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**Abstract**

We characterize the Miquelian Laguerre planes of odd order by the existence of shift groups in affine derivations.

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**Keywords:** Laguerre plane, translation plane, shift plane, shift group, Miquelian Laguerre plane.

## Introduction

A finite Laguerre plane $\mathscr{L} = (P, \mathscr{C}, \mathscr{G})$ of order $n$ consists of a set $P$ of $n(n+1)$ points, a set $\mathscr{C}$ of $n^3$ circles and a set $\mathscr{G}$ of $n+1$ generators, where both circles and generators are subsets of $P$, such that the following three axioms are satisfied.

(G) $\mathscr{G}$ partitions $P$, each generator contains $n$ points, and there are $n+1$ generators.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator are joined by a unique circle.

Circles through $x$ are called *touching in $x$* if they are equal or have no other point in common. The set of all circles through a given point $x$ is denoted by $\mathscr{C}_x$. The *derived affine plane* $\mathbb{A}_x = (P \smallsetminus [x], \mathscr{C}_x \cup \mathscr{G} \smallsetminus \{[x]\})$ at a point $x \in P$ has the collection of all points not on the generator $[x]$ through $x$ as point set and, as lines, all circles passing through $x$ (without the point $x$) and all generators apart from $[x]$. The axioms above easily yield that $\mathbb{A}_x$ is an affine plane, indeed. We refer to the generators as *vertical lines* in $\mathbb{A}_x$. Circles that touch each other in $x$ give parallel lines in $\mathbb{A}_x$. A line $W$ is introduced to obtain the projective completion $\mathbb{P}_x$ of $\mathbb{A}_x$; the common point of the verticals will be denoted by $v \in W$.

The group $\mathrm{Aut}(\mathscr{L})$ of all automorphisms of a Laguerre plane $\mathscr{L}$ acts on the set $\mathscr{G}$ of generators. We call $\mathscr{L}$ an *elation Laguerre plane* if the kernel $\Delta$ of that action acts transitively on the set $\mathscr{C}$ of circles. It is known (see [5, 1.3]) that in every finite elation Laguerre plane the group $\Delta$ has a (unique) regular normal subgroup $E$ called the *elation group*. For more details on elation Laguerre planes, we refer the reader to the introduction in [6].

In the present note, we only use a weaker transitivity assumption on $\Delta$ but combine this with additional assumptions. Our results can (and will) be applied to elation Laguerre planes with additional homogeneity assumptions, e.g. in [7] (cf. 2.3 below).

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# 1 Translation planes

**1.1 Theorem.** *Let $\mathbb{P}$ be a finite projective plane of order $n$. Assume that a subgroup $D \le \mathrm{Aut}(\mathbb{P})$ fixes each point on some line $L$. If $n^2$ divides the order of $D$ then $D$ contains a subgroup $T$ of order $n^2$ consisting of elations with axis $L$. In particular, the plane $\mathbb{P}$ is a translation plane, and the order $n$ is a prime power.*

*Proof.* For each non-trivial element $\delta \in D$ there is a (unique) center $c_\delta$; i.e. a point $c_\delta$ such that $\delta$ fixes each line through $c_\delta$ ([1], see [2, Thm. 4.9]). The elations in $D$ are just those in the set $T := \{\mathrm{id}\} \cup \left\{\tau \in D \setminus \{\mathrm{id}\} \mid c_\tau \in L\right\}$; that set forms a normal subgroup of $D$ (see [2, Thm 4.13]).

For any point $x$ outside $L$, the stabilizer $D_x$ consists of id and elements with center $x$. The order of any element of $D_x$ divides $n-1$. So the order of $D_x$ and the number $n^2$ of points outside $L$ are co-prime, and $D$ acts transitively on the set $A$ of points outside $L$. For each $\delta \in D \setminus T$ we have $c_\delta \notin L$, and $\delta \in D_{c_\delta}$ yields that the order of $\delta$ divides $n-1$, and is co-prime to $n^2$.

Let $\mathscr{B}$ denote the set of $T$-orbits in $A$. Then $D$ acts on $\mathscr{B}$, and so does $D/T$ because $T \trianglelefteq D$ acts trivially on $\mathscr{B}$. Transitivity of $D$ on $A$ implies that $D/T$ is transitive on $\mathscr{B}$. Now $|\mathscr{B}| = n^2/|T|$ divides $|D/T|$. The latter order is co-prime to $n^2$ because each member of the quotient has a representative of order co-prime to $n^2$. So $|\mathscr{B}| = 1$, and transitivity of $T$ is proved. □

**1.2 Theorem.** *Let $\mathscr{L}$ be a Laguerre plane of finite order $n$. If $\infty$ is a point such that $n^2$ divides the order of the stabilizer $\Delta_\infty$ then the derived projective plane $\mathbb{P}_\infty$ is a dual translation plane, and the order $n$ is a prime power.*

*Proof.* The group $D$ induced by $\Delta_\infty$ on the dual $\mathbb{P}$ of $\mathbb{P}_\infty$ satisfies the assumptions of 1.1. □

**1.3 Theorem.** *Let $\mathscr{L}$ be a Laguerre plane of finite order $n$, and assume that there is a point $\infty$ such that $n^2$ divides the order of the stabilizer $\Delta_\infty$. If there exist a circle $K \in \mathscr{C}_\infty$ and a subgroup $H \le \mathrm{Aut}(\mathscr{L})_\infty$ such that $H$ fixes each circle touching $K$ in $\infty$ and $H$ acts transitively on $K \setminus \{\infty\}$, then $\mathbb{P}_\infty$ has Lenz type V (at least), and is coordinatized by a semifield.*

*Proof.* From 1.2 we know that $\mathbb{P}_\infty$ is a dual translation plane. The translation axis in the dual of $\mathbb{P}_\infty$ is the common point $v$ for the generators in the projective closure of $\mathbb{A}_\infty$. The elations of $\mathbb{P}_\infty$ with center $v$ and axis $W$ form a group of order $n$; we denote that group by $V$ and note that $V$ is a group of translations of $\mathbb{A}_\infty$.

Our assumptions on $H$ secure that $H$ induces a group of translations of $\mathbb{A}_\infty$; the common center is the point at infinity for the "horizontal line" $K \setminus \{\infty\}$. We obtain a transitive group $HV$ of translations on $\mathbb{A}_\infty$. So $\mathbb{P}_\infty$ is also a translation plane, and has Lenz type V at least. □

# 2 Shift groups

Recall that a shift group on a projective plane is a group of automorphisms fixing an incident point-line pair $(x, Y)$ and acting regularly both on the set of points outside $Y$ and on the set of lines not through $x$.

**2.1 Theorem.** *Let $\mathscr{L}$ be a finite elation Laguerre plane of odd order, and assume that there exists a point $u$ and a subgroup $S \le \mathrm{Aut}(\mathscr{L})_u$ such that $S$ induces a transitive group of translations on the affine plane $\mathbb{A}_u$.*

1. *If $s \in [u] \setminus \{u\}$ is fixed by $S$ then $S$ induces a shift group on $\mathbb{P}_s$.*
2. *If $S$ fixes a point $t$ of $\mathscr{L}$ and induces a transitive group of translations on $\mathbb{A}_t$ then $t = u$.*

*Proof.* Let $n$ denote the order of $\mathscr{L}$. Assume that $s \in [u] \smallsetminus \{u\}$ is fixed by $S$. Then $S$ induces a group of automorphisms of $\mathbb{P}_s$; we have to exhibit an incident point-line pair $(x, Y)$ such that $S$ acts regularly both on the set of points outside $Y$ and on the set of lines not through $x$.

It is obvious that $S$ acts regularly on the set of affine points in $\mathbb{P}_s$ because that set coincides with the set of points of $\mathbb{A}_u$. We let the line $W$ at infinity play the role of $Y$. Also, the set of vertical lines (induced by generators) is invariant under $S$, we let their point at infinity play the role of $x$ (so $x = v \in W$).

It remains to show that $S$ acts regularly on the set of non-vertical lines of $\mathbb{A}_s$; these lines are induced by the circles through $s$. Assume that $\tau \in S$ fixes a circle $C$ through $s$. Our assumption that $n$ be odd implies that the translation of $\mathbb{A}_u$ induced by $\tau$ does not have any orbit of length 2, and we obtain that $\tau$ is trivial if there is a set of one or two points outside $[u]$ invariant under $\tau$.

As $\tau$ induces a translation on $\mathbb{A}_u$, there exists $D \in \mathscr{C}_u$ such that $\tau$ fixes each circle touching $D$ in $u$ (these circles induce the parallels to the line induced by $D$ on $\mathbb{A}_u$). Pick a point $z \in C$, and let $D'$ be the circle through $z$ touching $D$ in $u$. Then $\tau$ leaves the intersection $D' \cap C$ invariant. This is a set with one ore two elements, and we find that $\tau$ is trivial. So the orbit of $C$ under $S$ has length $|S| = n^2$, and fills all of $\mathscr{C}_s$. Thus $S$ acts regularly on the set of non-vertical lines of $\mathbb{A}_s$, as required.

Now assume that $S$ fixes $t$ and induces a transitive group of translations on $\mathbb{A}_t$. Then $t \in [u]$ because $S$ acts regularly on the set of points outside $[u]$. For any circle $C \in \mathscr{C}_t$, we pick two points $a, b \in C \smallsetminus \{t\}$. Then there exists $\tau \in S$ such that $\tau(a) = b$. As $\tau$ is a translation both of $\mathbb{A}_u$ and of $\mathbb{A}_t$, the orbit of $a$ under $\langle\tau\rangle$ is contained both in the line $C$ of $\mathbb{A}_t$ and in some line $B$ of $\mathbb{A}_u$, that is, in some circle $B$ through $u$. Since $n$ is odd, that orbit has at least three points, and $B = C$. This yields $t = u$, as claimed. □

**2.2 Theorem.** *Assume that $\mathscr{L}$ is a Laguerre plane of odd order $n$, and let $\infty$ be a point. Let $U$ denote the set of all points $u \in [\infty] \smallsetminus \{\infty\}$ such that there exists a subgroup $S_u \le \mathrm{Aut}(\mathscr{L})$ of order $n^2$ fixing both $\infty$ and $u$ and acting as a group of translations on $\mathbb{A}_u$. Then the following hold:*

1. *There are at least $|U|$ many different shift groups on $\mathbb{P}_\infty$.*

2. *If $|U| > 1$ then $\mathbb{A}_\infty$ is a translation plane.*

3. *If $\mathbb{A}_\infty$ is a translation plane and $U$ is not empty then $\mathbb{P}_\infty$ has Lenz type V at least, can be coordinatized by a commutative semifield, and the middle nucleus of such a coordinatizing semifield has order at least $|U| + 1$.*

4. *If $|U| > \sqrt{n}$ then $\mathbb{P}_\infty$ is Desarguesian.*

*Proof.* Using 2.1 we see for any $u \in U$ that $S_u$ is a shift group on $\mathbb{A}_\infty$, and different points $t, u \in U$ yield different groups $S_t$ and $S_u$. This gives the first assertion. All these shift groups have the same fixed flag in $\mathbb{P}_\infty$.

If a finite projective plane admits more than one shift group, it is a translation plane, see [3, 10.2]. If a translation plane admits at least one shift group then it can be coordinatized by a commutative semifield ([3, 9.12], [4]) and the different shift groups with the same fixed flag are parameterized by the non-zero elements of the middle nucleus of such a semifield, see [3, 9.4].

The additive group of the coordinatizing semifield forms a vector space over the middle nucleus (see [2, p. 170]). If the middle nucleus has more than $\sqrt{n}$ elements then that vector space has dimension one, and the middle nucleus coincides with the semifield. This means that the semifield is a field, and the plane is Desarguesian. □

In [7], our present result 2.2 is used to prove the following:

**2.3 Theorem.** *Let $\mathscr{L}$ be an elation Laguerre plane of odd order. If there exists a point $\infty$ such that* $\mathrm{Aut}(\mathscr{L})_\infty$ *acts two-transitively on* $\mathscr{G} \smallsetminus \{[\infty]\}$ *then the affine plane* $\mathbb{A}_\infty$ *is Desarguesian, and* $\mathscr{L}$ *is Miquelian.* □

**2.4 Remark.** If $\mathbb{P}$ is a projective plane of even order then a shift group on $\mathbb{P}$ will never be elementary abelian, see [3, 1.5, 5.8]. Thus a shift group on such a plane will not act as transitive group of translations on any other affine plane (of the same order).

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