminus \{0\}$, the subspace $H := K + Ka$ is a quaternion algebra. Pick a non-trivial element $w \in (H + Kb)^\perp$. Then $\mathbb{O} = H + Hw$ is obtained by doubling (cf. 1.1.(k)), and $H^\perp = Hw = Kw + K(aw)$. We use the $K$-basis $1, w, a, aw$.

As $b \in K^\perp \cap (Kw)^\perp = Ka + K(aw)$, there exist $x, y \in K$ with $b = xa + y(aw)$, and $N(b) = N(a)(N(x) + N(yw))$. Using the multiplication formula 1.1.(k) and $a \in \{1, x, y\}^\perp$, we obtain $b = xa + (ay)w = xa + (\bar{y}a)w$ and compute $\gamma(1) = ba = -N(a)(x - \bar{y}w)$, $\gamma(w) = b(aw) = -N(a)(yN(w) + \bar{x}w)$, $\gamma(a) = b(aa) = -N(a)(xa + y(aw))$, and $\gamma(aw) = b(a(aw)) = -N(a)(-\bar{y}N(w)a + \bar{x}(aw))$. Thus $\gamma$ is described by the matrix

$$-N(a)\begin{pmatrix} x & yN(w) & 0 & 0 \\ -\bar{y} & \bar{x} & 0 & 0 \\ 0 & 0 & x & -\bar{y}N(w) \\ 0 & 0 & y & \bar{x} \end{pmatrix},$$

and we find $\det_K(\gamma) = N(a)^4(N(x) + N(yw))^2 = N(a)^2N(b)^2 = N(ba)^2$, as claimed.

It follows from Artin's Theorem (1.1.(e)) that $\rho_u$ is $K$-linear. From 2.8 we know that $\rho_u$ is a similitude of $g$, with multiplier $N(u)$. In order to find the determinant, we note that $au = \bar{u}a$ holds for each $a \in K^\perp$. This means that $\rho_u$ has the characteristic root $\bar{u}$ with multiplicity three, and the characteristic root $u$ (with multiplicity one). This yields $\det_K(\rho_u) = \bar{u}^3 u$. As the elements of $\mathrm{P}_K$ can be simultaneously diagonalized, we obtain $\rho_u \circ \rho_v = \rho_{uv}$, and $\mathrm{P}_K$ is indeed closed under multiplication and inversion. □

**6.3 Theorem.** *The group $\Lambda^+_{K^\perp}$ coincides with $\Xi := \left\{\xi \in \mathrm{GU}_K(\mathbb{O}, g) \,\middle|\, \det_K(\xi) = \mu_\xi^2\right\}$. We have $\Lambda_F < \Lambda^+_{K^\perp}$. The quotients are $\Lambda^+_{K^\perp}/\Lambda_F \cong \mathrm{O}^+(K^\perp, N|_{K^\perp})$ and $\Lambda_{K^\perp}/\Lambda_F \cong \mathrm{O}(K^\perp, N|_{K^\perp})$; the latter is an orthogonal group in 6 variables.*

*Proof.* From 6.2.(b) we see that $\Lambda^+_{K^\perp}$ is contained in $\Xi$. As $\Lambda^+_{K^\perp}$ is transitive on $\mathbb{O}^*$ (see 1.2), it only remains to show that $\Lambda^+_{K^\perp}$ contains the stabilizer $\Xi_1$. By 5.9, that stabilizer is the intersection $\mathrm{Aut}_K(\mathbb{O})$ of $\mathrm{Aut}(\mathbb{O})$ with the group $\mathrm{GL}_K(\mathbb{O})$ of $K$-linear bijections of $\mathbb{O}$, and by 5.7 it is contained in $\Lambda^+_{K^\perp}$ .

From 4.3 we know that $\Lambda^+_{K^\perp}$ contains $\Lambda_F = \ker(\mathrm{pr}^2 \circ \eta)$. From 4.5.(b) we know that $\mathrm{pr}^2_{K^\perp}(\eta(\Lambda^+_{K^\perp})) = \mathrm{pr}^2_{K^\perp}(\Gamma^+_{K^\perp}) = \mathrm{O}^+(K^\perp, N|_{K^\perp})$, and that $\mathrm{pr}^2_{K^\perp}(\Gamma^+_{K^\perp})$ acts trivially on $K = (K^\perp)^\perp$. Therefore, the kernel of $\mathrm{pr}^2_{K^\perp}$ coincides with the kernel of $\mathrm{pr}^2$. Thus $\Lambda^+_{K^\perp}/\Lambda_F \cong$

$\mathrm{O}^+(K^\perp, N|_{K^\perp})$. Pick any $a \in K^\perp \setminus \{0\}$, from 4.5.(a) we then know $\mathrm{pr}^2(\eta(\lambda_a)) = \sigma_a \circ \kappa$, and $\mathrm{pr}^2_{K^\perp}(\eta(\lambda_a)) = -\sigma_a|_{K^\perp} \in \mathrm{O}(K^\perp, N|_{K^\perp}) \setminus \mathrm{O}^+(K^\perp, N|_{K^\perp})$. This yields $\mathrm{O}(K^\perp, N|_{K^\perp}) = \mathrm{pr}^2(\eta(\Lambda_{K^\perp})) \cong \Lambda_{K^\perp}/\Lambda_F$, as claimed. □

We have met a similar situation in the group $\mathrm{GO}^+(\mathbb{O}, N)$ of direct similitudes, see 3.13.(f). The full group of similitudes of the the form $g$ can also be generated by multiplications:

**6.4 Theorem.** *The group $\mathrm{GU}_K(\mathbb{O}, g)$ is the product $\mathrm{P}_K \circ \Lambda^+_{K^\perp}$, where $\mathrm{P}_K = \{\rho_c \mid c \in K^*\}$. The intersection $\mathrm{P}_K \cap \Lambda^+_{K^\perp}$ equals $\Lambda_F$. So we obtain that $\mathrm{GU}_K(\mathbb{O}, g)/\Lambda_F$ splits as a semidirect product $K^*/F^* \ltimes \mathrm{O}^+(K^\perp, N|_{K^\perp})$.*

*Proof.* Let $\psi \in \mathrm{GU}_K(\mathbb{O}, g)$ have multiplier $\mu_\psi$. Comparing determinants of Gram matrices, we see $\det_K(\psi)\,\overline{\det_K(\psi)} = \mu_\psi^4$. So $\mu_\psi^{-2}\det_K(\psi)$ has norm $1$. By Hilbert's Theorem 90 (see [24, VI 6.1]) there exists $c \in K$ with $\bar{c}^{-1}c = \mu_\psi^{-2}\det_K(\psi)$. Using 6.2.(c) we see that $\rho_c \circ \psi \in \mathrm{GU}_K(\mathbb{O}, g)$ satisfies $\det_K(\rho_c \circ \psi) = \bar{c}^3 c \det_K(\psi) = \bar{c}^2 c^2 \mu_\psi^2 = (\bar{c}c\mu_\psi)^2 = \mu^2_{\rho_c\circ\psi}$. This gives $\rho_c \circ \psi \in \Xi = \Lambda^+_{K^\perp}$, and $\psi \in \mathrm{P}_K \circ \Lambda^+_{K^\perp}$, as claimed.

Now consider $\rho_c \in \mathrm{P}_K \cap \Lambda^+_{K^\perp}$. Then $\bar{c}^3 c = \det_K(\rho_c) = \mu^2_{\rho_c} = N(c)^2 = \bar{c}^2 c^2$ yields $\bar{c} = c$. This gives $\mathrm{P}_K \cap \Lambda^+_{K^\perp} = \mathrm{P}_{K\cap\mathrm{Fix}(\kappa)} = \mathrm{P}_F = \Lambda_F$; if $\operatorname{char} F = 2$ we use that $K$ is assumed to be separable. The rest is clear from 6.3. □

**6.5 Remark.** The reader may wonder why we use right multiplications in 6.4. In fact, the group $\Lambda_K \circ \Lambda^+_{K^\perp}$ consists of products $\lambda_c \circ (\lambda_b \circ \lambda_a)$ of $K$-linear maps with determinant $c^4\,N(ba)^2$ (cf. 6.2), and will in general be smaller than $\mathrm{GU}_K(\mathbb{O}, g)$. Using methods as in 6.4, one sees that $\Lambda_K \circ \Lambda^+_{K^\perp}$ in fact coincides with $\{\xi \in \mathrm{GU}_K(\mathbb{O}, g) \mid \det_K \xi \text{ is a square in } K\}$.

**6.6 Theorem.** *Let $W \leqslant \mathbb{O}$ be a vector subspace with $1 \in W \nleqslant W^\perp$ and $\dim W = 3$. Then $\Lambda^+_{W^\perp} = \langle \lambda_u \circ \lambda_v \mid u, v \in W^\perp \setminus \{0\}\rangle$ coincides with $\mathrm{GU}_H(\mathbb{O}, h)$, where the quaternion field $H := M_W$ and the hermitian form $h \in \{h_W, h_W^\perp\}$ are constructed as in 2.10 and in 2.12 or 2.13, respectively.*

- **(a)** *We have $\Lambda_F < \Lambda^+_{W^\perp}$ and $\Lambda^+_{W^\perp}/\Lambda_F \cong \mathrm{pr}^2_{W^\perp}(\Gamma^+_{W^\perp}) = \mathrm{pr}^2_{W^\perp}(\Gamma_{W^\perp}) = \mathrm{SO}(W^\perp, N|_{W^\perp})$; this is a special orthogonal group in 5 variables.*
- **(b)** *The group $\Lambda_{W^\perp}$ is a direct product of $\Lambda^+_{W^\perp}$ and a cyclic group of order two; so $\Lambda_{W^\perp}/\Lambda_F \cong \mathrm{O}(W^\perp, N|_{W^\perp})$ if $\operatorname{char} F \neq 2$ but $\Lambda_{W^\perp}/\Lambda_F \cong \mathrm{O}(W^\perp, N|_{W^\perp}) \times \mathrm{C}_2$ if $\operatorname{char} F = 2$.*
- **(c)** *The group $\langle \Lambda_W \cup \Lambda^+_{W^\perp}\rangle = \Lambda_W \circ \Lambda^+_{W^\perp}$ coincides with the group $\mathrm{Aut}_F(H) \ltimes \mathrm{GU}_H(\mathbb{O}, h)$ of all semi-similitudes of $h$ with $F$-linear companions.*

*Proof.* The group $\Lambda^+_{W^\perp}$ is transitive on $\mathbb{O}^*$ (see 1.2) and a subgroup of $\mathrm{GU}_H(\mathbb{O}, h)$ by 2.10 and 2.12. For $(\alpha|\beta|\gamma) \in \Gamma^+_{W^\perp}$, we have $\beta(1) = 1$. Each element of the stabilizer $(\Lambda^+_{W^\perp})_1$ acts trivially on $H$. The embedding $\eta|_{\Lambda_{W^\perp}} : \Lambda^+_{W^\perp} \to \Gamma^+_{W^\perp} \leqslant \Delta$ maps the stabilizer $(\Lambda^+_{W^\perp})_1$ into the group $\Psi := \{(\alpha|\alpha|\alpha) \mid \alpha \in \mathrm{Aut}_H(\mathbb{O})\}$ because $\eta\left((\Lambda^+_{W^\perp})_1\right)$ consists of autotopisms $(\alpha|\beta|\gamma)$ where both $\beta$ and $\gamma$ (and then also $\alpha$) fix $1$. Conversely, we know $\Psi \leqslant \Gamma^+_{H^\perp} \leqslant \Gamma^+_{W^\perp}$ from 5.7, and $\mathrm{pr}^3(\Psi) \leqslant \Lambda^+_{W^\perp}$ yields $\Lambda^+_{W^\perp} = \mathrm{GU}_H(\mathbb{O}, h)$.

From 4.3 we know that $\Lambda^+_{W^\perp}$ contains $\Lambda_F = \ker(\mathrm{pr}^2 \circ \eta)$. There exist $a_1, a_2, a_3, a_4, a_5 \in W^\perp \setminus \{0\}$ such that $-\mathrm{id}_{W^\perp} = \sigma_{a_1} \circ \sigma_{a_2} \circ \sigma_{a_3} \circ \sigma_{a_4} \circ \sigma_{a_5}|_{W^\perp}$ (see 2.5.(d) for the case where $\operatorname{char} F = 2$). From $\mathrm{pr}^2(\eta(\lambda_{a_j})) = \sigma_{a_j} \circ \kappa$ (cf. 4.5.(a)) and $\kappa|_{W^\perp} = -\mathrm{id}$ we then infer $\psi :=$

$\lambda_{a_1} \circ \lambda_{a_2} \circ \lambda_{a_3} \circ \lambda_{a_4} \circ \lambda_{a_5} \in \ker(\mathrm{pr}^2_{W^\perp} \circ \eta)$. As $\psi \in \Lambda_{W^\perp}$ has companion $\iota_c$, we have $\psi \notin \Lambda^+_{W^\perp}$ and $\psi \notin \ker(\mathrm{pr}^2 \circ \eta)$, but $\psi^2 \in \Lambda^+_{W^\perp} \cap \ker(\mathrm{pr}^2 \circ \eta)$. So $\ker(\mathrm{pr}^2_{W^\perp} \circ \eta) = \Lambda_F \circ \langle\psi\rangle$, and $\ker(\mathrm{pr}^2_{W^\perp} \circ \eta) \cap \Lambda^+_{W^\perp} = \Lambda_F$. (See also 4.5.(d) and 4.5.(e).)

From 4.5.(b) and 4.5.(c) we know that $\mathrm{pr}^2_{K^\perp}(\eta(\Lambda^+_{K^\perp})) = \mathrm{pr}^2_{K^\perp}(\Gamma^+_{K^\perp}) = \mathrm{pr}^2_{K^\perp}(\Gamma_{K^\perp}) = \mathrm{SO}(K^\perp, N|_{K^\perp})$; recall from 2.5.(b) that $\mathrm{SO}(K^\perp, N|_{K^\perp}) = \mathrm{O}^+(K^\perp, N|_{K^\perp})$ if $\operatorname{char} F \neq 2$ and from 2.5.(d) that $\mathrm{SO}(K^\perp, N|_{K^\perp}) = \mathrm{O}(K^\perp, N|_{K^\perp}) = \mathrm{O}^+(K^\perp, N|_{K^\perp})$ if $\operatorname{char} F = 2$.

Clearly, every element of $\langle \Lambda_W \cup \Lambda_{W^\perp}\rangle$ is $F$-linear, so $\langle \Lambda_W \cup \Lambda^+_{W^\perp}\rangle$ is contained in the group $\Gamma_F\mathrm{U}_H(\mathbb{O}, h) = \mathrm{Aut}_F(H) \ltimes \mathrm{GU}_H(\mathbb{O}, h)$ of all semi-similitudes of $h$ with $F$-linear companions. Every $F$-linear automorphism of the quaternion field $H$ is inner (by the Skolem-Noether Theorem, see [2, Cor. 7.2D] or [17, § 4.6, Cor. to Th. 4.9]). We introduce coordinates with respect to any $H$-basis in the left vector space; then $\Lambda_W$ consists of all multiplications by scalars (from the left) by 2.10.(b), and $\mathrm{GU}_H(\mathbb{O}, h)$ contains the scalar multiples of the identity matrix (acting as multiplications by scalars from the right on our coordinates). So each inner automorphism of $H$ occurs as the companion of some element of $\langle \Lambda_W \cup \Lambda^+_{W^\perp}\rangle$, and that group in fact coincides with $\Gamma_F\mathrm{U}_H(\mathbb{O}, h)$. □

## References


[1] E. Artin, *Geometric algebra*, Interscience Publishers, Inc., New York-London, 1957. MR 0082463 (18,553e). Zbl 0077.02101.

[2] E. Artin, C. J. Nesbitt, and R. M. Thrall, *Rings with Minimum Condition*, University of Michigan Publications in Mathematics, no. 1, University of Michigan Press, Ann Arbor, Mich., 1944. MR 0010543 (6,33e). Zbl 0060.07701.

[3] R. Baer, *Linear algebra and projective geometry*, Academic Press Inc., New York, 1952. MR 0052795 (14,675j). Zbl 0049.38103.

[4] A. Blunck, N. Knarr, B. Stroppel, and M. J. Stroppel, *Clifford parallelisms defined by octonions*, Preprint 2017-006, Fachbereich Mathematik, Universität Stuttgart, Stuttgart, 2017, `http://www.mathematik.uni-stuttgart.de/preprints/downloads/2017/2017-006`.

[5] T. De Medts, *A characterization of quadratic forms of type $E_6$, $E_7$, and $E_8$*, J. Algebra **252** (2002), no. 2, 394–410, ISSN 0021-8693, `doi:10.1016/S0021-8693(02)00064-9`. MR 1925144 (2003f:11046). Zbl 1012.11029.

[6] P. Dembowski, *Finite geometries*, Ergebnisse der Mathematik und ihrer Grenzgebiete 44, Springer-Verlag, Berlin, 1968, ISBN 3-540-61786-8, `doi:10.1007/978-3-642-62012-6`. MR 0233275 (38 #1597). Zbl 0865.51004.

[7] J. A. Dieudonné, *Sur les groupes classiques*, Actualités Sci. Ind., no. 1040 = Publ. Inst. Math. Univ. Strasbourg (N.S.) no. 1 (1945), Hermann et Cie., Paris, 1948. MR 0024439 (9,494c). Zbl 0037.01304.

[8] J. A. Dieudonné, *La géométrie des groupes classiques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.) 5, Springer-Verlag, Berlin, 1955. MR 0072144 (17,236a). Zbl 0221.20056.

[9] R. H. Dye, *A geometric characterization of the special orthogonal groups and the Dickson invariant*, J. London Math. Soc. (2) **15** (1977), no. 3, 472–476, ISSN 0024-6107, doi:10.1112/jlms/s2-15.3.472. MR 0453639 (56 #11899). Zbl 0358.20054.

[10] M. J. Ganley, *Polarities in translation planes*, Geom. Dedicata **1** (1972), no. 1, 103–116, ISSN 0046-5755, doi:10.1007/BF00147384. MR 0307036 (46 #6157). Zbl 0241.50024.

[11] L. C. Grove, *Classical groups and geometric algebra*, Graduate Studies in Mathematics 39, American Mathematical Society, Providence, RI, 2002, ISBN 0-8218-2019-2. MR 1859189 (2002m:20071). Zbl 0990.20001.

[12] T. Grundhöfer, *Über Projektivitätengruppen affiner und projektiver Ebenen unter besonderer Berücksichtigung von Moufangebenen*, Geom. Dedicata **13** (1983), no. 4, 435–458, ISSN 0046-5755, doi:10.1007/BF00147542. MR 699886 (84i:51010). Zbl 0523.51006.

[13] H. Hähl, $\mathrm{SU}_4(\boldsymbol{C})$ *als Kollineationsgruppe in sechzehndimensionalen lokalkompakten Translationsebenen*, Geom. Dedicata **23** (1987), no. 3, 319–345, ISSN 0046-5755, doi:10.1007/BF00181316. MR 900284 (88j:51023). Zbl 0622.51008.

[14] H. Hähl, *Sechzehndimensionale lokalkompakte Translationsebenen mit* $\mathrm{Spin}(7)$ *als Kollineationsgruppe*, Arch. Math. (Basel) **48** (1987), no. 3, 267–276, ISSN 0003-889X, doi:10.1007/BF01195360. MR 880087 (88d:51012). Zbl 0639.51017.

[15] N. Jacobson, *A note on hermitian forms*, Bull. Amer. Math. Soc. **46** (1940), 264–268, ISSN 0002-9904, doi:10.1090/S0002-9904-1940-07187-3. MR 0001957 (1,325d). Zbl 0024.24503. JfM 66.0048.03.

[16] N. Jacobson, *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo (2) **7** (1958), 55–80, ISSN 0009-725X, doi:10.1007/BF02854388. MR 0101253. Zbl 0083.02702.

[17] N. Jacobson, *Basic algebra. II*, W. H. Freeman and Company, New York, 2nd edn., 1989, ISBN 0-7167-1933-9. MR 1009787 (90m:00007). Zbl 0694.16001.

[18] N. Knarr, *Translation planes*, Lecture Notes in Mathematics 1611, Springer-Verlag, Berlin, 1995, ISBN 978-3-540-44724-5, doi:10.1007/BFb0096312. MR 1439965 (98e:51019). Zbl 0843.51004.

[19] N. Knarr and M. Stroppel, *Unitals over composition algebras*, Forum Math. **26** (2014), no. 3, 931–951, ISSN 0933-7741, doi:10.1515/forum-2011-0139. MR 3200355. Zbl 06342615.

[20] N. Knarr and M. J. Stroppel, *Polarities of shift planes*, Adv. Geom. **9** (2009), no. 4, 577–603, ISSN 1615-715X, doi:10.1515/ADVGEOM.2009.028. MR 2574140. Zbl 1181.51003.

[21] N. Knarr and M. J. Stroppel, *Polarities and planar collineations of Moufang planes*, Monatsh. Math. **169** (2013), no. 3-4, 383–395, ISSN 0026-9255, doi:10.1007/s00605-012-0409-6. MR 3019290. Zbl 06146027.

[22] N. Knarr and M. J. Stroppel, *Heisenberg groups over composition algebras*, Beitr. Algebra Geom. **57** (2016), 667–677, ISSN 0138-4821, doi:10.1007/s13366-015-0276-0.

[23] N. Knarr and M. J. Stroppel, *Subforms of norm forms of octonion fields*, Preprint 2017-005, Fachbereich Mathematik, Universität Stuttgart, Stuttgart, 2017, http://www.mathematik.uni-stuttgart.de/preprints/downloads/2017/2017-005.pdf.

[24] S. Lang, *Algebra*, Graduate Texts in Mathematics 211, Springer-Verlag, New York, 3rd edn., 2002, ISBN 0-387-95385-X. MR 1878556 (2003e:00003). Zbl 0984.00001.

[25] G. Pickert, *Projektive Ebenen*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete LXXX, Springer-Verlag, Berlin, 1955. MR 0073211 (17,399e). Zbl 0066.38707.

[26] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, and M. J. Stroppel, *Compact projective planes*, de Gruyter Expositions in Mathematics 21, Walter de Gruyter & Co., Berlin, 1995, ISBN 3-11-011480-1. MR 1384300 (97b:51009). Zbl 0851.51003.

[27] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000, ISBN 3-540-66337-1. MR 1763974 (2001f:17006). Zbl 1087.17001.

[28] M. J. Stroppel, *Orthogonal polar spaces and unitals*, Innov. Incidence Geom. **12** (2011), 167–179, ISSN 1781-6475, http://iig.ugent.be/online/12/volume-12-article-11-online.pdf. MR 2942723. Zbl 06225476.

[29] D. E. Taylor, *The geometry of the classical groups*, Sigma Series in Pure Mathematics 9, Heldermann Verlag, Berlin, 1992, ISBN 3-88538-009-9. MR 1189139 (94d:20028). Zbl 0767.20001.

[30] J. Tits and R. M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, ISBN 3-540-43714-2. MR 1938841 (2003m:51008). Zbl 1010.20017.

Andrea Blunck
Dept. of Mathematics
Universität Hamburg
Bundesstr. 55
D-20146 Hamburg
Germany

Norbert Knarr
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

Bernhild Stroppel
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

Markus J. Stroppel
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

## Erschienene Preprints ab Nummer 2016-001

Komplette Liste: `http://www.mathematik.uni-stuttgart.de/preprints`

2017-007 *Blunck, A.; Knarr, N.; Stroppel, B.; Stroppel, M.:* Transitive groups of similitudes generated by octonions

2017-006 *Blunck, A.; Knarr, N.; Stroppel, B.; Stroppel, M.:* Clifford parallelisms defined by octonions

2017-005 *Knarr, N.; Stroppel, M.:* Subforms of Norm Forms of Octonion Fields

2017-004 *Apprich, C.; Dieterich, A.; Höllig, K.; Nava-Yazdani, E.:* Cubic Spline Approximation of a Circle with Maximal Smoothness and Accuracy

2017-003 *Fischer, S.; Steinwart, I.:* Sobolev Norm Learning Rates for Regularized Least-Squares Algorithm

2017-002 *Farooq, M.; Steinwart, I.:* Learning Rates for Kernel-Based Expectile Regression

2017-001 *Bauer, B.; Devroye, L; Kohler, M.; Krzyzak, A.; Walk, H.:* Nonparametric Estimation of a Function From Noiseless Observations at Random Points

2016-006 *Devroye, L.; Györfi, L.; Lugosi, G.; Walk, H.:* On the measure of Voronoi cells

2016-005 *Kohls, C.; Kreuzer, C.; Rösch, A.; Siebert, K.G.:* Convergence of Adaptive Finite Elements for Optimal Control Problems with Control Constraints

2016-004 *Blaschzyk, I.; Steinwart, I.:* Improved Classification Rates under Refined Margin Conditions

2016-003 *Feistauer, M.; Roskovec, F.; Sändig, AM.:* Discontinuous Galerkin Method for an Elliptic Problem with Nonlinear Newton Boundary Conditions in a Polygon

2016-002 *Steinwart, I.:* A Short Note on the Comparison of Interpolation Widths, Entropy Numbers, and Kolmogorov Widths

2016-001 *Köster, I.:* Sylow Numbers in Spectral Tables