Universität Stuttgart

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Norbert Knarr
Markus J. Stroppel

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Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik.uni-stuttgart.de/preprints

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Abstract

We classify embeddings (i.e., "labeled drawings") of the Pappus configuration in projective planes over commutative fields, up to projective equivalence. Using pairs of field elements, we parameterize the space of classes of projectively equivalent embeddings, and then explicitly determine the group of ambient automorphisms (or dualities) for any given parameter pair, i.e., the subgroup of the group of all automorphisms (and dualities) of the abstract configuration that are induced by projective collineations (or dualities) leaving invariant the image under under any embedding in the given class. It turns out that the existence of an ambient duality implies an ambient polarity.

We show that these parameter pairs can be interpreted as pairs of cross ratios associated in a rather natural way with the embedded configuration. The number of equivalence classes of embeddings in a projective plane over a given finite field is determined.

The groups that occur as full ambient groups are identified in the subgroup lattice of the full automorphism group of the abstract configuration.

Finally, we use our results to understand embeddings of the Möbius-Kantor configuration.

MSC 2010: 05B30, 51A10, 51E30.

Keywords: Pappus configuration, Möbius-Kantor configuration, automorphism, embedding, duality, polarity.

Introduction

The Pappus configuration is (together with the Desargues configuration) one of the most important configurations in the foundations of (projective) geometry; it is used to secure that coordinates from a commutative field can be introduced for a given projective space.

Hilbert and Cohn-Vossen ([11, p. 132]) refer to the Pappus configuration as occurring in a special case of Pascal's theorem, writing[1] "Therefore we may say that Pascal's theorem is the only significant theorem on incidence in the plane and that the configuration $(9_3)_1$ is the most important figure in plane geometry." The earliest known formulation of Pappus' theorem is contained in Propositions 138, 139, 141, and 143 of Book VII of Pappus' Collection, see [21, 7.(206)–7.(211), pp. 270–277] (or the older edition [20, pp. 888–893]).

In the present paper, we consider embeddings of the abstract configuration into pappian projective planes (i.e., projective planes over commutative fields), and study those automorphisms of the abstract configuration that extend to projective collineations of the ambient plane. Analogous

[1] In the German original ([10, p. 117]): "[…] so können wir sagen, daß der Satz von Pascal der einzig wesentliche Schnittpunktsatz der Ebene ist, daß also die Konfiguration $(9_3)_1$ die wichtigste Figur der ebenen Geometrie darstellt."

studies for the Desargues configuration have been documented in [13]. Our present study extends observations by Coxeter ([4], [5], [6]) and Mielants [17]; see the more detailed remarks below.

The Pappus configuration Π is obtained as follows. Points of Π are the elements of the set $\{0,1,2,3,4,5,03,14,25\}$ of nine symbols, blocks are the sets $\{0,2,4\}$, $\{1,3,5\}$, $\{03,14,25\}$, and $\{j,k,uv\}$ with $j,k \in \{0,1,2,3,4,5\}$ such that $|k-j| = 3$ and $\{j,k\} \cap \{u,v\} = \emptyset$. See Figures 2, 3 and 8 below for various graphic representations of Π in the real projective (or the euclidean) plane; the blocks are indicated by straight line segments.

Each one of the three sets $\{0,3,03\}$, $\{1,4,14\}$, and $\{2,5,25\}$, respectively, consists of three points such that no two of them are joined by a line of Π. We refer to these sets as the *triads* of the Pappus configuration; they will play an important role in the study of embeddings of Π into projective planes. Dually, we have three *parallel classes* in Π (i.e., sets of three pairwise disjoint lines of Π).

The earliest study of Π as a configuration in its own right that we know of is in Levi's monograph [16, pp. 108 ff]. Coxeter ([4], [5], [6]) studies embeddings of Π into $\mathbb{P}_2(\mathbb{F})$ for $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, concentrating on the case where at least one of the triads is not collinear under the embedding. In those papers, the (essentially unique, see 2.1 below) embedding into $\mathbb{P}_2(\mathbb{F}_3)$ is employed to understand the full group of automorphisms and dualities of Π. A clearer (and much earlier) statement about $\operatorname{Aut}(\Pi)$ is proved in [16, pp. 110]. In the present paper, we consider embeddings into planes over arbitrary (commutative) fields, without any restrictions regarding collinearity of triads or confluence of parallel classes.

In Section 9, we use our present results in order to study related questions for the Möbius-Kantor configuration which is closely related to the Pappus configuration.

1 Automorphisms, embeddings and dualities

1.1 Definitions. Let $\mathbb{A}_0 = (A_0, B_0, *_0)$ and $\mathbb{A}_1 = (A_1, B_1, *_1)$ be incidence structures, with point sets A_j, block (or line) sets B_j, and incidence relations $*_j$, respectively. Without loss of generality, we assume that the sets A_j and B_j are disjoint, and consider the incidence relations as *symmetric* relations.

An *embedding* from $\mathbb{A}_0$ into $\mathbb{A}_1$ is an injective map $\varepsilon\colon A_0 \dot\cup B_0 \to A_1 \dot\cup B_1$ such that $A_0^\varepsilon \subseteq A_1$, $B_0^\varepsilon \subseteq B_1$, and $a *_0 b \iff a^\varepsilon *_1 b^\varepsilon$ holds for all $(a,b) \in A_0 \times B_0$. An *isomorphism* from $\mathbb{A}_0$ onto $\mathbb{A}_1$ is a surjective embedding, and an *automorphism* of $\mathbb{A}_0$ is an isomorphism of $\mathbb{A}_0$ onto itself.

A *duality* from $\mathbb{A}_0$ onto $\mathbb{A}_1$ is a bijection $\delta\colon A_0 \dot\cup B_0 \to B_1 \dot\cup A_1$ such that $A_0^\varepsilon = B_1$, $B_0^\varepsilon = A_1$, and $a *_0 b \iff a^\delta *_1 b^\delta$ holds for all $(a,b) \in A_0 \times B_0$. (Note that we use symmetry of the incidence relation here.) A *polarity* of $\mathbb{A}_0$ is an involutory duality of $\mathbb{A}_0$ onto itself. A *dual embedding* (or *dumbedding*, for short) from $\mathbb{A}_0$ into $\mathbb{A}_1$ is an injective map $\varepsilon\colon A_0 \dot\cup B_0 \to A_1 \dot\cup B_1$ such that $A_0^\varepsilon \subseteq B_1$, $B_0^\varepsilon \subseteq A_1$, and $a *_0 b \iff a^\varepsilon *_1 b^\varepsilon$ holds for all $(a,b) \in A_0 \times B_0$. In other words, a dumbedding is the concatenation of a duality and an embedding, or vice versa.

If every block of $\mathbb{A}$ is determined by the set of points incident with it, any embedding (or dumbedding) is determined by its restriction on the point set of $\mathbb{A}$. Abusing notation, we will sometimes use the same name for the embedding and for its restriction to the point set.

1.2 Remark. In general, our definition of embeddings imposes more severe restrictions than just requiring an injective map of points such that collinear point sets are mapped into collinear point sets. (E.g., minors of the Desargues configuration occur in this way, see [13, Sect. 2].) For embeddings of the Pappus configuration into projective spaces, however, we do not lose any interesting examples. In fact, dropping the injectivity condition on the block set would only add injective maps from the point set of Π into a single line of $\mathbb{P}_2(\mathbb{F})$, see [26, 1.8]. Moreover, the injectivity conditions on both the point and the line map imply that the image of a non-incident point-line pair will not be incident because the incidence graph of Π has diameter 4, see Figure 5.

1.3 Examples. Let $\mathbb{F}$ be a commutative field, and let n be a positive integer. We consider the vector space $\mathbb{F}^{n+1}$ of *rows* and the dual space of linear forms, written as *columns*. The projective space $\mathbb{P}_n(\mathbb{F})$ of (projective) dimension n over $\mathbb{F}$ will be interpreted as the incidence structure $(P, L, <)$ with P the set of all one-dimensional vector subspaces, and L the set of all two-dimensional vector subspaces of $\mathbb{F}^{n+1}$, respectively; incidence is containment. Every linear bijection of $\mathbb{F}^{n+1}$ onto itself induces an automorphism of $\mathbb{P}_n(\mathbb{F})$; such automorphisms are called *projective transformations* of $\mathbb{P}_n(\mathbb{F})$. We write $\Phi := \mathrm{PGL}_3(\mathbb{F})$ for the group of all projective transformations of the projective plane $\mathbb{P}_2(\mathbb{F})$ over $\mathbb{F}$.

If $n = 2$ then interchanging $\mathbb{F}(x, y, z)$ with $\ker(x, y, z)^\intercal = \left\{(u, v, w) \in \mathbb{F}^3 \,\middle|\, ux + vy + wz = 0\right\}$ is a duality (in fact, a polarity) θ of $\mathbb{P}_2(\mathbb{F})$ onto itself. A duality of $\mathbb{P}_2(\mathbb{F})$ onto itself is called a *projective duality* (see [15, IV.7, pp. 124 f]) if it is of the form $\theta\varphi$ with a projective transformation φ.

1.4 Definitions. Two embeddings ε and ε' from $\mathbb{A}_0 = (A_0, B_0, *_0)$ into $\mathbb{A}_1 = (A_1, B_1, *_1)$ are called *equivalent* if there exists an automorphism γ_1 of $\mathbb{A}_1$ such that $\varepsilon = \varepsilon'\gamma_1$. The embeddings are called *quasi-equivalent* if there exists an automorphism γ_0 of $\mathbb{A}_0$ such that $\gamma_0\varepsilon$ and ε' are equivalent, i.e., if there exists a pair (γ_0, γ_1) of automorphisms γ_j of $\mathbb{A}_j$ such that $\varepsilon = \gamma_0^{-1}\varepsilon'\gamma_1$.

Two embeddings ε, ε' from $\mathbb{A}_0$ into $\mathbb{P}_n(\mathbb{F})$ are called *projectively equivalent* if there exists a projective transformation $\varphi \in \Phi$ such that $\varepsilon = \varepsilon'\varphi$, and the embeddings are called *projectively quasi-equivalent* if there exist an automorphism γ of $\mathbb{A}_0$ and a projective transformation φ such that $\gamma\varepsilon = \varepsilon'\varphi$.

1.5 Definitions. For a given incidence structure $\mathbb{A} = (A, B, *)$, we abbreviate $\Gamma := \mathrm{Aut}(\mathbb{A})$, and write $\overline{\Gamma}$ for the group of all automorphisms and dualities of $\mathbb{A}$.

Let $\varepsilon\colon \mathbb{A} \to \mathbb{P}_2(F)$ be a an embedding. An automorphism $\gamma \in \Gamma$ is called *ambient* (under ε) if there exists $\varphi \in \Phi$ such that $\gamma\varepsilon = \varepsilon\varphi$; i.e., if the embeddings $\gamma\varepsilon$ and ε are projectively equivalent. Loosely speaking, an ambient automorphism extends to a projective collineation. A duality $\delta \in \overline{\Gamma} \smallsetminus \Gamma$ is called *ambient* (under ε) if there exists $\varphi \in \Phi$ such that $\delta\varepsilon = \varepsilon\theta\varphi$; i.e., if the dumbeddings $\delta\varepsilon$ and $\varepsilon\theta$ are projectively equivalent.

The set of all ambient automorphisms with respect to ε forms a subgroup $\Gamma_{\mathrm{amb}} := \Gamma^{\varepsilon}_{\mathrm{amb}} \le \Gamma$, and that subgroup together with the set of all ambient dualities with respect to ε forms a subgroup $\overline{\Gamma}_{\mathrm{amb}} := \overline{\Gamma}^{\varepsilon}_{\mathrm{amb}} \le \overline{\Gamma}$.

1.6 Lemma. **(a)** *We have actions* $\big(\varepsilon, (\gamma, \varphi)\big) \mapsto \gamma^{-1}\varepsilon\varphi$ *of the direct product* $\Gamma \times \Phi$ *(from the right) on the set of all embeddings of* $\mathbb{A}$ *into* $\mathbb{P}_2(\mathbb{F})$*, and* $\big(\delta, (\gamma, \varphi)\big) \mapsto \gamma^{-1}\delta\varphi$ *on the set of all dumbeddings from* $\mathbb{A}$ *into* $\mathbb{P}_2(\mathbb{F})$*.*

(b) *The classes of embeddings of* $\mathbb{A}$ *into* $\mathbb{P}_2(\mathbb{F})$ *modulo projective equivalence are orbits* $\varepsilon\Phi$ *(under the group* $\{\mathrm{id}\} \times \Phi \cong \Phi$*), while the classes modulo projective quasi-equivalence are orbits* $\Gamma\varepsilon\Phi$ *under* $\Gamma \times \Phi$*.* □

1.7 Lemma. *Let* $\mathbb{A}_0 = (A_0, B_0, *_0)$ *and* $\mathbb{A}_1 = (A_1, B_1, *_1)$ *be incidence structures.*

(a) *Let* $\iota\colon \mathbb{A}_0 \to \mathbb{A}_1$ *and* $\eta\colon \mathbb{A}_0 \to \mathbb{A}_1$ *be both embeddings or both dumbeddings. Then* $(A_0 \cup B_0)^\iota = (A_0 \cup B_0)^\eta$ *holds if, and only if, there exists* $\gamma \in \Gamma$ *such that* $\eta = \gamma\iota$*.*

(b) *Let* $\iota\colon \mathbb{A}_0 \to \mathbb{A}_1$ *be an embedding, and let* $\psi\colon \mathbb{A}_0 \to \mathbb{A}_1$ *be a dumbedding. Then* $(A_0 \cup B_0)^\iota = (A_0 \cup B_0)^\psi$ *holds if, and only if, there exists a duality* $\delta \in \overline{\Gamma} \smallsetminus \Gamma$ *such that* $\psi = \delta\iota$*.*

Proof. Consider first ι and η as in assertion (a). For each $x \in A_0 \cup B_0$ we then have a unique element $x^\gamma \in A_0 \cup B_0$ such that $(x^\gamma)^\iota = x^\eta$. This defines a bijection γ of $A_0 \cup B_0$ onto itself such that $A_0^\gamma = A_0$ and $B_0^\gamma = B_0$. For $a \in A_0$ and $b \in B_0$ we find $a *_0 b \iff a^\eta *_1 b^\eta \iff (a^\gamma)^\iota *_1 (b^\gamma)^\iota \iff a^\gamma *_0 b^\gamma$. This shows that γ is an automorphism of $\mathbb{A}_0$, as claimed.

The proof of assertion (b) is completely analogous; using the bijection δ such that $(x^\delta)^\iota = x^\psi$; here $A_0^\delta = B_0$ and $B_0^\delta = A_0$. □

1.8 Remark. In 1.7, it is crucial that we consider the image of an incidence structure (consisting of points and lines) and not just the image of the point set.

For example, every embedding of the Pappus configuration Π into the projective plane $\mathbb{P}_2(\mathbb{F}_3)$ of order 3 is projectively equivalent to the one given in 2.1 below. The image of the point set of Π is thus the complement of some line W in the point set of $\mathbb{P}_2(\mathbb{F}_3)$, while the image of the line set is the union of three classes of parallel lines in the affine plane obtained by deleting W from $\mathbb{P}_2(\mathbb{F}_3)$; one parallel class is missing. The embedding is determined (up to an automorphism of Π) by the choice of that parallel class.

1.9 Definition. Let $\varepsilon\colon \Pi \to \mathbb{P}_2(\mathbb{F})$ be an embedding of the Pappus configuration into the projective plane over a commutative field $\mathbb{F}$. Then the image Π^ε (considered as a set of 9 points and 9 lines) is called the *Pappus figure* obtained by ε. Two Pappus figures Π^ε and Π^ι are called *projectively equivalent* if there exists $\varphi \in \Phi$ such that $\Pi^\varepsilon = \Pi^{\iota\varphi}$.

From 1.7 we infer that the same Pappus figure is obtained by two embeddings ε and ι if, and only if, there exists $\gamma \in \mathrm{Aut}(\Pi)$ such that $\varepsilon = \gamma\iota$. Consequently, two Pappus figures are projectively equivalent precisely if they are obtained by projectively quasi-equivalent embeddings.

2 Automorphisms of the Pappus configuration

We start with an embedding of the Pappus configuration in the projective plane $\mathbb{P}_2(\mathbb{F}_3)$ of order 3. This embedding is very convenient (and, in fact, special — see 3.5 below) because it exhibits *all* automorphisms and all dualities of the abstract configuration.

2.1 The Pappus configuration in the affine plane of order three. Up to projective equivalence, there is only one embedding of Π into $\mathbb{P}_2(\mathbb{F}_3)$, i.e. only one way to “draw” the Pappus configuration in the projective plane of order three; see [26, 3.1], cp. also our result in 6.1 below.

In order to give such an embedding explicitly, we define a map $p \mapsto \overline{p}$ from the set of points of the Pappus configuration to the projective plane $\mathbb{P}_2(\mathbb{F}_3)$, as follows:

$$\begin{aligned}
\overline{0} &= \mathbb{F}_3(1,0,0), & \overline{2} &= \mathbb{F}_3(1,1,0), & \overline{4} &= \mathbb{F}_3(1,2,0),\\
\overline{03} &= \mathbb{F}_3(1,0,1), & \overline{25} &= \mathbb{F}_3(1,1,1), & \overline{14} &= \mathbb{F}_3(1,2,1),\\
\overline{1} &= \mathbb{F}_3(1,2,2), & \overline{3} &= \mathbb{F}_3(1,0,2), & \overline{5} &= \mathbb{F}_3(1,1,2);
\end{aligned}$$

the effect on the line set can be seen from Figure 1.

The points of the Pappus figure $\overline{\Pi}$ obtained by this embedding are just those *not* on the line $\mathbb{F}_3(0,1,0)+\mathbb{F}_3(0,0,1)$, and the blocks are induced by the lines *not* through the point $\mathbb{F}_3(0,0,1)$.

The stabilizer of the figure $\overline{\Pi}$ in $\mathrm{PGL}_3(\mathbb{F}_3)$ is thus induced by (and isomorphic to) the subgroup $\left\{\left.\begin{pmatrix} 1 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}\right|\ a_{jk}\in\mathbb{F}_3, a_{22}a_{33}\neq 0\right\}$ of $\mathrm{GL}_3(\mathbb{F}_3)$. Note that this stabilizer is a Borel subgroup (i.e., a minimal parabolic subgroup) in $\mathrm{PGL}_3(\mathbb{F}_3)$.

This group has order $2^2{\cdot}3^3 = 3!{\cdot}2{\cdot}3^2$, and induces[2] the full group $\Gamma := \mathrm{Aut}(\Pi)$ of all automorphisms of the (abstract) Pappus configuration Π. In other words, every automorphism of Π is ambient under the present embedding into $\mathbb{P}_2(\mathbb{F}_3)$.

[2] See [26, 3.1]. The order of $\mathrm{Aut}(\Pi)$ has already been determined by Schoenflies [25, p. 59]; its isomorphism type has been noted in Levi’s 1929 monograph [16, pp. 108 ff]. A different description of the group is given in [4, Section 4, pp. 261–266].

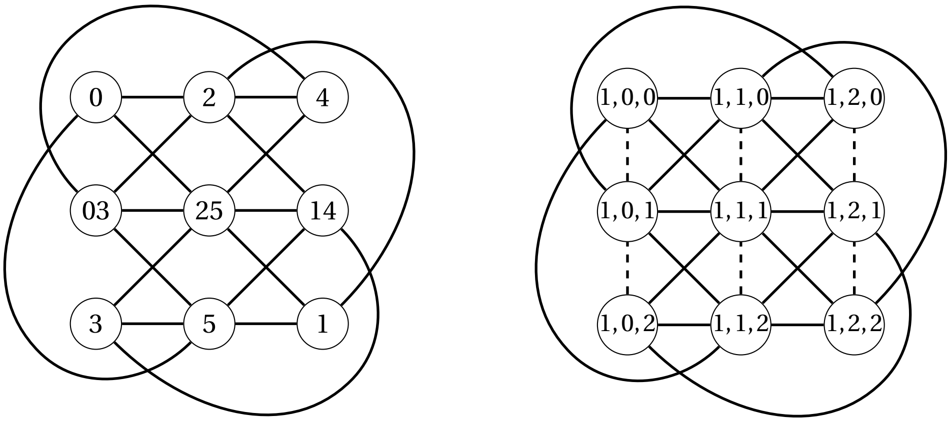


Figure 1: The Pappus configuration (left), and a Pappus figure in $\mathbb{A}_2(\mathbb{F}_3)$ (right).

2.2 Example. The Pappus configuration admits dualities, and each one of those dualities extends to a duality of $\mathbb{P}_2(\mathbb{F}_3)$. For instance, let $\tilde{\pi}$ be the polarity of $\mathbb{P}_2(\mathbb{F}_3)$ interchanging $\mathbb{F}_3(x, y, z)$ with the kernel of the linear map with matrix $(z, -y, x)^\intercal$. Then $\tilde{\pi}$ interchanges the point $\mathbb{F}_3(0,0,1)$ with the line $\mathbb{F}_3(0,1,0) + \mathbb{F}_3(0,0,1)$ and thus (via the embedding given in 2.1) induces a polarity π on the abstract Pappus configuration Π. The set of all dualities of Π is then the coset $\pi\Gamma$, and $\overline{\Gamma} = \Gamma \cup \pi\Gamma$ is the group of all automorphisms and dualities of Π.

2.3 Definition. By a *bow tie* in Π we mean an ordered quintuple (a, b, c, d, e) of five different points of Π such that $\{a, b\}$, $\{b, c, d\}$, $\{d, e\}$, and $\{e, c, a\}$ are collinear in Π.

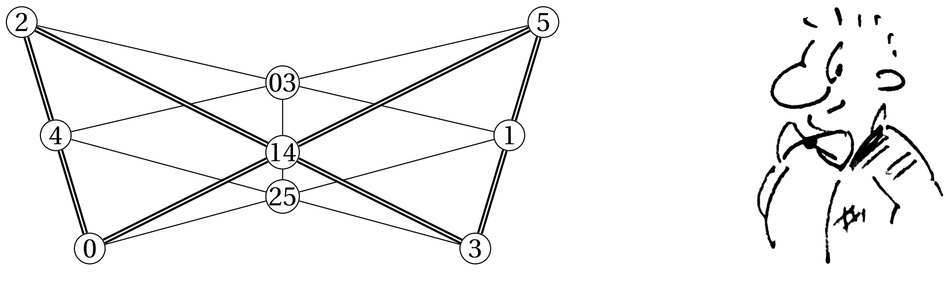


Figure 2: The bow tie $(0, 2, 14, 3, 5)$ in Π (left), and a more formal bow tie (right)

2.4 Lemma. *The bow ties in* Π *form a single orbit under* Γ.

Proof. From the representation in $\mathbb{A}_2(\mathbb{F}_3)$ we see that Γ is transitive on collinear pairs (i.e., pairs of points joined by a line in Π). So we may assume without loss of generality that $(a, b) = (0, 2)$. The point c is joined to both a and b, so $c \in \{1, 14\}$. Applying a reflection with axis $\{0, 2, 4\}$ we can interchange the two, and may assume $c = 14$. Now d and e are determined as the remaining points on the lines $\{2, 14, 3\}$ and $\{0, 14, 5\}$. □

2.5 Remark. There are four orbits of (ordered) quadrangles under Γ, represented by $(0, 2, 1, 5)$ (quadrangles with first diagonal), by $(2, 1, 5, 0)$ (quadrangles with second diagonal), by $(0, 2, 3, 5)$ (quadrangles with first dual diagonal), and by $(0, 5, 3, 2)$ (quadrangles with second dual diagonal), respectively. Only the quadrangles with first dual diagonal can be extended to bow ties by inserting a central point between the second and third entry.

3 Embeddings, and ambient automorphisms

The following embeddings will serve as representatives for the classes of projectively equivalent embeddings of Π in a projective plane $\mathbb{P}_2(\mathbb{F})$.

3.1 Definitions. Let $\mathbb{F}$ be any commutative field. For $(x,y)\in(\mathbb{F}\smallsetminus\{0,1\})^2$ we define $\eta_{(x,y)}\colon \Pi\to\mathbb{P}_2(\mathbb{F})$ by

$$\begin{array}{rclrclrcl}
0^{\eta_{(x,y)}} &=& \mathbb{F}(1,0,0), & 1^{\eta_{(x,y)}} &=& \mathbb{F}(0,1,y), & 2^{\eta_{(x,y)}} &=& \mathbb{F}(1,0,1),\\
3^{\eta_{(x,y)}} &=& \mathbb{F}(0,1,0), & 4^{\eta_{(x,y)}} &=& \mathbb{F}(1,0,x), & 5^{\eta_{(x,y)}} &=& \mathbb{F}(0,1,1),\\
14^{\eta_{(x,y)}} &=& \mathbb{F}(1,1,1), & 25^{\eta_{(x,y)}} &=& \mathbb{F}(y,x,xy), & 03^{\eta_{(x,y)}} &=& \mathbb{F}(y-1,x-1,xy-1)\\
\{0,1,25\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}0\\y\\-1\end{pmatrix}, & \{0,2,4\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}0\\1\\0\end{pmatrix}, & \{0,5,14\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}0\\1\\-1\end{pmatrix},\\
\{1,2,03\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}1\\y\\-1\end{pmatrix}, & \{1,3,5\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}1\\0\\0\end{pmatrix}, & \{2,3,14\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}1\\0\\-1\end{pmatrix},\\
\{3,4,25\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}x\\0\\-1\end{pmatrix}, & \{4,5,03\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}x\\1\\-1\end{pmatrix}, & \{03,14,25\}^{\eta_{(x,y)}} &=& \ker\begin{pmatrix}x-xy\\xy-y\\y-x\end{pmatrix}.
\end{array}$$

Then $\eta_{(x,y)}$ is an embedding of Π into $\mathbb{P}_2(\mathbb{F})$, cp. Figure 3.

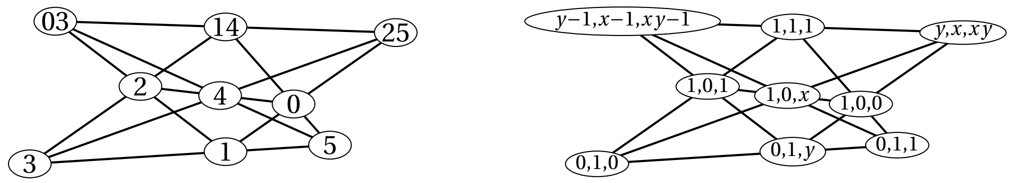


Figure 3: The embedding $\eta_{(x,y)}$.

3.2 Theorem. *For every embedding $\varepsilon\colon\Pi\to\mathbb{P}_2(\mathbb{F})$ there exists a unique parameter pair $\widehat{\varepsilon}\in(\mathbb{F}\smallsetminus\{0,1\})^2$ such that ε is projectively equivalent to the embedding $\eta_{\widehat{\varepsilon}}$. Two embeddings $\varepsilon,\iota\colon\Pi\to\mathbb{P}_2(\mathbb{F})$ are projectively equivalent precisely if $\widehat{\varepsilon}=\widehat{\iota}$.*

Proof. Consider any embedding $\varepsilon\colon\Pi\to\mathbb{P}_2(\mathbb{F})$. As the points 0^ε, 2^ε, 3^ε, and 5^ε form a quadrangle in $\mathbb{P}_2(\mathbb{F})$, there exists a unique element $\varphi\in\Phi$ such that $0^{\varepsilon\varphi}=\mathbb{F}(1,0,0)$, $2^{\varepsilon\varphi}=\mathbb{F}(1,0,1)$, $3^{\varepsilon\varphi}=\mathbb{F}(0,1,0)$, and $5^{\varepsilon\varphi}=\mathbb{F}(0,1,1)$. Then $14^{\varepsilon\varphi}=\mathbb{F}(1,1,1)$. The lines $0^{\varepsilon\varphi}+2^{\varepsilon\varphi}$ and $3^{\varepsilon\varphi}+5^{\varepsilon\varphi}$ meet in the point $\mathbb{F}(0,0,1)$. This implies that this point is not an image under $\varepsilon\varphi$. Therefore, there exist $x,y\in\mathbb{F}\smallsetminus\{0,1\}$ such that $4^{\varepsilon\varphi}=\mathbb{F}(1,0,x)$ and $1^{\varepsilon\varphi}=\mathbb{F}(0,1,y)$. Straightforward computations yield $25^{\varepsilon\varphi}=\mathbb{F}(y,x,xy)$ and $03^{\varepsilon\varphi}=\mathbb{F}(y-1,x-1,xy-1)$. So ε is projectively equivalent to $\eta_{\widehat{\varepsilon}}$, where $\widehat{\varepsilon}=(x,y)$.

The representative $\varepsilon\varphi$ for the orbit $\varepsilon\Phi$ is uniquely determined because the images 0^ε, 2^ε, 3^ε, and 5^ε form a quadrangle in $\mathbb{P}_2(\mathbb{F})$. Thus the parameter pair $\widehat{\varepsilon}$ is also uniquely determined by $\varepsilon\Phi$.

If ε and ι are embeddings with $\widehat{\varepsilon}=\widehat{\iota}$ then both are projectively equivalent to $\eta_{\widehat{\varepsilon}}=\eta_{\widehat{\iota}}$, and thus projectively equivalent to each other. □

3.3 Remark. The parameter pair $\widehat{\varepsilon}$ for an embedding ε can be computed as a pair of cross ratios, see 8.1 below.

Via pre-composition, the group $\overline{\Gamma}$ acts (from the right) on the orbits of $\Phi=\mathrm{PGL}_3(\mathbb{F})$ on the set of all embeddings of Π into $\mathbb{P}_2(\mathbb{F})$: every such orbit is of the form $\varepsilon\Phi=\{\iota\mid\widehat{\iota}=\widehat{\varepsilon}\}$, and $\gamma\in\overline{\Gamma}$ moves it to $\gamma^{-1}\varepsilon\Phi$. We translate that action of Γ on the set of all classes of projectively equivalent embeddings into an action on the space $(\mathbb{F}\smallsetminus\{0,1\})^2$ of parameters, namely

$$(\mathbb{F}\smallsetminus\{0,1\})^2\times\overline{\Gamma}\to(\mathbb{F}\smallsetminus\{0,1\})^2\colon(\widehat{\varepsilon},\gamma)\mapsto\widehat{\varepsilon}^{\gamma}:=\widehat{\gamma^{-1}\varepsilon}.$$

We collect our findings so far:

3.4 Proposition. *Consider any commutative field* $\mathbb{F}$.

(a) *The orbits of* $\Phi = \mathrm{PGL}_3(\mathbb{F})$ *on the set of Pappus figures in* $\mathbb{P}_2(\mathbb{F})$ *are in one-to-one correspondence to the orbits of* Γ *on* $(\mathbb{F}\smallsetminus\{0,1\})^2$.

(b) *A class of projectively equivalent Pappus figures contains the images of just one class of projectively equivalent embeddings if, and only if, every automorphism of* Π *is ambient for the embeddings involved.*

(c) *For each automorphism* $\gamma\in\Gamma$ *and each parameter pair* $u\in(\mathbb{F}\smallsetminus\{0,1\})^2$, *there exists a unique parameter pair* u^γ *and a unique element* $\tilde\gamma_u\in\Phi$ *such that* $\gamma\eta_{u^\gamma}=\eta_u\tilde\gamma_u$.

(d) *For each duality* $\delta\in\overline{\Gamma}\smallsetminus\Gamma$ *of* Π *and each parameter pair* $u\in(\mathbb{F}\smallsetminus\{0,1\})^2$, *there exists a unique parameter pair* u^δ *and a unique element* $\tilde\delta_u\in\theta\Phi$ *such that* $\delta\eta_{u^\delta}=\eta_u\tilde\delta_u$.

(e) *The map* $\omega\colon(\mathbb{F}\smallsetminus\{0,1\})^2\times\overline{\Gamma}\to(\mathbb{F}\smallsetminus\{0,1\})^2\colon(u,\beta)\mapsto u^\beta$ *is an action of* $\overline{\Gamma}$ *on* $(\mathbb{F}\smallsetminus\{0,1\})^2$. □

3.5 Proposition. *Let* $(x,y)\in(\mathbb{F}\smallsetminus\{0,1\})^2$; *we determine* $(x,y)^\gamma$ *for some automorphisms* $\gamma\in\Gamma$.

(a) *For* $\tau=(0,03,3)(1,4,14)(2,25,5)\in\Gamma$ *we have* $(x,y)^\tau=(x,y)$.

(b) *For* $\sigma=(0,5,1)(2,4,03)(3,25,14)\in\Gamma$ *we have* $(x,y)^\sigma=\left(x,\frac{1}{1-y}\right)$.

(c) *For* $\sigma^*=(0,5,4)(1,3,25)(2,14,03)\in\Gamma$ *we have* $(x,y)^{\sigma^*}=\left(\frac{1}{1-x},y\right)$.

(d) *For* $\mu=(0,2,4)(1,3,5)(03,25,14)\in\Gamma$ *we have* $(x,y)^\mu=\left(\frac{x-1}{x},\frac{y-1}{y}\right)$.

(e) *For* $\mu^*=(0,03,3)(2,5,25)\in\Gamma$ *we have* $(x,y)^{\mu^*}=\left(\frac{1}{1-x},\frac{y-1}{y}\right)$.

(f) *For central reflections* ζ *in the* $\langle\tau\rangle$*-orbit* $\zeta_0^{\langle\tau\rangle}=\left\{\zeta_0,\zeta_0^\tau,\zeta_0^{\tau^2}\right\}$ *of* $\zeta_0=(1,25)(2,4)(3,03)(5,14)$, *we have* $(x,y)^\zeta=\left(\frac{1}{y},\frac{1}{x}\right)$.
For ζ *in the* $\langle\tau\rangle$*-orbit of* $\zeta_1=(0,25)(2,03)(3,5)(4,14)$, *we have* $(x,y)^\zeta=(1-y,1-x)$.
For ζ *in the* $\langle\tau\rangle$*-orbit of* $\zeta_2=(0,4)(1,03)(3,14)(5,25)$, *we have* $(x,y)^\zeta=\left(\frac{y}{y-1},\frac{x}{x-1}\right)$.

(g) *For axial reflections* α *in the* $\langle\tau\rangle$*-orbit of* $\alpha_0=(2,5)(3,03)(4,14)$, *we have* $(x,y)^\alpha=\left(\frac{y-1}{y},\frac{1}{1-x}\right)$.
For α *in the* $\langle\tau\rangle$*-orbit of* $\alpha_1=(0,3)(4,14)(5,25)$, *we have* $(x,y)^\alpha=\left(\frac{1}{1-y},\frac{x-1}{x}\right)$.
For α *in the* $\langle\tau\rangle$*-orbit of* $\alpha_2=(1,14)(3,03)(5,25)$, *we have* $(x,y)^\alpha=(y,x)$.

(h) *The triad reflections are* $\alpha_0\zeta_0$, $\alpha_1^{\tau^2}\zeta_0$, $\alpha_2\zeta_0$, $\alpha_0\zeta_1$, $\alpha_1\zeta_1$, $\alpha_2^{\tau^2}\zeta_1$, $\alpha_0^{\tau^2}\zeta_2$, $\alpha_1\zeta_2$, *and* $\alpha_2\zeta_2$, *respectively. Each one of those commutes with* τ. *We obtain*

$$\begin{array}{rclrclrcl}
(x,y)^{\alpha_0\zeta_0} &=& \left(1-x,\frac{y}{y-1}\right), & (x,y)^{\alpha_2^{\tau^2}\zeta_1} &=& (1-x,1-y), & (x,y)^{\alpha_1\zeta_2} &=& \left(1-x,\frac{1}{y}\right),\\
(x,y)^{\alpha_1\zeta_1} &=& \left(\frac{1}{x},\frac{y}{y-1}\right), & (x,y)^{\alpha_0^{\tau^2}\zeta_2} &=& \left(\frac{1}{x},1-y\right), & (x,y)^{\alpha_2\zeta_0} &=& \left(\frac{1}{x},\frac{1}{y}\right),\\
(x,y)^{\alpha_2\zeta_2} &=& \left(\frac{x}{x-1},\frac{y}{y-1}\right), & (x,y)^{\alpha_1^{\tau^2}\zeta_0} &=& \left(\frac{x}{x-1},1-y\right), & (x,y)^{\alpha_0\zeta_1} &=& \left(\frac{x}{x-1},\frac{1}{y}\right).
\end{array}$$

Proof. Let $u = (x, y) \in (\mathbb{F} \smallsetminus \{0,1\})^2$ be an arbitrary parameter pair. In order to prove the claims of the proposition for $\gamma \in \{\tau, \sigma, \sigma^*, \mu, \mu^*\}$ and for selected involutions, we give the respective projective transformation $\tilde{\gamma}_u$ such that $\gamma\eta_{u^\gamma} = \eta_u\tilde{\gamma}_u$ (where square brackets around matrices indicate that we consider the induced projective transformation), and leave it to the reader to verify these statements by straightforward computations:

$$\tilde{\tau}_{(x,y)} = \begin{bmatrix} x(1-y) & x(1-x) & x(1-xy) \\ (x-1)y & 0 & 0 \\ y-x & 0 & x(y-1) \end{bmatrix}, \quad \tilde{\sigma}_{(x,y)} = \begin{bmatrix} 0 & x(y-1) & x(y-1) \\ -y & xy(y-1) & -xy \\ y & x(1-y) & x \end{bmatrix}, \quad \tilde{\sigma}^*_{(x,y)} = \begin{bmatrix} 0 & x & x \\ (1-x)y & 1 & y \\ x-1 & -1 & -1 \end{bmatrix},$$

$$\tilde{\mu}_{(x,y)} = \begin{bmatrix} x & 0 & x \\ 0 & y & y \\ 0 & 0 & -1 \end{bmatrix}, \quad \tilde{\mu}^*_{(x,y)} = \begin{bmatrix} x-1 & xy & xy-1 \\ (x-1)y & 0 & 0 \\ 1-x & -y & 1-y \end{bmatrix}.$$

The $\langle\tau\rangle$-conjugates of ζ_0 are $\zeta_0^\tau := \tau^{-1}\zeta_0\tau = (0,3)(1,2)(4,5)(14,25)$ and $\zeta_0^{\tau^2} = (0,03)(1,5)(2,14)(4,25)$. We have $(\tilde{\zeta}_0^\tau)_{(x,y)} = \left[\begin{smallmatrix} 0 & x & 0 \\ y & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$; any two $\langle\tau\rangle$-conjugates of ζ_0 have the same effect on the parameter pairs (by assertion (a)). The observations $\zeta_1 = \zeta_0^{\tau^2\mu^2}$ and $\zeta_2 = \zeta_0^\mu$ allow to compute the effect of the $\langle\tau\rangle$-conjugates of ζ_1, and of ζ_2, respectively.

For $\alpha_2^\tau = (0,3)(1,4)(2,5)$ we verify $(x, y)^{\alpha_2^\tau} = (y, x)$ using $(\tilde{\alpha}_2^\tau)_{(x,y)} = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$; the effect of the conjugates of α_0 can then be computed using information that we already have.

The values for the action of triad reflections can be computed from the values for the central and axial reflections, using $u^{\alpha\zeta} = (u^\alpha)^\zeta$. □

3.6 Corollary. *For any embedding, the automorphism τ (leaving invariant each triad and each dual triad, and generating the center of the Sylow 3-subgroup of Γ) is ambient.* □

3.7 Definitions. The transformations $m_0 := x \mapsto \frac{1}{x}$, $m_1 := x \mapsto 1 - x$, and $m_2 := x \mapsto \frac{x}{x-1}$ are involutions on $\mathbb{F} \smallsetminus \{0,1\}$; they generate a dihedral subgroup M of order 6 in the stabilizer of $\{0,1,\infty\}$ in the group of Moebius transformations on $\mathbb{P}_1(\mathbb{F}) = \mathbb{F} \cup \{\infty\}$; apart from $m_0^2 = m_1^2 = m_2^2 = \mathrm{id}$ and the generating involutions m_0, m_1, $m_2 = m_0m_1m_0$, that subgroup contains the elements $m_0m_1 = m_1m_2 = m_2m_0 = x \mapsto \frac{x-1}{x}$ and $m_1m_0 = m_2m_1 = m_0m_2 = x \mapsto \frac{1}{1-x}$ of order 3.

We let $M \times M$ act (from the right) on the cartesian product $\mathbb{P}_1(\mathbb{F}) \times \mathbb{P}_1(\mathbb{F})$, and also consider the involution $\rho\colon (x, y) \mapsto (y, x)$. Thus we obtain the semidirect product $\langle\rho\rangle \ltimes (M \times M)$; the multiplication is given by

$$\begin{aligned} (\mathrm{id},(a,b))(\mathrm{id},(c,d)) &= (\mathrm{id},(ac,bd)), & (\mathrm{id},(a,b))(\rho,(c,d)) &= (\rho,(bc,ad)), \\ (\rho,(a,b))(\mathrm{id},(c,d)) &= (\rho,(ac,bd)), & (\rho,(a,b))(\rho,(c,d)) &= (\mathrm{id},(bc,ad)). \end{aligned}$$

3.8 Remark. The action ω of $\overline{\Gamma}$ on the space of parameter pairs may be interpreted as a homomorphism $\hat{\omega}$ from $\overline{\Gamma}$ into the group $\langle\rho\rangle \ltimes (M \times M)$. From 3.5 (cp. 3.6) we know that $\tau \in \ker\hat{\omega}$ holds for *any*[3] embedding, and

$$\begin{aligned} \sigma^{\hat{\omega}} &= (\mathrm{id},(\mathrm{id},m_1m_0)), & (\sigma^*)^{\hat{\omega}} &= (\mathrm{id},(m_1m_0,\mathrm{id})), & \mu^{\hat{\omega}} &= (\mathrm{id},(m_0m_1,m_0m_1)), \\ (\zeta_0)^{\hat{\omega}} &= (\rho,(m_0,m_0)), & (\zeta_1)^{\hat{\omega}} &= (\rho,(m_1,m_1)), & (\zeta_2)^{\hat{\omega}} &= (\rho,(m_2,m_2)), \\ (\alpha_0)^{\hat{\omega}} &= (\rho,(m_0m_1,m_1m_0)), & (\alpha_2)^{\hat{\omega}} &= (\rho,(m_1m_0,m_0m_1)), & (\alpha_1)^{\hat{\omega}} &= (\rho,(\mathrm{id},\mathrm{id})). \end{aligned}$$

It follows that $\ker\hat{\omega}$ coincides with $\langle\tau\rangle$. The group $\overline{\Gamma}^{\hat{\omega}}$ has order $108 \cdot 2/3 = 72$, while $\Gamma^{\hat{\omega}}$ has order $108/3 = 36$. So $\overline{\Gamma}^{\hat{\omega}} = \langle\rho\rangle \ltimes (M \times M)$, and $\Gamma^{\hat{\omega}}$ has index 2 in the latter group. Note that $\Gamma^{\hat{\omega}}$ differs from the normal subgroup $M \times M$.

[3] For embeddings into $\mathbb{P}_2(\mathbb{R})$ or $\mathbb{P}_2(\mathbb{C})$, this has been observed already by Levi [16, pp. 116], see also [5, p. 276]. For embeddings into $\mathbb{P}_2(\mathbb{R})$ such that no triad is collinear and no parallel class is confluent, Kommerell [14, p. 32] reports that he has not found any ambient automorphisms apart from those in $\langle\tau\rangle$.

We interpret the group M as the symmetric group on the three-element set $\{0,1,\infty\}$, and use the corresponding sign function. The elements of Γ considered in 3.5 suffice to generate all of Γ. So we read off from 3.5:

3.9 Proposition. *The group* $\Gamma^{\hat{\omega}}$ *equals* $\left\{(r,(m,n)) \,\middle|\, r \in \langle\rho\rangle, m,n \in M, \mathrm{sign}(mn) = 1\right\}$. □

If one chooses $x \in \mathbb{F} \smallsetminus \{0,1\}$ and searches for $y \in \mathbb{F} \smallsetminus \{0,1\}$ such that the embedding $\eta_{(x,y)}$ has no ambient involutions, one just has to avoid the orbit of x under M. That orbit has length 6, in general; it will be shorter if x is fixed by at least one of the non-trivial elements of M.

3.10 Lemma. *Let* $\mathbb{F}$ *be any commutative field, and let* W *be the (possibly empty) set of roots of the polynomial* $X^2 - X + 1$ *in* $\mathbb{F}$*. The orbit decomposition of* $\mathbb{P}_1(\mathbb{F})$ *under the group* M *is the following.*

- **(a)** *In any case, the set* $\{0,1,\infty\}$ *is an orbit of length* 3*. The set* W *is either empty, or an orbit.*
- **(b)** *If* $\mathrm{char}(\mathbb{F}) = 2$ *then the orbit of* $2 = 0$ *is* $\{0,1,\infty\} = \{0,1,\infty\}$*. The set* W *forms an orbit of length* 2 *unless it is empty. All other orbits are regular (of length* 6*).*
- **(c)** *If* $\mathrm{char}(\mathbb{F}) = 3$ *then* $W = \{2\}$*. Every element outside* $\{0,1,2,\infty\}$ *has a regular orbit (of length* 6*).*
- **(d)** *If* $\mathrm{char}(\mathbb{F}) \notin \{2,3\}$ *then the orbit of* 2 *is* $\{2, \frac{1}{2}, -1\}$*. The set* W *is either empty, or forms an orbit of length* 2*. All other orbits are regular (of length* 6*).*

Proof. A straightforward computation yields that $\{0,1,\infty\}$ is an orbit under M. The fixed points of $m_1 m_2$ and $m_2 m_0 = (m_1 m_0)^{-1}$ are the roots of $X^2 - X + 1$ in $\mathbb{F}$ (if any). The sets of fixed points of m_0, of m_1, and of m_2, respectively, are $\{1,-1\}$, $\{\infty, h\}$, and $\{0,2\}$, where $2h = -1$. Note that these sets are singletons if $\mathrm{char}(\mathbb{F}) = 2$. In particular, each element of $\mathbb{F} \smallsetminus (W \cup \{0,1,-1,2,h\})$ has trivial stabilizer in M, and a regular orbit under M. □

3.11 Remarks. The group $M \cong \mathrm{Sym}_3$ acts on the projective line $\mathbb{P}_1(\mathbb{F})$, and $\langle\rho\rangle \ltimes (M \times M)$ acts on $\mathbb{P}_1(\mathbb{F}) \times \mathbb{P}_1(\mathbb{F})$. On a generic orbit, this action of M is regular (i.e., sharply transitive; these orbits have length 6). See 3.10 for a discussion of the orbits.

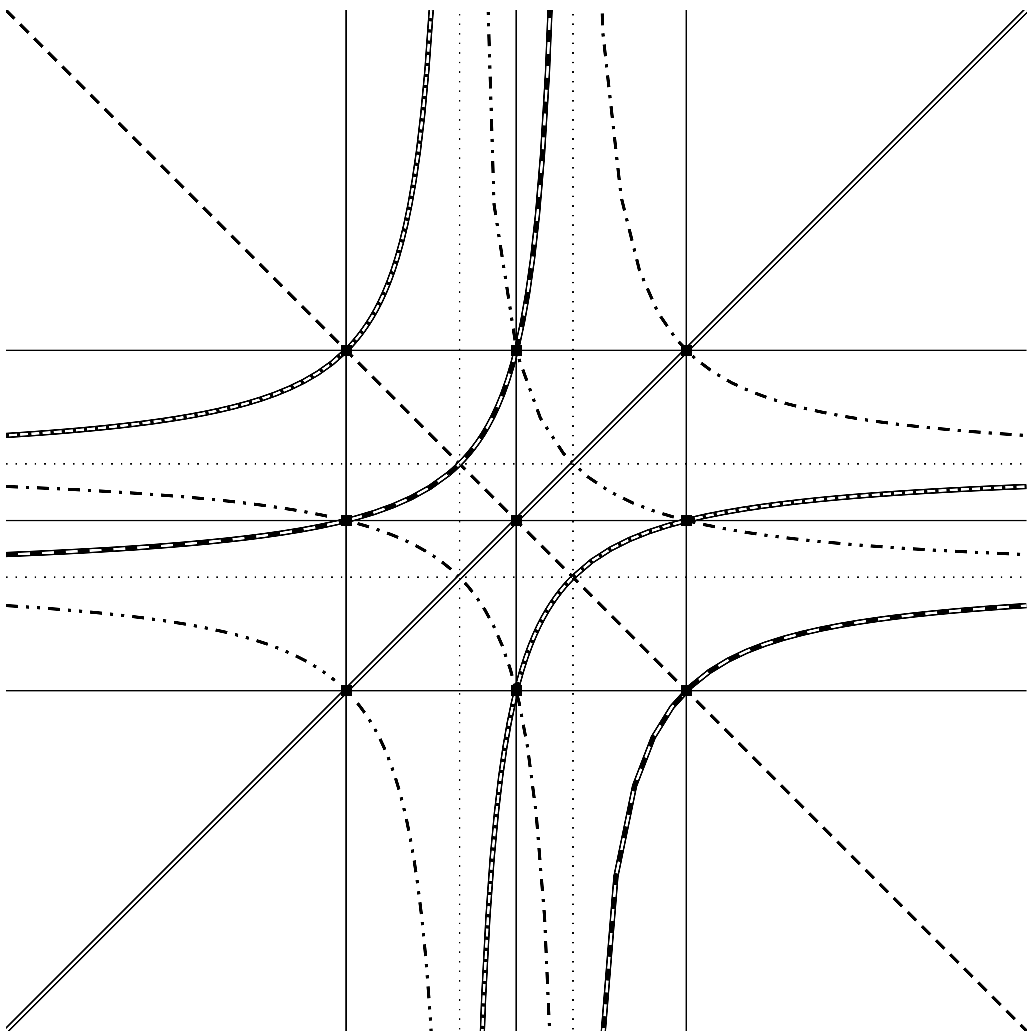
On $\mathbb{P}_1(\mathbb{F}) \times \mathbb{P}_1(\mathbb{F})$, we have generic orbits of length 36, and orbits of length 9, and 18. Depending on $\mathrm{char}(\mathbb{F})$ and the existence of roots of $X^2 - X + 1$, we also have orbits of length 1, 2, 3, 4, or 6, respectively.

The generic orbits on $\mathbb{P}_1(\mathbb{F}) \times \mathbb{P}_1(\mathbb{F})$ contain parameter pairs (x,y) such that the embedding $\eta_{(x,y)}$ has an ambient group of order 3 (generated by τ: this group is the center of the Sylow 3-subgroup of Γ).

Over $\mathbb{F} = \mathbb{R}$, we visualize the distribution of orbits in Figure 4. Note that then M acts by homeomorphisms of the projective line $\mathbb{P}_1(\mathbb{R})$ (which, topologically, is a circle). Deleting the non-regular orbits $\{0,1,\infty\}$ (of values that are not admitted in parameter pairs) and $\{2, 1/2, -1\}$ we obtain six disjoint intervals; these intervals are permuted by the action of M. Each one of these intervals forms a set of representatives for the generic orbits under M.

This decomposition of $\mathbb{P}_1(\mathbb{R})$ yields a decomposition of $\mathbb{P}_1(\mathbb{R}) \times \mathbb{P}_1(\mathbb{R})$ into 36 sets (roughly, of triangular shape); each one of those forms a set of representatives for the generic orbits under $M \times M$. The orbits not covered by these representatives are the orbits of points on the quadrics (hyperbolas and lines, respectively) that contain the parameter pairs for embeddings with ambient groups of order 6 or 12 (the latter occur for pairs on the intersection of two quadrics).

The embeddings with at least one parameter in the orbit $\{2, 1/2, -1\}$ are just those where we have ambient polarities, see 4.5 below.



pairs not suited as parameters (involving 0 or 1)

parameter pairs involving the orbit $\{2, 1/2, -1\}$ (ambient polarities)

$y = x$ (α_2 ambient) $y = \frac{1}{x}$ (ζ_0 ambient) $y = \frac{x}{x-1}$ (ζ_2 ambient)

$y = 1 - x$ (ζ_1 ambient) $y = \frac{1}{1-x}$ (α_0 ambient) $y = \frac{x-1}{x}$ (α_1 ambient)

Figure 4: The space of parameter pairs.

3.12 Remark. For the special case of embeddings of Π into $\mathbb{P}_2(\mathbb{R})$ one knows that at least one triad is not collinear (e.g., see [26, 3.7]). Levi [16, § 5] and Coxeter [5, pp. 273–278] use parameterizations for those embeddings, differing from our parameterization which also covers the cases where more than one triad is collinear.

The picture given by Levi [16, p. 115] is similar to our Figure 4, with the crucial difference that Levi ignores dualities (and thus obtains only 36 cells, while we have 72), and his picture lives in the projective plane (and some of the maps on the parameter space are rational transformations that are not defined in every point of that plane) while we consider the space of parameters as contained in a torus.

4 Polarities and Dualities

The Pappus configuration Π admits 108 dualities, among them 18 polarities (see [26, 7.1]). Some of these dualities can easily be seen from pictures of the incidence graph, as in Figure 5.

4.1 Theorem. *There are* 5 *conjugacy classes of dualities in* $\overline{\Gamma}$. *These classes are represented by dualities of order* 2, 4, 6, *and* 12, *respectively; only the elements of order* 12 *fall in different classes.*

With respect to the labeling in the drawing on the left in Figure 5 (where the polarity π interchanges two vertices of different color if they carry the same label), the conjugacy classes of dualities are represented by the following:

(a) *A polarity π (of order* 2*), given by* $0^\pi = \{0,2,4\}$, $1^\pi = \{1,2,03\}$, $2^\pi = \{0,1,25\}$, $3^\pi = \{03,14,25\}$, $4^\pi = \{0,5,14\}$, $5^\pi = \{4,5,03\}$, $03^\pi = \{1,3,5\}$, $14^\pi = \{3,4,25\}$, *and* $25^\pi = \{2,3,14\}$.

(b) *The duality* $(\alpha_0\pi)^3 = (0,14^\pi,1,3^\pi)(14,0^\pi,3,1^\pi)(03,4^\pi,4,03^\pi)(2,25^\pi)(5,5^\pi)(25,2^\pi)$ *of order* 4.

(c) *The duality* $\sigma\pi = (0,5^\pi,1,0^\pi,5,1^\pi)(2,4^\pi,03,2^\pi,4,03^\pi)(3,25^\pi,14,3^\pi,25,14^\pi)$ *of order* 6.

(d) *The duality* $\alpha_0\pi = (0,0^\pi,4,14^\pi,3,03^\pi,1,1^\pi,03,3^\pi,14,4^\pi)(2,5^\pi,25,25^\pi,5,2^\pi)$ *of order* 12.

(e) *The duality* $(\alpha_0\pi)^5 = (0,03^\pi,14,14^\pi,03,0^\pi,1,4^\pi,3,3^\pi,4,1^\pi)(2,2^\pi,5,25^\pi,25,5^\pi)$ *of order* 12.

Each one of the corresponding conjugacy classes has size 18*, except for the class containing dualities of order* 6*, which has size* 36.

Proof. The picture of the incidence graph of Π in Figure 5 (on the left) exhibits one of the polarities as the half turn π; the labels for the lines are chosen in such a way that π interchanges each point with the line carrying the same label. In coordinates as used in 2.1, the polarity π maps the point $\mathbb{F}_3(x,y,z)$ to the kernel of the linear form with matrix $\left(z,-y,x\right)^\intercal$; cp. 2.2. On the left hand side in Figure 5, one sees three more polarities, namely, the reflections π', π'', π''' in axes through midpoints of edges of the outer hexagon. Note that those three polarities form a coset modulo $\langle\sigma\rangle$ in the centralizer of π, here $\sigma = (0,5,1)(2,4,03)(3,25,14)$ is the automorphism considered in 3.5 (b), showing up as a counterclockwise rotation by 120 degrees in the picture.

This drawing of the incidence graph also shows that the centralizer $C_\Gamma(\pi)$ contains a dihedral subgroup of order 6, and that $C_{\overline{\Gamma}}(\pi)$ contains a dihedral subgroup of order 12. As every polarity of Π belongs to the conjugacy class π^Γ and that set has size $18 = |\Gamma|/6$ (see [26, 7.1]), those subgroups indeed coincide with the centralizers.

Rotation of the hexagon shows dualities of order 6, in fact, the clockwise rotation is

$$\sigma\pi \quad = \quad (0,5^\pi,1,0^\pi,5,1^\pi)(2,4^\pi,03,2^\pi,4,03^\pi)(3,25^\pi,14,3^\pi,25,14^\pi).$$

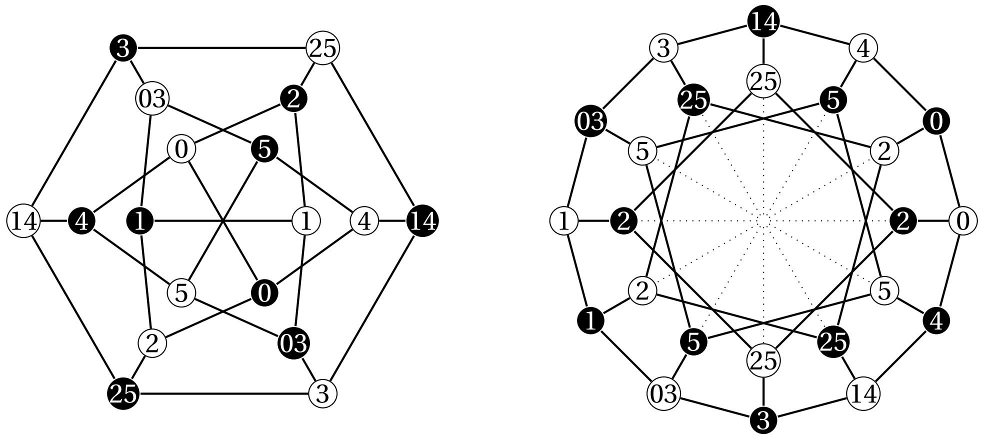


Figure 5: Two representations of the incidence graph of the Pappus configuration: points are white vertices, lines are black vertices (each of those labeled by the label of its image under the polarity π which is obvious in the drawing on the left). In the drawing on the right, we identify vertices (labeled by $x \in \{2,5,25\}$); the identifications are indicated by dotted lines.

The third power of that duality is the polarity π, so $C_{\overline{\Gamma}}(\sigma\pi) \le C_{\overline{\Gamma}}(\pi)$. However, the polarity $\pi' \in C_{\overline{\Gamma}}(\pi)$ does not centralize $\sigma\pi$. Thus $C_{\overline{\Gamma}}(\sigma\pi) = \langle\sigma\pi\rangle$ has order 6, and the conjugacy class of $\sigma\pi$ in $\overline{\Gamma}$ has size 36.

The drawing on the right in Figure 5 exhibits the duality

$$(\alpha_0\pi)^3 \quad = \quad (0,14^\pi,1,3^\pi)(14,0^\pi,3,1^\pi)(03,4^\pi,4,03^\pi)(2,25^\pi)(5,5^\pi)(25,2^\pi) \quad = \quad \pi\alpha_1$$

of order 4, and the dualities

$$\begin{aligned} \alpha_0\pi &= (0,0^\pi,4,14^\pi,3,03^\pi,1,1^\pi,03,3^\pi,14,4^\pi)(2,5^\pi,25,25^\pi,5,2^\pi) \\ (\alpha_0\pi)^5 &= (0,03^\pi,14,14^\pi,03,0^\pi,1,4^\pi,3,3^\pi,4,1^\pi)(2,2^\pi,5,25^\pi,25,5^\pi) \end{aligned}$$

of order 12.

We note $\alpha_0\pi = \pi\zeta_2$, and $(\alpha_0\pi)^6 = \alpha_0^{\tau^2}\zeta_2$, so $C_\Gamma((\alpha_0\pi)^6) = \langle\alpha_0^{\tau^2},\zeta_2\rangle$ and $C_{\overline{\Gamma}}((\alpha_0\pi)^6) = \langle\pi,\zeta_2\rangle$.

For $\delta \in \{\alpha_0\pi,(\alpha_0\pi)^5,(\alpha_0\pi)^3\}$, we have $(\alpha_0\pi)^6 \in \{\delta^6,\delta^2\}$ and thus $\langle\alpha_0\pi\rangle \le C_{\overline{\Gamma}}(\delta) \le C_{\overline{\Gamma}}((\alpha_0\pi)^6) = \langle\pi,\zeta_2\rangle$. The latter group is dihedral of order 24, but π does not belong to $C_{\overline{\Gamma}}(\delta)$ because π acts as a reflection on the drawing. So $C_{\overline{\Gamma}}(\delta) = \langle\alpha_0\pi\rangle$ is a cyclic group of order 12, and the conjugacy class of δ has size 18.

Any element conjugating $\alpha_0\pi$ into $(\alpha_0\pi)^5$ would belong to the centralizer $\langle\pi,\zeta_2\rangle$ of $(\alpha_0\pi)^6$. However, the two elements are not conjugates in that dihedral group. Thus $\alpha_0\pi$ and $(\alpha_0\pi)^5$ represent different conjugacy classes in $\overline{\Gamma}$. Summing up the sizes of conjugacy classes of dualities found so far, we see that our study of the two representations of the incidence graph has indeed revealed all the dualities of Π, up to conjugacy. □

4.2 Lemma. *Let δ be any duality of Π. Then $\langle\delta\rangle$ either contains a polarity, or a triad reflection.*

Proof. We know the dualities (up to conjugacy) from 4.1. If δ has order 2 or 6 then δ^3 is a polarity. If δ has order 12 then δ^3 has order 4. So it remains to note that the square of any duality of order 4 is a triad reflection. □

4.3 Proposition. *As in 4.1, let π be the polarity of Π with*

$$\begin{array}{rclrclrcl} 0^\pi &=& \{0,2,4\}, & 2^\pi &=& \{0,1,25\}, & 4^\pi &=& \{0,5,14\}, \\ 14^\pi &=& \{3,4,25\}, & 25^\pi &=& \{2,3,14\}, & 03^\pi &=& \{1,3,5\}, \\ 3^\pi &=& \{03,14,25\}, & 5^\pi &=& \{4,5,03\}, & 1^\pi &=& \{1,2,03\}. \end{array}$$

(a) *We have $\pi^{\hat\omega} = (\mathrm{id},(m_0,\mathrm{id}))$; i.e., $(x,y)^\pi = \left(\frac{1}{x},y\right)$ holds for each $(x,y) \in (\mathbb{F} \smallsetminus \{0,1\})^2$ (see 3.8).*

(b) *We have $\psi^{\hat\omega} \in \big\{(\mathrm{id},(\mathrm{id},m_0)),(\mathrm{id},(\mathrm{id},m_1)),(\mathrm{id},(\mathrm{id},m_2)),(\mathrm{id},(m_0,\mathrm{id})),(\mathrm{id},(m_1,\mathrm{id})),(\mathrm{id},(m_2,\mathrm{id}))\big\}$ for every polarity ψ of Π.*

Proof. Lines will be given as kernels of linear forms, and the latter will be given as matrices of the form $(u,v,w)^\intercal$, consisting of a single column.

For each pair $(x,y) \in (\mathbb{F} \smallsetminus \{0,1\})^2$, we determine the images under $\pi\,\eta_{(x,y)}$ as

$$\begin{array}{rclrclrcl} 0^{\pi\,\eta_{(x,y)}} &=& \ker(0,1,0)^\intercal, & 2^{\pi\,\eta_{(x,y)}} &=& \ker(0,y,-1)^\intercal, & 4^{\pi\,\eta_{(x,y)}} &=& \ker(0,1,-1)^\intercal, \\ 14^{\pi\,\eta_{(x,y)}} &=& \ker(x,0,-1)^\intercal, & 25^{\pi\,\eta_{(x,y)}} &=& \ker(1,0,-1)^\intercal, & 03^{\pi\,\eta_{(x,y)}} &=& \ker(1,0,0)^\intercal, \\ 3^{\pi\,\eta_{(x,y)}} &=& \ker(x(y-1),(1-x)y,x-y)^\intercal, & 5^{\pi\,\eta_{(x,y)}} &=& \ker(x,1,-1)^\intercal, & 1^{\pi\,\eta_{(x,y)}} &=& \ker(1,y,-1)^\intercal. \end{array}$$

Let $\tilde\pi_{(x,y)}$ be the (projective) duality of $\mathbb{P}_2(\mathbb{F})$ mapping the point $\mathbb{F}(a,b,c)$ to the line

$$\ker\left(\begin{pmatrix} 0 & 1-y & 0 \\ x(y-1) & (1-x)y & x-y \\ 0 & xy-1 & 1-x \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right).$$

For each pair $(x,y) \in (\mathbb{F} \smallsetminus \{0,1\})^2$ we then verify $\pi\,\eta_{(x,y)} = \eta_{(\frac{1}{x},y)}\,\tilde\pi_{(x,y)} = \eta_{(x,y)^\pi}\,\tilde\pi_{(x,y)}$, as required.

The polarities of Π form a single conjugacy class. So $\psi^{\hat\omega}$ lies in the conjugacy class of $(\mathrm{id},(m_0,\mathrm{id}))$ in $\langle\rho\rangle \ltimes (M \times M)$ if ψ is any polarity. As any two involutions in M are conjugates, this conjugacy class is $\big\{(\mathrm{id},(\mathrm{id},m_0)),(\mathrm{id},(\mathrm{id},m_1)),(\mathrm{id},(\mathrm{id},m_2)),(\mathrm{id},(m_0,\mathrm{id})),(\mathrm{id},(m_1,\mathrm{id})),(\mathrm{id},(m_2,\mathrm{id}))\big\}$. □

4.4 Remark. For any polarity of Π, one can use the homomorphism $\hat\omega\colon \overline{\Gamma} \to \langle\rho\rangle \ltimes (M \times M)$ in order to compute the effect of that polarity on the parameters.

4.5 Theorem. *An embedding ε of Π into $\mathbb{P}_2(\mathbb{F})$ admits an ambient polarity precisely if at least one of the entries in the parameter pair $\widehat{\varepsilon}$ lies in the orbit of 2 under M in $\mathbb{F}$. In particular, ambient polarities can occur only if $\operatorname{char}\mathbb{F} \neq 2$.*

Proof. From 4.3 we know the effect of a polarity on the space of parameter pairs. The polarity in question is ambient if the parameter pair is fixed. This means that at least one of the two parameters is fixed by an involution in M, and thus lies in the orbit of 2. □

4.6 Proposition. *If a given embedding has an ambient duality then it also has an ambient polarity.*

Proof. Let ε be an embedding with an ambient duality δ. Without loss of generality, we may assume $\varepsilon = \eta_{(x,y)}$ for $(x,y) = \widehat{\varepsilon}$.

The group $\langle\delta\rangle$ consists of ambient dualities and automorphisms. According to 4.2, it contains either a polarity or a triad reflection. According to 3.5 (h), we have $(x,y) \in \{2,1/2,-1\}^2$ in the latter case, and infer from 4.3 that some polarities are ambient. □

Ambient polarities under embeddings of Π into $\mathbb{P}_2(\mathbb{R})$ are in the focus of [4] and [5], together with the corresponding conics in $\mathbb{P}_2(\mathbb{R})$. Kommerell [14, p. 20] searched for ambient polarities (under embeddings into $\mathbb{P}_2(\mathbb{R})$) but did not find any; this seems to be due to the fact that he only considered embeddings such that no triad becomes collinear.

4.7 Remarks. Using the bijection $0 \leftrightarrow 3\mathbb{Z}$, $1 \leftrightarrow 1+3\mathbb{Z}$, $\infty \leftrightarrow -1+3\mathbb{Z}$, we translate the action of $\overline{\Gamma}$ on $\{0,1,\infty\}^2 \subseteq \mathbb{P}_1(\mathbb{F})^2$ into an action on $\mathbb{F}_3^2 = (\mathbb{Z}/(3\mathbb{Z}))^2$ which is in fact an action by automorphisms of $\mathbb{A}_2(\mathbb{F}_3)$. The kernel of that action is $Z := \langle\tau\rangle$.

More explicitly, the elements σ and σ^* in Γ induce translations of $\mathbb{A}_2(\mathbb{F}_3)$ (in the "vertical" and "horizontal" directions, respectively). Each one of the central or axial reflections in Γ induces a reflection in a ("diagonal") line of the affine plane; the triad reflections induce reflections in points of the affine plane. The polarities in $\overline{\Gamma}\smallsetminus\Gamma$ act as reflections in lines parallel to one of the coordinates axes. Each duality of order 4 or 12 in $\overline{\Gamma}\smallsetminus\Gamma$ induces an automorphism of order 4 fixing one point of $\mathbb{A}_2(\mathbb{F}_3)$ but no line; the dualities of order 6 act as glide reflections.

Assume $\operatorname{char}\mathbb{F} \notin \{2,3\}$, and identify $(j,k) \in \{0,1,-1\}^2$ with $(-(-2)^j, -(-2)^k)$ on the one hand, and with a pair in $\mathbb{F}_3^2$ on the other hand. This translates the action of $\overline{\Gamma}/Z$ on $\{2,1/2,-1\}$ into an action on the affine plane $\mathbb{A}_2(\mathbb{F}_3)$. That action on $\mathbb{A}_2(\mathbb{F}_3)$ is equivalent to the action discussed in the previous paragraph.

5 Existence of ambient automorphisms or dualities

We have seen in 3.5 (a) that the group generated by τ (i.e., the center of the Sylow 3-subgroup of Γ) is ambient under every embedding of Π into a projective plane over a commutative field.

The actions of the groups M and $\langle\rho\rangle \ltimes (M\times M)$ introduced in 3.7 and in 3.8 play a crucial role in the understanding of ambient automorphisms. From 3.5 and 4.3 we can actually derive explicit conditions for ambiance of any given element of Γ, as follows.

5.1 Theorem. *Let $\widehat{\varepsilon} = (x,y) \in (\mathbb{F}\smallsetminus\{0,1\})^2$ be the parameter pair for an embedding ε of Π into the projective plane over a given commutative field $\mathbb{F}$. We consider conjugacy classes under Γ.*

- **(a)** *An ambient conjugate of α_0 (i.e., an ambient axial reflection in one of the lines of Π) exists precisely if y is in the orbit of x under the Sylow 3-subgroup of M (i.e., if y either equals x or is the image of x under one of the elements of order 3 in M).*
- **(b)** *An ambient conjugate of ζ_0 (i.e., an ambient central reflection in one of the points of Π) exists precisely if y is the image of x under one of the involutions in M.*
- **(c)** *An ambient conjugate of $\alpha_0\zeta_0$ (i.e., an ambient reflection in a triad) exists precisely if (x,y) is in the orbit of $(2,2)$ under $M\times M$. In particular, this will never happen if* $\operatorname{char}\mathbb{F} = 2$.
- **(d)** *An ambient involution — i.e., an ambient (central, axial, or triad) reflection — exists precisely if y is in the orbit of x under M.*
- **(e)** *An ambient element of order 3 in $\Gamma\smallsetminus\langle\tau\rangle$ exists precisely if the set $\{x,y\}$ contains a root of X^2-X+1. The whole Sylow 3-subgroup of Γ is ambient precisely if both x and y are roots of that polynomial.*
- **(f)** *The following are equivalent:*
  - *The ambient group Γ_{amb} is transitive on the point set of Π.*
  - *The ambient group Γ_{amb} is transitive on the line set of Π.*
  - *The set $\{x,y\}$ contains a root of X^2-X+1.*
- **(g)** *An ambient duality exists precisely if $\{x,y\}$ contains an element in the orbit $\{2,1/2,-1\}$ of 2 under M. In particular, this will never happen if* $\operatorname{char}\mathbb{F} = 2$. □

5.2 Corollary. *A central reflection in Γ is ambient under a given embedding if, and only if, the triad containing the center is collinear under the embedding. Dually, an axial reflection in Γ is ambient if, and only if, the parallel class containing the axis is confluent under the embedding.*

A central reflection in Γ is ambient if, and only if, each reflection in a center in the same triad is ambient. Dually, an axial reflection in Γ is ambient if, and only if, each reflection in an axis in the same parallel class is ambient. □

5.3 Examples. Assume that $\mathbb{F}$ contains a root w of $X^2 - X + 1$, and consider the embedding $\eta_{(w,\frac{1}{w})}$. Then the whole Sylow 3-subgroup of Γ is ambient, and every central reflection of Π is ambient. The embedding $\eta_{(w,\frac{1}{w})}$ then extends to an embedding of the affine plane $\mathbb{A}_2(\mathbb{F}_3)$.

However, an ambient triad reflection (under $\eta_{(w,\frac{1}{w})}$) will only exist if w is in the orbit of 2 under M; this happens if, and only if, the field $\mathbb{F}$ has characteristic three. As each axial reflection of Π is a product of an (ambient) central reflection and a triad reflection, we obtain an analogous restriction for the existence of ambient axial reflections.

However, if the field $\mathbb{F}$ admits an automorphism κ interchanging w with $\frac{1}{w}$ then the non-ambient axial reflection α_2^τ is induced by the semilinear bijection mapping (x, y, z) to $(y^\kappa, x^\kappa, z^\kappa)$. The triad reflections (which are conjugates of $\alpha_0\zeta_0$) are also not ambient, but induced by semilinear bijections in this case. (See also the discussion of the Möbius-Kantor configuration MK in Section 9 below.)

This observation is of interest because $\mathbb{A}_2(\mathbb{F}_3)$ is isomorphic to the unital of order 2, and that unital is embedded in $\mathbb{P}_2(\mathbb{F}_4)$ as the geometry induced on the set of absolute points of a unitary polarity. Every automorphism of the unital (and thus every automorphism of $\mathbb{A}_2(\mathbb{F}_3)$, in particular, the non-ambient axial reflection α_2^τ) is induced by some *semi-linear* bijection of $\mathbb{F}_4^3$.

More generally, embeddings of hermitian unitals into arbitrary pappian projective planes have been studied in [9]. The existence of embeddings that do not correspond to embeddings of coordinatizing fields is a unique feature of the smallest unital, viz., $\mathbb{A}_2(\mathbb{F}_3)$.

5.4 Remarks. Not every field containing roots of $X^2 - X + 1$ admits automorphisms interchanging those roots; trivial examples are prime fields of order $p \equiv 1 \pmod 3$.

Here is a less obvious example[4]. In the field $\mathbb{C}$ of complex numbers, consider the roots $w = \frac{1+i\sqrt{3}}{2}$ and $\frac{1}{w} = \frac{1-i\sqrt{3}}{2} = 1 - w$ of the polynomial $X^2 - X + 1$, and a root a of $X^2 - w + 2$. Then $w - 2$ is not a square in $\mathbb{Q}(w)$ because the norm $(w-2)\overline{(w-2)} = 3$ is not a square in $\mathbb{Q}$. So $1, a$ are linearly independent over $\mathbb{Q}(w)$. Aiming at a contradiction, we assume that there exists $\alpha \in \mathrm{Aut}(\mathbb{Q}(a))$ such that $\alpha(w) = 1 - w$. There exist $x, y \in \mathbb{Q}(w)$ with $\alpha(a) = x + ya$, and $y \neq 0$ because $\alpha(a) \notin \mathbb{Q}(w)$. We compute $-1-w = \alpha(w-2) = \alpha(a^2) = \alpha(a)^2 = (x+ya)^2 = x^2+2xya+y^2a^2 = x^2+y^2(w-2)+2xya$. Comparing coefficients, we obtain $x = 0$ (since $2y \neq 0$) and $-1 - w = y^2(w-2)$. So $y^2 = \frac{-1-w}{w-2} = w$ would be a square in $\mathbb{Q}(w)$. But then $w = -w^4$ yields that -1 is a square in $\mathbb{Q}(w)$, and there exist $r, s \in \mathbb{Q}$ such that $-1 = (r+sw)^2 = r^2 - s^2 + (2rs+s^2)w$. As $s \neq 0$, this leads to $s = -2r$ and $-1 = -3r^2$, contradicting the fact that 3 is not a square in $\mathbb{Q}$.

A quite general result ([24], see also [7, Cor. 2]) asserts that there exists an extension field $\mathbb{K}$ of $\mathbb{Q}[X]/(X^2 - X + 1)$ such that $\mathrm{Aut}(\mathbb{K})$ is isomorphic to any given group: in particular, one can have the trivial group, or a finite group without elements of even order.

In Table 1, we collect information about ambient automorphisms and dualities under an embedding $\eta_{(x,y)}$, depending on the lengths of the two orbits x^M and y^M. The first column gives the set of lengths of those orbits. Note that the orbits necessarily coincide if they have the same length and that length is less than 6; only in the last case we need a further distinction. The second column

[4] The authors are grateful to Peter Müller at Würzburg for providing this example plus the theory necessary to understand it in context.

(marked "duality") says whether there is at least one ambient duality (from 4.5 and 4.6 we know that this happens if at least one of the parameters is in the orbit of 1, which has length 1 or 3 depending on $\operatorname{char}\mathbb{F}$). The orders of the groups $\Gamma^{\eta_{(x,y)}}_{\mathrm{amb}}$ and $\overline{\Gamma}^{\eta_{(x,y)}}_{\mathrm{amb}}$ of all ambient automorphisms and of all ambient automorphisms and dualities, respectively, are given in the last two columns; we indicate by "=" the cases where no ambient dualities exist (and the two groups coincide). The columns marked "#2" and "#3" give the size of the Sylow 2- and 3-subgroups in $\Gamma^{\eta_{(x,y)}}_{\mathrm{amb}}$, respectively. The entries "projective" and "affine" indicate those embeddings that extend to embeddings of $\mathbb{P}_2(\mathbb{F}_3)$ or of $\mathbb{A}_2(\mathbb{F}_3)$ (up to duality), respectively.

orbit lengths $\{\lvert x^M\rvert, \lvert y^M\rvert\}$	ambient duality?	#2	#3	conditions		remarks	$\lvert\overline{\Gamma}^{\eta_{(x,y)}}_{\mathrm{amb}}\rvert$	$\lvert\Gamma^{\eta_{(x,y)}}_{\mathrm{amb}}\rvert$
$\{1\}$	yes	4	27	$\operatorname{char}\mathbb{F}=3$,	$\lvert\mathbb{F}\rvert\ge 3$	projective	216	108
$\{1,6\}$	yes	1	9	$\operatorname{char}\mathbb{F}=3$,	$\lvert\mathbb{F}\rvert\ge 9$		18	9
$\{2\}$	no	2	27	$\operatorname{char}\mathbb{F}\ne 3$,	$\lvert\mathbb{F}\rvert\ge 4$	affine	=	54
$\{2,3\}$	yes	1	9	$\operatorname{char}\mathbb{F}\notin\{2,3\}$,	$\lvert\mathbb{F}\rvert\ge 7$		18	9
$\{2,6\}$	no	1	9	$\operatorname{char}\mathbb{F}\ne 3$,	$\lvert\mathbb{F}\rvert\ge 13$		=	9
$\{3\}$	yes	4	3	$\operatorname{char}\mathbb{F}\notin\{2,3\}$	$\lvert\mathbb{F}\rvert\ge 5$		24	12
$\{3,6\}$	yes	1	3	$\operatorname{char}\mathbb{F}\notin\{2,3\}$,	$\lvert\mathbb{F}\rvert\ge 11$		6	3
$\{6\}$	no	2	3	same orbit,	$\lvert\mathbb{F}\rvert\ge 8$		=	6
	no	1	3	different orbits,	$\lvert\mathbb{F}\rvert\ge 16$		=	3

Table 1: Ambient automorphisms under $\eta_{(x,y)}$ (depending on orbit lengths)

6 Counting

6.1 Theorem. *Let q be a power of a prime, let $t\in\{0,1\}$ denote the representative of $q \pmod 2$, and let $r\in\{0,1,2\}$ denote the representative of $q+1 \pmod 3$. Then the number of equivalence classes (up to projective transformations of subsets in the projective plane) of Pappus figures in $\mathbb{P}_2(\mathbb{F}_q)$ is $N_q=\frac{1}{36}\left(q^2+(2+4r)q+4r(r+1)+9t-8\right)$.*

The number of equivalence classes up to projective transformations and projective dualities is $DN_q:=\frac{1}{2}N_q+\frac{t}{12}(q+2r+1)=\frac{1}{72}\left(q^2+(2+4r+6t)q+4r(r+1)+(12r+15)t-8\right)$.

Proof. From 3.4(a) we know that the equivalence classes are the orbits under $\Gamma\times\mathrm{PGL}_3(\mathbb{F})$, and thus correspond to the orbits of Γ/Z on the parameter space $(\mathbb{F}\smallsetminus\{0,1\})^2$.

In order to count the parameter pairs fixed under some element of Γ, we note that t gives the number of fixed points of an involution in M on $\mathbb{F}_q\smallsetminus\{0,1\}$, and r counts the roots of X^2-X+1 in $\mathbb{F}_q$, i.e., the number of fixed points of an element of order 3 in M.

Every element of Γ/Z is represented[5] by a conjugate of an element occurring in the following table. That table also gives the number of fixed points of the representative ψ (in the action on the parameter space $(\mathbb{F}_q\smallsetminus\{0,1\})^2$) and the length c_ψ of the conjugacy class of $\psi\mathrm{Z}$ in Γ/Z.

ψ	id	σ	μ	μ^*	ζ_0	α_2	$\alpha_2\zeta_0$	$\alpha_2\mu$
$(x,y)^\psi$	(x,y)	$(x,\frac{1}{1-y})$	$(\frac{x-1}{x},\frac{y-1}{y})$	$(\frac{1}{1-x},\frac{y-1}{y})$	$(\frac{1}{y},\frac{1}{x})$	(y,x)	$(\frac{1}{x},\frac{1}{y})$	$(\frac{y-1}{y},\frac{x-1}{x})$
$\lvert\mathrm{Fix}(\psi)\rvert$	$(q-2)^2$	$r(q-2)$	r^2	r^2	$q-2$	$q-2$	t	r
c_ψ	1	4	2	2	3	3	9	12

[5] The action of $\overline{\Gamma}/\mathrm{Z}$ on $\mathbb{A}_2(\mathbb{F}_3)$ may be helpful here, see 4.7.

A general lemma (that is not Burnside's but due to Cauchy and Frobenius [8, p. 287], see [19]) now gives the formula for the number of orbits as

$$N_q = \frac{1}{36}\sum_{\psi} c_\psi \, |\mathrm{Fix}(\psi)| = \frac{1}{36}\left(q^2 + (2+4r)q + 4r(r+1) + 9t - 8\right).$$

Now we turn to equivalence of figures up to projective dualities and projective collineations; viz., to the orbits under $\overline{\Gamma} \times \mathrm{PGL}_3(\mathbb{F})$, and thus to the orbits of $\overline{\Gamma}/\mathrm{Z}$ on the parameter space $(\mathbb{F} \smallsetminus \{0,1\})^2$. We use the following table (where d_δ denotes the length of the conjugacy class of $\delta\mathrm{Z}$ in $\overline{\Gamma}$) together with the previous one (note that the Γ-conjugacy classes of μ and μ^*, and those of ζ_0 and α_2, respectively, are fused in the larger group $\overline{\Gamma}$, but this does not affect the outcome of the calculation):

δ	π	$(\alpha_0\pi)^3$	$\sigma\pi$	$\alpha_0\pi$	$(\alpha_0\pi)^5$
$(x,y)^\delta$	$(\frac{1}{x}, y)$	$(\frac{y-1}{y}, \frac{x}{x-1})$	$(\frac{1}{x}, \frac{1}{1-y})$	$(\frac{y}{y-1}, \frac{1}{1-x})$	$(\frac{y}{y-1}, \frac{1}{1-x})$
$\lvert\mathrm{Fix}(\delta)\rvert$	$(q-2)t$	t	rt	t	t
d_δ	6	6	12	6	6

This gives the number of orbits under $\overline{\Gamma}/\mathrm{Z}$ as $DN_q = \frac{1}{72}\left(\sum_\psi c_\psi \, |\mathrm{Fix}(\psi)| + \sum_\delta d_\delta \, |\mathrm{Fix}(\delta)|\right) = \frac{1}{2}N_q + \frac{t}{12}(q+2r+1) = \frac{1}{72}\left(q^2 + (2+4r+6t)q + 4r(r+1) + 12rt + 15t - 8\right)$, as claimed. □

6.2 Remark. In [2], the total number of "PAPPOS-Konfigurationen" in the projective plane over a given finite field of order n is determined as

$$2n^2(n-1)(n-2)(n^2+n+1)\binom{n+1}{4} = \tfrac{1}{12}(n+1)n^3(n-1)^2(n-2)^2(n^2+n+1).$$

The formula given in [2, Satz 2] does not give the number of Pappus figures as defined in the present paper (see 1.9); for instance, it claims that there are 468 "PAPPOS-Konfigurationen" in the projective plane $\mathbb{P}_2(\mathbb{F}_3)$ over the field of order 3, but it is clear from 2.1 that the number of Pappus figures in $\mathbb{P}_2(\mathbb{F}_3)$ equals the number $13 \cdot 4 = 52$ of incident point–line pairs.

However, a "PAPPOS-Konfiguration" as considered in [2] is, in our present terminology, a Pappus figure with a specified line p (which defines a set of two other lines of the figure, namely the two that together with $\{p\}$ form the parallel class of the latter in the abstract configuration). Thus the formula in [2, Satz 2] gives nine times the number of Pappus figures in $\mathbb{P}_2(\mathbb{F})$.

7 Subgroups

We study (the subgroups of) $\Gamma = \mathrm{Aut}(\Pi)$ in more detail. This group is isomorphic to the group

$$\left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \middle| \; a_{jk} \in \mathbb{F}_3, a_{22}a_{33} \neq 0 \right\},$$

cf. 2.1. From this representation it is obvious that the Sylow 3-subgroup (given by $a_{22} = a_{33} = 1$) is normal, and thus characteristic in Γ; in fact, it is the commutator group of Γ. The center Z of the Sylow 3-subgroup is thus also a characteristic subgroup of Γ. We note that the non-trivial elements of Z are just the automorphisms of order three leaving invariant each triad, viz. the permutation $\tau = (0,03,3)(1,4,14)(2,25,5)$ of the point set of Π, and the inverse of that permutation (cf. 3.5 (a)).

We have seen in 3.5 (a) that these automorphisms are ambient under *any* embedding. It remains to discuss the subgroups containing $\mathrm{Z} = \langle\tau\rangle$; i.e., the subgroups of the quotient Γ/Z. That quotient is isomorphic to a semidirect product $C_2^2 \ltimes C_3^2$. The normal subgroup can be identified with the vector space $\mathbb{F}_3^2$, and the complement acts by the matrices $I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $-J$, and $-I = J(-J)$. There are four subgroups of order 3 in $\mathbb{F}_3^2$. Of these, the groups $T := \mathbb{F}_3(1,0)$ and $T^* := \mathbb{F}_3(0,1)$ are normalized by the operators in the complement, but $U := \mathbb{F}_3(1,1)$ and $U^* := UJ = \mathbb{F}_3(1,-1)$ are interchanged by J (and also by $-J$), and are invariant only under $\pm I$.

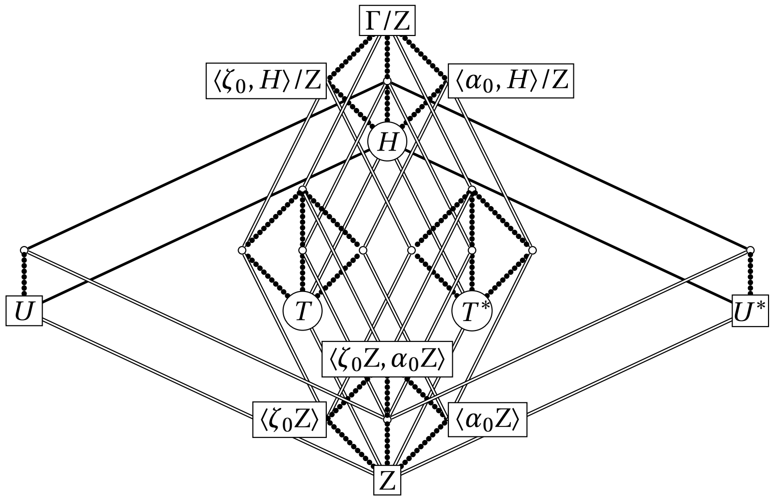


Figure 6: The subgroup lattice of $\Gamma/\mathrm{Z} \cong C_2^2 \ltimes C_3^2$, where Z is the center of the normal Sylow 3-subgroup H. Not all conjugates of 2-subgroups are shown in the picture. Dotted edges indicate subgroups of index 2.

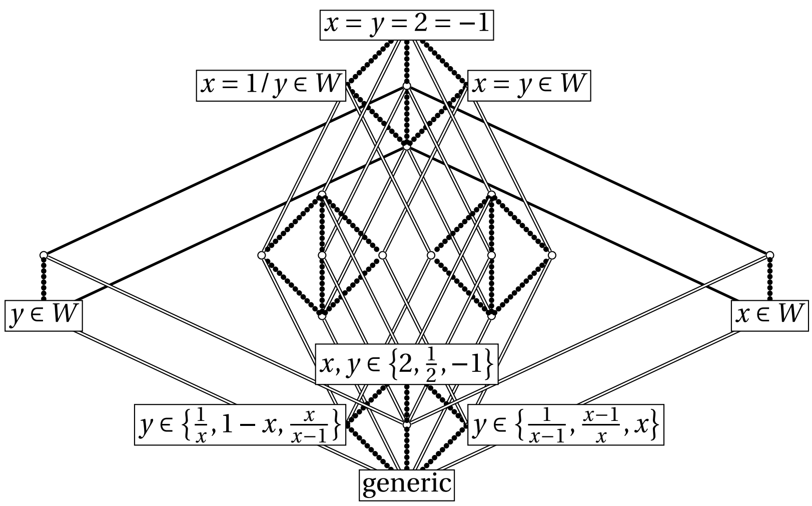


Figure 7: The conditions on the parameters singling out subgroups of Γ/Z (cp. Figure 6) that do occur as full ambient groups. Conjugates of 2-subgroups are shown in the same place.

Any subgroup of Γ/Z is a conjugate of a subgroup of the complement (i.e., subgroups of Sylow 2-subgroups), or contains a non-trivial subgroup of $\mathbb{F}_3^2$ and is contained in the normalizer of that subgroup of $\mathbb{F}_3^2$. The subgroup lattice of Γ/Z is indicated in Figure 6. That picture does not contain all the Sylow 2-subgroups. However, the non-trivial elements of those groups are involutions, and we understand those from 3.5 (cp. 5.1).

In Figure 6, labels in rectangles indicate those subgroups of Γ/Z that correspond to full groups of ambient automorphisms with respect to suitable embeddings. In Figure 7, the relevant conditions on the parameters replace the labels in rectangles. (Note that some of these imply conditions of the field $\mathbb{F}$.)

7.1 Theorem. *Let an embedding* $\varepsilon\colon \Pi \to \mathbb{P}_2(\mathbb{F})$ *be given, with parameter pair* $(x,y) := \widehat{\varepsilon} \in (\mathbb{F}\smallsetminus\{0,1\})^2$.

(a) *If* $\mathbb{F}$ *does not contain any root of* X^2-X+1 *then the Sylow* 3*-subgroup of the ambient group* $\Gamma^{\varepsilon}_{\mathrm{amb}}$ *is just* Z*, for any embedding of the Pappus configuration into* $\mathbb{P}_2(\mathbb{F})$.
Depending on the value of (x,y)*, the ambient group* $\Gamma^{\varepsilon}_{\mathrm{amb}}$ *will be a conjugate of* Z, $\langle\zeta_0\rangle Z$, $\langle\alpha_0\rangle Z$*, or* $\langle\zeta_0,\alpha_0\rangle Z$*, having order* 3, 6, 6*, or* 12*, respectively.*

(b) *If the order of the ambient group is divisible by* 9 *then the set* $\{x,y\}$ *contains a root of* X^2-X+1*. The automorphism* σ *is ambient if* y *is a root of* X^2-X+1*, and* σ^* *is ambient if* x *is such a root. The automorphisms* μ *or* μ^* *are ambient if both* x *and* y *are such roots.*
Depending on $\operatorname{char}\mathbb{F}$ *and the choice of* (x,y)*, the ambient group will be one of* $\langle\sigma,\tau\rangle$, $\langle\sigma^*,\tau\rangle$, $\langle\zeta_0,\sigma,\sigma^*\rangle$, $\langle\alpha_0,\sigma,\sigma^*\rangle$*, or* Γ*, having order* 9, 9, 54, 54*, or* 108*, respectively.* □

See Figure 8 for embeddings into $\mathbb{P}_2(\mathbb{R})$ realizing the three possibilities[6] in 7.1 (a).

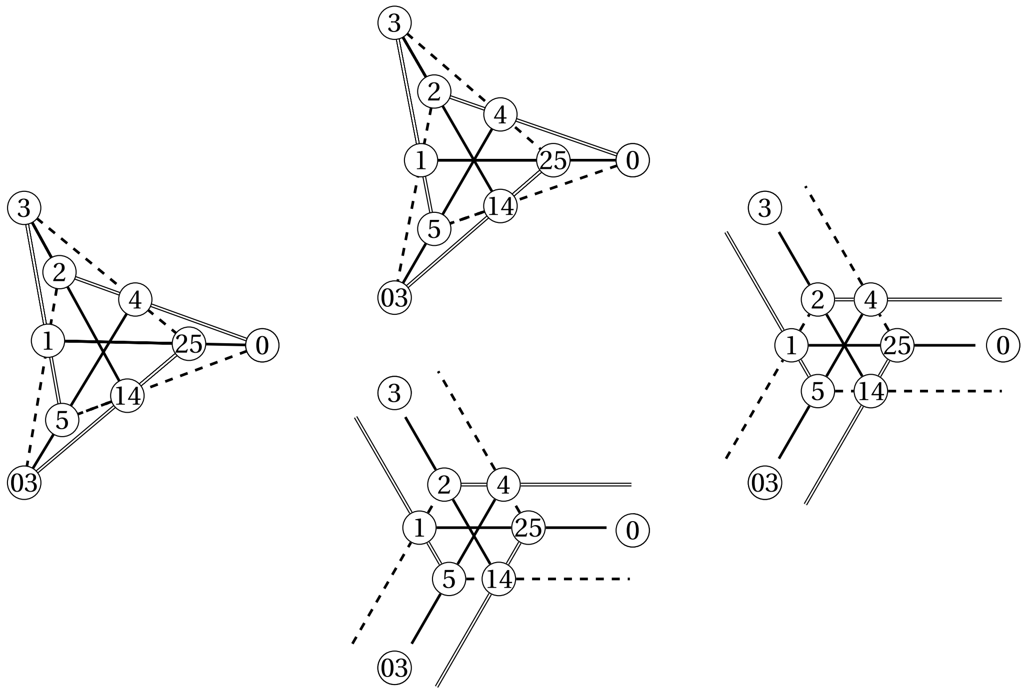


Figure 8: Four different embeddings of Π into $\mathbb{P}_2(\mathbb{R})$, with ambient group C_3 (left: no collinear triad, no confluent parallel class), ambient group D_3 (center top: no collinear triad, one confluent parallel class; center bottom: one collinear triad, no confluent parallel class), and ambient group D_6 (right: one confluent parallel class and one collinear triad $\{0,3,03\}$ at infinity), respectively.

[6] Some of these embeddings into $\mathbb{P}_2(\mathbb{R})$ have already been noted by Coxeter in [4, p. 267–269] and [5, p. 270–273].

8 Cross Ratios

Let a,b,c,d be four points on a projective line over $\mathbb{F}$. In order to determine the *cross ratio* $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ of (a,b,c,d),we introduce homogeneous coordinates on the projective line in such a way that $a = \mathbb{F}(1,0)$, $b = \mathbb{F}(0,1)$, $c = \mathbb{F}(1,u)$, and $d = \mathbb{F}(1,1)$. Then $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] = u$, see [1, p. 72], or [22, 18, S. 118]. Note that the concept of cross ratios was already introduced by Pappus ([21, Book VII, Prop. 129], see https://archive.org/details/pappialexandrin01hultgoog/page/n409), who noted its invariance under projection and used it for his proof of the theorem named after him.

8.1 Proposition. *If we know the embedding ε, we can find the corresponding parameter pair $\widehat{\varepsilon}$ as*

$$\widehat{\varepsilon} = \left(\begin{bmatrix} 0^\varepsilon & \infty_\varepsilon \\ 4^\varepsilon & 2^\varepsilon \end{bmatrix}, \begin{bmatrix} 3^\varepsilon & \infty_\varepsilon \\ 1^\varepsilon & 5^\varepsilon \end{bmatrix} \right),$$

where ∞_ε is the intersection point of the lines $0^\varepsilon + 2^\varepsilon$ and $3^\varepsilon + 5^\varepsilon$.

Proof. As cross ratios are invariant under projective transformations, we may assume $\varepsilon = \eta_{(x,y)}$, where $(x,y) = \widehat{\varepsilon}$. Then the definition of that embedding in 3.1 immediately gives $\infty_\varepsilon = \mathbb{F}(0,0,1)$; moreover, we find

$$\begin{bmatrix} 0^\varepsilon & \infty_\varepsilon \\ 4^\varepsilon & 2^\varepsilon \end{bmatrix} = \begin{bmatrix} \mathbb{F}(1,0,0) & \mathbb{F}(0,0,1) \\ \mathbb{F}(1,0,x) & \mathbb{F}(1,0,1) \end{bmatrix} = x, \quad \text{and} \quad \begin{bmatrix} 3^\varepsilon & \infty_\varepsilon \\ 1^\varepsilon & 5^\varepsilon \end{bmatrix} = \begin{bmatrix} \mathbb{F}(0,1,0) & \mathbb{F}(0,0,1) \\ \mathbb{F}(0,1,y) & \mathbb{F}(0,1,1) \end{bmatrix} = y. \qquad \square$$

8.2 Definition. If $F = \Pi^\varepsilon$ is an arbitrary Pappus figure in $\mathbb{P}_2(\mathbb{F})$, we pick any two lines K, L of F such that their intersection point ∞ does not belong to F. Taking the points a,c,d of F on K in arbitrary (but fixed) order, we note that for each $u \in \{a,c,d\}$ there is a unique point u' of F on L such that u and u' are not joined by a line of F. The pair $\left(\left[\begin{smallmatrix} a & \infty \\ c & d\end{smallmatrix}\right], \left[\begin{smallmatrix} a' & \infty \\ c' & d'\end{smallmatrix}\right]\right)$ is called a *cross ratio pair associated to the figure F with respect to (a,d,a',d').*

Recall (cp. [23, 2.3]) that four points a, b, c, d on a projective line over a commutative field $\mathbb{F}$ form an *equianharmonic tetrad* if $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ is a root of $X^2 - X + 1$. If $\operatorname{char}\mathbb{F} \neq 2$ then the four points form a *harmonic tetrad* if $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ lies in the orbit $\{2, \frac{1}{2}, -1\}$. In our present context, we obtain:

8.3 Theorem. **(a)** *The ambient group Γ_{amb} has a Sylow 3-subgroup of order greater than 3 if, and only if, the points a, ∞, c, d form an equianharmonic tetrad for at least one choice of K, L as in 8.2.*

(b) *There exists an ambient duality if, and only if, the points a, ∞, c, d form a harmonic tetrad for at least one choice of K, L.* $\square$

8.4 Theorem. *The set of cross ratio pairs associated to a given Pappus figure F in $\mathbb{P}_2(\mathbb{F})$ is just one Γ-orbit in $(\mathbb{F} \smallsetminus \{0,1\})^2$.*

Proof. Consider any embedding $\varepsilon\colon \Pi \to \mathbb{P}_2(\mathbb{F})$ such that $\Pi^\varepsilon = F$. Then $\widehat{\varepsilon}$ is the cross ratio pair associated to F with respect to $(0^\varepsilon, 2^\varepsilon, 3^\varepsilon, 5^\varepsilon)$; see 8.1.

For each cross ratio pair associated to F with respect to (a,d,a',d'), we let z denote the intersection point of $a+d'$ and $d+a'$, and then note that the pre-image of (a,d,z,a',d') under ε is a bow tie in Π. From 2.4 we know that there exists $\gamma \in \Gamma$ mapping $(0,2,14,3,5)$ to that bow tie. Thus $(a,d,a',d') = (0^{\gamma\varepsilon}, 2^{\gamma\varepsilon}, 3^{\gamma\varepsilon}, 5^{\gamma\varepsilon})$, and $\left(\left[\begin{smallmatrix} a & \infty_\varepsilon \\ c & d\end{smallmatrix}\right], \left[\begin{smallmatrix} a' & \infty_\varepsilon \\ c' & d'\end{smallmatrix}\right]\right) = \gamma(\widehat{\varepsilon})$, as required. $\square$

8.5 Theorem. *Two Pappus figures in $\mathbb{P}_2(\mathbb{F})$ are projectively equivalent if, and only if, the sets of cross ratio pairs associated to them are equal, i.e., if any cross ratio pair associated to the second figure lies in the Γ-orbit of any cross ratio pair associated to the first one.*

There exists a projective duality mapping the first figure to the second if, and only if, any cross ratio pair of the second figure is the image of a pair for the first under an element of the coset $\overline{\Gamma} \smallsetminus \Gamma$.

A Pappus figure admits an ambient duality if, and only if, the set of cross ratio pairs is invariant under $\overline{\Gamma}$ (and not just under Γ). □

8.6 Remark. One can compute a cross ratio pair associated to the dual figure with the help of coordinates for lines in the original figure F, as follows: Pick two points of F not joined by a line of the figure, let their joining line in $\mathbb{P}_2(\mathbb{F})$ play the role of ∞, and pair each line L through the first point with the unique line L' through the second point that does not meet L in a point of F.

Explicitly, for $F = \Pi^\varepsilon$ with $\varepsilon = \eta_{(x,y)}$ we could use the points $3^\varepsilon = \mathbb{F}(0,1,0)$ and $0^\varepsilon = \mathbb{F}(1,0,0)$, then $\infty = \ker(0,0,1)^\intercal$, and the pairing of lines is given by $A := 3^\varepsilon + 5^\varepsilon = \ker(-1,0,0)^\intercal$, $A' = 0^\varepsilon + 2^\varepsilon = \ker(0,-1,0)$, $C := 3^\varepsilon + 4^\varepsilon = \ker(-1,0,\frac{1}{x})^\intercal$, $C' = 0^\varepsilon + 5^\varepsilon = \ker(0,-1,1)$, $D := 3^\varepsilon + 2^\varepsilon = \ker(-1,0,1)^\intercal$, $D' = 0^\varepsilon + 1^\varepsilon = \ker\bigl(0,-1,\frac{1}{y}\bigr)^\intercal$. Then $\bigl(\bigl[\begin{smallmatrix} A & \infty \\ C & D \end{smallmatrix}\bigr], \bigl[\begin{smallmatrix} A' & \infty \\ C' & D' \end{smallmatrix}\bigr]\bigr) = \bigl(\frac{1}{x}, y\bigr) = \pi'(x,y)$.

8.7 Corollary. *Let L_0, L_1 be two lines in $\mathbb{P}_2(\mathbb{F})$, intersecting in a point s. Consider three points $a_0, a_2, a_4 \in L_0 \smallsetminus \{s\}$ and three points $a_1, a_3, a_5 \in L_1 \smallsetminus \{s\}$. Construct a Pappus figure F by joining a_j to a_{j+1}, intersecting $a_j + a_{j+1}$ with $a_{j+3} + a_{j+4}$, and joining the three resulting points by a line, where the indices are taken modulo* 6.

For permutations β_0 and β_1 of $\{0,2,4\}$ and $\{1,3,5\}$ respectively, consider the Pappus figure $F^{(\beta_0,\beta_1)}$ obtained by replacing a_j on L_k by $a_{j\beta_k}$. Then the following hold.

- **(a)** *The figures F and $F^{(\beta_0,\beta_1)}$ are projectively equivalent if β_0 and β_1 have the same sign.*
- **(b)** *If β_0 and β_1 have different sign then there exists a projective duality mapping F to $F^{(\beta_0,\beta_1)}$.*
- **(c)** *If F and $F^{(\beta_0,\beta_1)}$ are projectively equivalent but do not admit an ambient duality then β_0 and β_1 have the same sign.*
- **(d)** *If F and $F^{(\beta_0,\beta_1)}$ are projectively equivalent and admit an ambient duality then the signs of β_0 and β_1 are arbitrary.*

9 The Möbius-Kantor configuration

Starting with the affine plane $\mathbb{A}_2(\mathbb{F}_3)$ of order 3, we obtain a configuration MK with 8 points and 3 points per line, and dually 8 lines and 3 lines per point, see Figure 9 (left). One can show (see [16, Kap. 3, § 3]) that these parameters (together with the assumption that no more than one line joins any two given points) uniquely determine the configuration, up to isomorphism. This configuration is called the Möbius-Kantor configuration, see [18] and [12], cp. [3, § 5]. The obvious action of $\mathrm{GL}_2(\mathbb{F}_3)$ on our model in $\mathbb{A}_2(\mathbb{F}_3)$ already gives the full group Aut(MK), see [16, Kap. 3, § 3].

In the configuration MK, each point p is joined to 6 points different from p, so there remains exactly one point $p^\#$ *opposite* p (i.e., not joined to p). The complement of a pair of opposite points in the point set of the configuration is the union of two disjoint lines. In the picture on the left in Figure 9, the numbering of the points by elements of $\mathbb{Z}/(8\mathbb{Z})$ is such that points p_j and p_{j+4} are opposite, and $p_j \mapsto p_{j+1}$ gives an automorphism ρ of order 8. That automorphism corresponds to an element of $\mathrm{GL}_2(\mathbb{F}_3)$ of order 8, with determinant -1 and trace in $\{1,-1\}$. (Note that

$\operatorname{trace}(\rho^{-1}) = -\operatorname{trace}(\rho)$ here; so a change of orientation of the 8-cycle used for the numbering of points interchanges the two possible values of the trace.) Another automorphism that will be relevant to our discussion is $\beta = (p_1, p_7)(p_0, p_4)(p_3, p_5)$, this involution also corresponds to an element of determinant -1 (and trace 0) in $\mathrm{GL}_2(\mathbb{F}_3)$.

Using Pappus' theorem, we see that every embedding of MK into a pappian plane extends to an embedding of $\mathbb{A}_2(\mathbb{F}_3)$, see Figure 9 (center). That extension then restricts to an embedding of Π such that at least two triads are collinear, see Figure 9 (right). More explicitly, we have the following:

9.1 Theorem. *An embedding of* MK *into* $\mathbb{P}_2(\mathbb{F})$ *exists if, and only if, the field* $\mathbb{F}$ *contains a root* w *of* $X^2 - X + 1$. *If this is the case then* $\frac{1}{w}$ *is also a root of that polynomial, and we have the following.*

(a) *The embedding extends to an embedding of* $\mathbb{A}_2(\mathbb{F}_3)$.

(b) *The embedding is projectively equivalent to* ι *or to* ι', *where*

$$\begin{array}{llllllllllll} p_0^\iota &=& \mathbb{F}(1,0,w), & p_2^\iota &=& \mathbb{F}(\frac{1}{w},w,1), & p_4^\iota &=& \mathbb{F}(0,w,1), & p_6^\iota &=& \mathbb{F}(w,\frac{1}{w},0), \\ p_1^\iota &=& \mathbb{F}(1,0,0), & p_3^\iota &=& \mathbb{F}(1,0,1), & p_5^\iota &=& \mathbb{F}(0,1,1), & p_7^\iota &=& \mathbb{F}(0,1,0), \end{array}$$

and

$$\begin{array}{llllllllllll} p_0^{\iota'} &=& \mathbb{F}(1,0,\frac{1}{w}), & p_2^{\iota'} &=& \mathbb{F}(w,\frac{1}{w},1), & p_4^{\iota'} &=& \mathbb{F}(0,1,w), & p_6^{\iota'} &=& \mathbb{F}(\frac{1}{w},w,0), \\ p_1^{\iota'} &=& \mathbb{F}(1,0,0), & p_3^{\iota'} &=& \mathbb{F}(1,0,1), & p_5^{\iota'} &=& \mathbb{F}(0,1,1), & p_7^{\iota'} &=& \mathbb{F}(0,1,0), \end{array}$$

respectively.

(c) *If* $\operatorname{char}\mathbb{F} = 3$ *then* $w = 2 = \frac{1}{w}$, *every automorphism of* MK *is ambient, and the embedding is unique (up to projective equivalence).*

(d) *If* $\operatorname{char}\mathbb{F} \neq 3$ *then the ambient automorphisms of* MK *form the subgroup* $\mathrm{SL}_2(\mathbb{F}_3)$ *of index* 2 *in* $\mathrm{Aut}(\mathrm{MK}) = \mathrm{GL}_2(\mathbb{F}_3)$. *There are two classes of projectively equivalent embeddings, represented by* ι *and* ι', *respectively.*
For example, the automorphisms β *and* ρ *are not ambient if* $\operatorname{char}\mathbb{F} \neq 3$.

(e) *If there exists an automorphism* κ *of the field* $\mathbb{F}$ *such that* $w^\kappa = \frac{1}{w}$ *then the semilinear transformation* $\hat\kappa\colon \mathbb{F}(x,y,z) \mapsto \mathbb{F}(y^\kappa, x^\kappa, z^\kappa)$ *interchanges the two classes of projectively inequivalent embeddings.*

Proof. Let $\varepsilon\colon \mathrm{MK} \to \mathbb{P}_2(\mathbb{F})$ be an embedding. The extension is obtained in the following way (see Figure 9). We choose two sets of opposite points, say $\{p_0, p_4\}$ and $\{p_2, p_6\}$. Let z denote the intersection point of the lines $p_1 + p_5$ and $p_3 + p_7$ joining the remaining opposite pairs. A Pappus configuration is now constructed by deleting the two disjoint lines in the complement of $\{p_0, p_4\}$ (but keep the points on them, those lines become two of the triads, the third one is $\{p_0, p_4, z\}$): the points p_6^ε, z, and p_2^ε are collinear by Pappus' theorem. Putting $0^\eta := p_1^\varepsilon$, $1^\eta := p_4^\varepsilon$, $2^\eta := p_3^\varepsilon$, $3^\eta := p_7^\varepsilon$, $4^\eta := p_0^\varepsilon$, $5^\eta := p_5^\varepsilon$, $03^\eta := p_6^\varepsilon$, $14^\eta := z$, and $25^\eta := p_2^\varepsilon$ we obtain an embedding $\eta\colon \Pi \to \mathbb{P}_2(\mathbb{F})$. The triads $\{0^\eta, 03^\eta, 3^\eta\} = \{p_1^\varepsilon, p_6^\varepsilon, p_7^\varepsilon\}$ and $\{2^\eta, 25^\eta, 5^\eta\} = \{p_3^\varepsilon, p_2^\varepsilon, p_5^\varepsilon\}$ are collinear by construction. Therefore, the third triad $\{1^\eta, 14^\eta, 4^\eta\} = \{p_4^\varepsilon, z, p_0^\varepsilon\}$ is collinear, as well (see [26, 3.5]), and we obtain the extension as claimed in assertion (a).

Up to a projective transformation, we may and will assume that $\eta = \eta_{(x,y)}$ for some parameter pair $(x,y) := \hat\eta \in (\mathbb{F} \smallsetminus \{0,1\})^2$. From 5.2 we now infer that $\zeta_1^{\tau^2}$ is an ambient automorphism, so $(x,y) =$

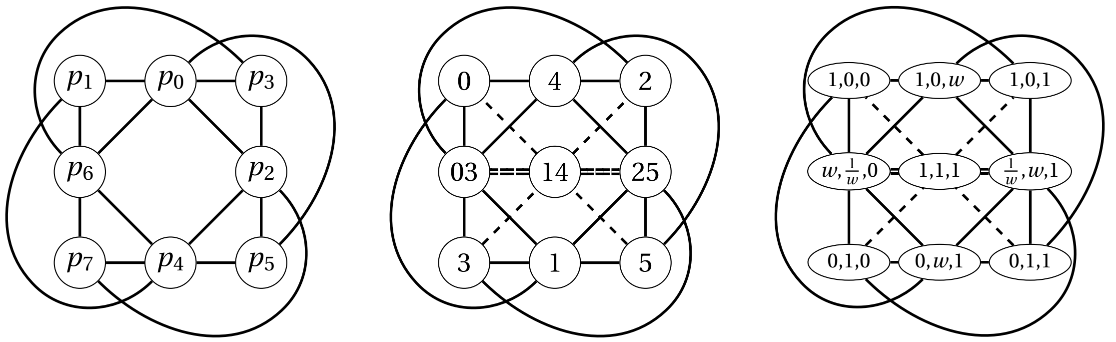


Figure 9: The configuration MK (left), its extension (center), and the embedding ι (right)

$(x,y)^{\zeta_1^{\tau^2}} = \big(1-y, 1-x\big)$ by 3.5. As ζ_0 is also ambient, we have $(x,y) = (x,y)^{\zeta_0} = \big(\frac{1}{y}, \frac{1}{x}\big)$. This leads to $x^2 - x + 1 = 0$ and $y = \frac{1}{x}$. The set of roots of $X^2 - X + 1$ in $\mathbb{F}$ is $\{w, \frac{1}{w}\}$, and we obtain that ε is projectively equivalent either to the restriction ι of $\eta_{(w,\frac{1}{w})}$ or to the restriction ι' of $\eta_{(\frac{1}{w},w)}$; the two roots (and thus the embeddings) coincide if $\operatorname{char}\mathbb{F} = 3$. This completes the proof of assertion (b), and of assertion (c).

From $(w, \frac{1}{w})^{\mu^*} = \big(\frac{1}{1-w}, \frac{\frac{1}{w}-1}{\frac{1}{w}}\big) = \big(w, \frac{1}{w}\big)$ we obtain that μ^* is an ambient automorphism. The automorphism $\varphi := (p_0, p_2, p_4, p_6)(p_1, p_3, p_5, p_7)$ of MK has order 4, under the given embedding ι it is induced by $\mathbb{F}(u,v,w) \mapsto \mathbb{F}(u+w, u, -u+v)$. Thus φ is ambient for the given embedding of MK but does not leave the Pappus configuration invariant (in fact, it permutes the possible choices of a pair of sets of opposite points in orbits of length two). We note that $\zeta_1^{\tau^2} = \varphi^2$.

Now $\langle \varphi, \mu^* \rangle$ is a subgroup of index 2 in $\operatorname{Aut}(\mathrm{MK}) = \mathrm{GL}_2(\mathbb{F}_3)$, and coincides with the commutator subgroup $\mathrm{SL}_2(\mathbb{F}_3)$.

The automorphism $\beta = (p_1, p_7)(p_0, p_4)(p_3, p_5)$ of MK is induced by $\alpha_2^\tau = (0,3)(1,4)(2,5) \in \Gamma$; it is ambient if, and only if, we have $\big(w, \frac{1}{w}\big) = \big(w, \frac{1}{w}\big)^{\alpha_2^\tau} = \big(\frac{1}{w}, w\big)$. Thus it is ambient precisely if $\operatorname{char}\mathbb{F} = 3$ (and $w = 2 = \frac{1}{w}$). In that case, every automorphism of MK is ambient. In all other cases, the ambient group is the commutator group of $\operatorname{Aut}(\mathrm{MK})$. The automorphism ρ is not in that commutator group because it is induced by an element of $\mathrm{GL}_2(\mathbb{F}_3)$ with determinant -1.

The last assertion is checked by straightforward computations. □

9.2 Remark. Assume that there exists an automorphism κ of the field $\mathbb{F}$ such that $w^\kappa = \frac{1}{w}$. The involutory automorphism $\beta^\iota = (p_0^\iota, p_4^\iota)(p_1^\iota, p_7^\iota)(p_3^\iota, p_5^\iota)$ of MK^ι is then induced by the semilinear collineation $\mathbb{F}(x,y,z) \mapsto \mathbb{F}(y^\kappa, x^\kappa, z^\kappa)$, and the automorphism $\rho^\iota = (p_0^\iota, p_1^\iota, p_2^\iota, p_3^\iota, p_4^\iota, p_5^\iota, p_6^\iota, p_7^\iota)$ of order 8 is induced by the semilinear collineation $\mathbb{F}(x,y,z) \mapsto \mathbb{F}(x^\kappa, y^\kappa, z^\kappa)\begin{pmatrix} w^\kappa & w & 1 \\ 1 & 0 & w \\ w^2 & w^\kappa & -w \end{pmatrix}$. These automorphisms fix $\mathbb{F}(1,1,1)$ and so extend to automorphisms of $\mathbb{A}_2(\mathbb{F}_3)^\iota$.

9.3 Remark. In any case, the projective transformation $\xi\colon \mathbb{F}(x,y,z) \mapsto \mathbb{F}(y,x,z)$ gives $\beta\iota\xi = \iota'$. Therefore, the embeddings ι and ι' are projectively *quasi*-equivalent, and the *images* of MK under ι and ι' are projectively equivalent. Recall, however, that the embeddings ι and ι' are *not* projectively equivalent if $\operatorname{char}\mathbb{F} \neq 3$.

9.4 Remark. For the case where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, our assertions (a) and (b) in 9.1 have been proved already in [16, Kap. 3, § 3], and later again in [3, § 5].

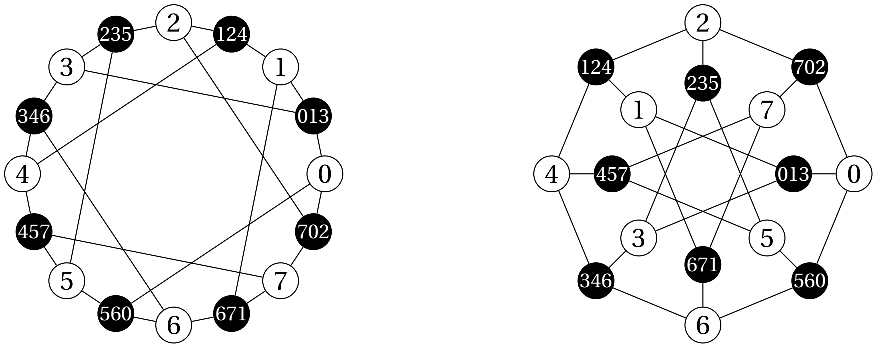


Figure 10: Two representations of the incidence graph of the Möbius-Kantor configuration

9.5 Examples. The representation of the incidence graph of MK on the left in Figure 10 exhibits the automorphism $\rho = (p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ of order 8, and representatives from two conjugacy classes of polarities of MK:

$$\begin{aligned}
\psi &= \left(p_0,\left\{\begin{smallmatrix}p_0\\p_1\\p_3\end{smallmatrix}\right\}\right)\left(p_1,\left\{\begin{smallmatrix}p_7\\p_0\\p_2\end{smallmatrix}\right\}\right)\left(p_2,\left\{\begin{smallmatrix}p_6\\p_7\\p_1\end{smallmatrix}\right\}\right)\left(p_3,\left\{\begin{smallmatrix}p_5\\p_6\\p_0\end{smallmatrix}\right\}\right)\left(p_4,\left\{\begin{smallmatrix}p_4\\p_5\\p_7\end{smallmatrix}\right\}\right)\left(p_5,\left\{\begin{smallmatrix}p_3\\p_4\\p_6\end{smallmatrix}\right\}\right)\left(p_6,\left\{\begin{smallmatrix}p_2\\p_3\\p_5\end{smallmatrix}\right\}\right)\left(p_7,\left\{\begin{smallmatrix}p_1\\p_2\\p_4\end{smallmatrix}\right\}\right),\\
\psi\rho &= \left(p_0,\left\{\begin{smallmatrix}p_1\\p_2\\p_4\end{smallmatrix}\right\}\right)\left(p_1,\left\{\begin{smallmatrix}p_0\\p_1\\p_3\end{smallmatrix}\right\}\right)\left(p_2,\left\{\begin{smallmatrix}p_7\\p_0\\p_2\end{smallmatrix}\right\}\right)\left(p_3,\left\{\begin{smallmatrix}p_6\\p_7\\p_1\end{smallmatrix}\right\}\right)\left(p_4,\left\{\begin{smallmatrix}p_5\\p_6\\p_0\end{smallmatrix}\right\}\right)\left(p_5,\left\{\begin{smallmatrix}p_4\\p_5\\p_7\end{smallmatrix}\right\}\right)\left(p_6,\left\{\begin{smallmatrix}p_3\\p_4\\p_6\end{smallmatrix}\right\}\right)\left(p_7,\left\{\begin{smallmatrix}p_2\\p_3\\p_5\end{smallmatrix}\right\}\right).
\end{aligned}$$

We note that ψ fixes two flags of MK, while $\psi\rho$ fixes four flags. So these two polarities are not conjugates under Aut(MK).

The polarity ψ is ambient in any case, in fact we have $\psi\iota = \iota\tilde{\psi}$, where $\tilde{\psi}\colon \mathbb{F}v \mapsto \left(v\begin{pmatrix} w & 0 & -1\\ 0 & -1 & w\\ -1 & w & \frac{1}{w}\end{pmatrix}\right)^{\intercal}$. From 9.1 we know that ρ is not ambient unless $w = 2$ (and $\operatorname{char}\mathbb{F} = 3$). So the polarity $\psi\rho$ is ambient precisely if $\operatorname{char}\mathbb{F} = 3$ and $w = 2$.

There are 12 conjugates of ψ (cp. [3, § 5]). The remaining elements in the coset $\mathrm{Aut}(\mathrm{MK})'\psi$ of ψ modulo the commutator group $\mathrm{Aut}(\mathrm{MK})'$ of Aut(MK) form two conjugacy classes of dualities of order 8. Representatives of those can be seen as rotations by 45 or 135 degrees, respectively, in the representation on the right in Figure 10:

$$\begin{aligned}
\delta &= \left(p_0,\left\{\begin{smallmatrix}p_7\\p_0\\p_2\end{smallmatrix}\right\},p_2,\left\{\begin{smallmatrix}p_1\\p_2\\p_4\end{smallmatrix}\right\},p_4,\left\{\begin{smallmatrix}p_3\\p_4\\p_6\end{smallmatrix}\right\},p_6,\left\{\begin{smallmatrix}p_5\\p_6\\p_0\end{smallmatrix}\right\}\right)\quad\left(p_1,\left\{\begin{smallmatrix}p_4\\p_5\\p_7\end{smallmatrix}\right\},p_3,\left\{\begin{smallmatrix}p_6\\p_7\\p_1\end{smallmatrix}\right\},p_5,\left\{\begin{smallmatrix}p_0\\p_1\\p_3\end{smallmatrix}\right\},p_7,\left\{\begin{smallmatrix}p_2\\p_3\\p_5\end{smallmatrix}\right\}\right)\\
\delta^3 &= \left(p_0,\left\{\begin{smallmatrix}p_1\\p_2\\p_4\end{smallmatrix}\right\},p_6,\left\{\begin{smallmatrix}p_7\\p_0\\p_2\end{smallmatrix}\right\},p_4,\left\{\begin{smallmatrix}p_5\\p_6\\p_0\end{smallmatrix}\right\},p_2,\left\{\begin{smallmatrix}p_3\\p_4\\p_6\end{smallmatrix}\right\}\right)\quad\left(p_1,\left\{\begin{smallmatrix}p_6\\p_7\\p_1\end{smallmatrix}\right\},p_7,\left\{\begin{smallmatrix}p_4\\p_5\\p_7\end{smallmatrix}\right\},p_5,\left\{\begin{smallmatrix}p_2\\p_3\\p_5\end{smallmatrix}\right\},p_3,\left\{\begin{smallmatrix}p_0\\p_1\\p_3\end{smallmatrix}\right\}\right)
\end{aligned}$$

In any case, the duality δ is ambient; we have $\delta\iota = \iota\tilde{\delta}$, where $\tilde{\delta}\colon \mathbb{F}v \mapsto \left(v\left(\begin{smallmatrix} 1 & 0 & 0\\ 1 & 1 & -1\\ -1 & 0 & 1\end{smallmatrix}\right)\right)^{\intercal}$. Consequently, the duality δ^3 is also ambient, over any field. The two are not conjugates because any conjugating element would normalize $\langle\delta\rangle = \langle\delta^3\rangle$ and then also $\langle\delta^2\rangle$, but the latter's normalizer in $\mathrm{GL}_2(\mathbb{F}_3)$ is the extension of the centralizer $\mathrm{C}_{\mathrm{Aut(MK)}}(\delta^2) = \langle\rho\rangle$ by an involution inducing inversion on $\langle\rho\rangle$, and does not map δ to δ^3.

9.6 Remark. The group of ambient automorphisms of dualities generated by the commutator group $\mathrm{Aut}(\mathrm{MK})'$ of Aut(MK) and the polarity ψ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$.

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Norbert Knarr
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

Markus J. Stroppel
LExMath
Fakultät für Mathematik und Physik
Universität Stuttgart
D-70550 Stuttgart
Germany

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