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Regularity results to semilinear Parabolic Initial Boundary Value Problems with Nonlinear Newton Boundary Conditions in a polygonal space-time cylinder

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Abstract

In this paper some semilinear parabolic problems with nonlinear Newton boundary conditions in a polygonal space-time cylinder are considered: a simple model problem, see [7], more advanced problems with nonlinear terms, which satisfy growth conditions and problems with nonlinear advection terms. We start with the corresponding stationary boundary value problems in a polygon, discuss the existence of weak solutions in $H^1(\Omega)$ using a theorem of Brézis about the surjectivity of pseudomonotone and coercive operators and derive regularity results in $W^{2,q}(\Omega)$. Here, q is given by a linearized problem in a polygon Ω moving the nonlinear terms to the right hand sides.

We introduce an abstract initial value problem, where the associated operator A maps $D(A) \subset L_q(\Omega) \to L_q(\Omega)$. We discuss the m-accretivity of A and use the semigroup theory in order to get solvability and regularity results in space and time. If the m-accretivity is not available, then we investigate the existence of weak solutions in Bochner spaces via pseudomonotonicity and semicoerciveness.

1 Introduction

Nonlinear parabolic initial boundary value problems describe e.g. the evolution of heat processes, regularized conservation laws, semiconductor devices or transport processes in porous electrodes, growing plants or biochemical reactions.

There are different papers, where nonlinear parabolic initial boundary problems are studied. In [13] there are considered nonlinear heat equations with nonlinear boundary conditions and equations based on regularized conservation laws with linear boundary conditions. In particular, there is investigated when weak solutions belong to Bochner spaces with the derivative order one. In [12] there are considered parabolic quasilinear equations with linear Neumann, Dirichlet or mixed type boundary conditions which simulate the current flow in real semiconductor devices. The focus of the paper [4] lies on numerical methods. There is assumed that the nonlinear boundary condition has a bounded derivative with respect to the the solution u. In [1] are studied existence, uniqueness and regularity of semilinear heat equations with nonlinear boundary conditions. Here growth conditions play an important role.

In [5] an Euler forward scheme is used for the numerical solution of nonlinear parabolic problems of p-Laplacian type.

In this paper we concentrate to semilinear parabolic problems in polygonal space-domains. In sections 2 we desrcibe our procedure for the simple boundary initial problem with a nonlinear Newton boundary condition. Since the operator to the weak formulated stationary boundary value problem $A : H^1(\Omega) \to H^1(\Omega)^*$ is strictly monotone, Lipschitz continuous and coercive, see [6], we need no growth conditions, in order to guarantee existence und uniqueness of a weak solution u. Due to regularity results for a linearized problem we get that $u \in W^{2,q}(\Omega)$ for sufficient smooth right hand sides, where q is determined by the maximal interior opening angle ω_0: $q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon < 2$ for $\omega_0 > \pi$, $q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon > 2$ for $\frac{\pi}{2} < \omega_0 < \pi$ and $q \geq 1$ is abitrary for $\omega_0 \leq \frac{\pi}{2}$. Here, $\varepsilon > 0$ is a small real number. We introduce an abstract initial value problem, where the associated operator A maps $D(A) \subset L_q(\Omega) \to L_q(\Omega)$. The m-accretivity of A can be shown and due to the semigroup theory we get solvability and regularity results in space and time of the type: $u \in C(I, D(A)) \cap W^{1,\infty,\infty}(I, L_q(\Omega), L_q(\Omega))$ and $u(t, \cdot) \in W^{2,q}(\Omega)\ \forall t \in I$.

In section 3 we study a more advanced problem with a nonlinear term $c(u)$ in the partial differential equation and a nonlinear term $b(u)$ in the boundary condition. At first we consider a weak formulation of the stationary boundary value problem described by an operator $A : H^1(\Omega) \to H^1(\Omega)^*$. We use a theorem of Brézis which says that pseudomonotone coercive operators are surjective. In order to ensure the pseudomonotonicity and coerciveness of the operator A we need growth conditions and a Carathéodory condition. Starting from a weak solution in $H^1(\Omega)$ and assuming that a growth condition is satisfied for the boundary term $b'(\cdot)$ additionally we get the same regularity result as for the simple model problem. For the application of the semigroup theory we have to guarantee the m-accretivity of $A : L_q(\Omega) \to L_q(\Omega)$. This can be done supposing that the functions $c(\cdot), b(\cdot)$ are monotone. In this way we get regularity results in space and time of the same quality as for the first model problem.

Section 4 is devoted to a semilinear problem with advection term $c(u)\nabla u$. We start with the stationary problem. The pseudomonotonicity of the operator belonging to the weak formulated problem can be shown, similar to the first two problems, whereas the coerciveness needs stronger growth conditions. The existence of weak solutions in $H^1(\Omega)$ follows again from the theorem of Brézis. A regularity result is formulated similar to the first two cases. But, the nonstationary problem can not be handled as before, since the m-accretivity can not be shown. Therefore, we introduce a weak formulated initial boundary value problem with the the operator $\mathcal{A} : L_2(I, V) \to L_2(I, V^*)$, where $V = H^1(\Omega)$. We discuss the boundedness of the operator $\mathcal{A}$ and get finally a result when a weak solution u belongs to the Bochner space $W^{1,2,2}(I, V, V^*) \cap C(I, L_2(\Omega))$.

2 The initial-boundary-value problem for an model problem

Let $\Omega \subset \mathbb{R}^2$ be a time-independent bounded polygonal domain with the maximal interior opening angle ω_0 and the boundary $\partial\Omega$. We set $Q = I \times \Omega, \Sigma = I \times \partial\Omega$, where $I = [0, T]$ is a fixed bounded time interval. By $\boldsymbol{n} = (n_1, n_2)$ we denote the unit outward normal to $\partial\Omega$. First, we consider the following initial-boundary-value problem as an model problem: Find $u = u(t, x)$ such that

$$\frac{\partial u}{\partial t} - \triangle u = f(t, x) \quad \text{for } (t, x) \in Q, \tag{2.1}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} + \kappa |u|^\alpha u = \varphi(t, x) \quad \text{for } (t, x) \in \Sigma, \tag{2.2}$$

$$u(0, x) = u_0(x) \quad \text{for } x \in \Omega, \tag{2.3}$$

where f and φ are given functions and $\kappa > 0$, $\alpha \geq 0$ are given constants.
In the sections 3 and 4 we prove solvability and regularity results for more general semilinear problems where the nonlinear terms satisfy growth conditions.

Besides the classsical formulation (2.1), (2.2), (2.3) we consider an abstract initial-value problem. For this purpose we interpret the function $u = u(t, x)$ as mapping from the time interval I into a Banach space X, $u : I \to X$. The quality of this mapping will be described by the belonging to Bochner spaces. Thus we consider the following spaces for $1 \leq p \leq \infty$:

$$L_p(I, X) = \{u : I \to X, \|u\|_{L_p(I,X)} = \left(\int_0^T \|u\|_X^p dt\right)^{\frac{1}{p}} < \infty\}, \quad \text{if } p \in [1, \infty), \tag{2.4}$$

$$L_\infty(I, X) = \{u : I \to X, \|u\|_{L_\infty(I,X)} = \text{ess sup}_{t \in I} \|u\|_X < \infty\}, \quad \text{if } p = \infty, \tag{2.5}$$

$$C(I, X) = \{u : I \to X \text{continuous}, \|u\|_{C(I,X)} = \max_{t \in I} \|u(t)\|_X\}, \tag{2.6}$$

and for two Banach spaces X_1, X_2 and $1 \leq p_i \leq \infty, i = 1, 2,$

$$W^{1,p_1,p_2}(I, X_1, X_2) = \{u \in L_{p_1}(I, X_1);\ \frac{du}{dt} \in L_{p_2}(I, X_2)\} \tag{2.7}$$

equipped with the norm

$$\|u\|_{W^{1,p_1,p_2}(I,X_1,X_2)} = \|u\|_{L_{p_1}(I,X_1)} + \|\frac{du}{dt}\|_{L_{p_2}(I,X_2)}. \tag{2.8}$$

The regularity of solutions to the stationary boundary value problem similar to (2.1), (2.2) was discussed in [8]. Starting from an uniquely defined weak solution $u \in W^{1,2}(\Omega)$ the following theorem was proved:

Theorem 1 *Let $u \in W^{1,2}(\Omega)$ a weak solution of the corresponding stationary boundary value problem in the polygonal domain Ω. If $f \in L_q(\Omega), \varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$, then $u \in W^{2,q}(\Omega)$, where*

$$q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon < 2 \quad \textit{for } \omega_0 > \pi, \tag{2.9}$$

$$q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon > 2 \quad \textit{for } \frac{\pi}{2} < \omega_0 < \pi, \tag{2.10}$$

$$q \geq 1 \textit{ is abitrary} \quad \textit{for } \omega_0 \leq \frac{\pi}{2}. \tag{2.11}$$

Here $\varepsilon > 0$ is a small real number.

This regularity result leads to the choice of the Banach space

$$X = L_q(\Omega), \tag{2.12}$$

where q is given by Theorem 1. We introduce the operator $A = -\triangle$, $A : \mathrm{dom}A = D(A) \subset X \to X$ where for given $\varphi = \varphi(t, \cdot)$

(2.13)
$$D(A) = \{u(t,\cdot) \in L_q(\Omega) : A(u(t,\cdot) = -\triangle u(t,\cdot) \in L_q(\Omega); \frac{\partial u}{\partial \boldsymbol{n}} + \kappa |u|^\alpha u = \varphi(t,\cdot) \in W^{1-\frac{1}{q},q}(\partial\Omega)\ \forall t \in I\}.$$

Thus we receive the formal initial boundary value problem: Find a solution $u \in C(I, D(A)) \cap W^{1,p_1,p_2}(I, X, X)$ for appropriate p_1, p_2 and $u_0 \in \overline{D(A)} \subset X$ such that

$$\frac{du}{dt} + A(u(t)) = f(t) \quad \text{for} \quad t \in I, \tag{2.14}$$

$$u(0) = u_0. \tag{2.15}$$

2.1 The m-accretivity of the operator A

We use the theory of semigroups in order to discuss the solvability and uniqueness of the solution of problem (2.14), (2.15) in appropriate Bochner spaces. The m-accretivity of the operator A is the key for getting such results, see [13], [9], [10]:

Definition 1 *Let X be a separable Banach space. The duality mapping $J : X \to X^*$ is defined by*

$$J(u) = \{u^* : u^* \in X^* \text{ with } \langle u^*, u\rangle = \|u\|_X^2 = \|u^*\|_{X^*}^2\}.$$

The operator $A : D(A) \subset X \to X$ is accretive, if for $u, v \in D(A)$ there exists an $(u-v)^* \in J(u-v)$ such that

$$\langle (u-v)^*, Au - Av\rangle \geq 0.$$

If additionally $R(Id + A) = X$, then A is called m-accretive.

Theorem 2 *The operator A, defined by (2.13), is m-accretive.*

Proof. As in [13], proposition 3.13, we choose to every $u \in X = L_q(\Omega)$ the following functional from $X^* = L_{\frac{q}{q-1}}(\Omega)$ defined as

$$u^* = \frac{u|u|^{q-2}}{\|u\|_{L_q(\Omega)}^{q-2}}. \tag{2.16}$$

A simple calculation shows that indeed $u^* \in J(u)$. Now, for arbitrary $u, v \in D(A), u \neq v$ we consider the dual pairing

$$\langle (u-v)^*, Au - Av\rangle = \frac{1}{\|u-v\|_{L_q(\Omega)}^{q-2}} \int_\Omega (u-v)|u-v|^{q-2}(-\triangle(u-v))dx. \tag{2.17}$$

Since $\nabla|u-v| = \text{sgn}(u-v)\nabla(u-v)$ exists, see e.g. [3], we can apply partial integration by Green's theorem and get

$$\int_\Omega (u-v)|u-v|^{q-2}(-\triangle(u-v))dx = \int_\Omega \nabla((u-v)|u-v|^{q-2})\cdot\nabla(u-v)dx \tag{2.18}$$
$$- \int_{\partial\Omega}((u-v)|u-v|^{q-2})\frac{\partial(u-v)}{\partial \boldsymbol{n}}ds = I_1 + I_2.$$

We estimate the first integral on the right-hand side of (2.18). We get

$$I_1 = \int_\Omega |u-v|^{q-2}\nabla(u-v)\cdot\nabla(u-v) + (q-2)\text{sgn}(u-v)(u-v)|u-v|^{q-3}\nabla(u-v)\cdot\nabla(u-v)dx$$
$$= \int_\Omega (q-1)|u-v|^{q-2}\nabla(u-v)\cdot\nabla(u-v)dx \geq 0.$$

In the second boundary integral we insert the nonlinear Newton condition (2.2). It follows that

$$I_2 = \kappa\int_{\partial\Omega}((u-v)|u-v|^{q-2})(|u|^\alpha u - |v|^\alpha v)ds \geq 0.$$

Here, we have used the uniform monotonicity of the mapping $\psi : \mathbb{R} \to \mathbb{R}; \psi(u) = |u|^\alpha u$, (see, e.g. ([8], proof of Lemma 5.8.), which means that $(u-v)(|u|^\alpha u - |v|^\alpha v) \geq 0$. This implies that (2.17) is nonnegative and the operator A is accretive.

Now, we show that $Id + A : D(A) \subset L_q(\Omega) \to L_q(\Omega)$ is a surjective mapping. We show even, that the boundary value problem

$$-\triangle u + u = F \quad \text{in } \Omega, \tag{2.19}$$

$$\frac{\partial u}{\partial \mathbf{n}} + \kappa|u|^\alpha u = \varphi \quad \text{on } \partial\Omega, \tag{2.20}$$

has for all $F \in L_q(\Omega)$ an uniquely defined solution in $D(A)$. We can proceed as in [7] and [8]. First we introduce a weak formulation considering the following relation on $H^1(\Omega)\times H^1(\Omega)$ for sufficiently smooth functions F and φ:

$$A(u,v) = \int_\Omega \nabla u\cdot\nabla v dx + \int_\Omega uvdx + \kappa\int_{\partial\Omega}|u|^\alpha uvdS \tag{2.21}$$

$$= \int_\Omega Fvdx + \int_{\partial\Omega}\varphi vdS. \tag{2.22}$$

Using the monotone operator theory it can be shown that exactly one solution in $H^1(\Omega)$ exists. Since $H^1(\Omega) \subset L_\gamma(\Omega)$ for all $\gamma \in [1,\infty)$, we can shift the term u in (2.19) to the right-hand side and consider the problem

$$-\triangle u = F - u = f \quad \text{in } \Omega, \tag{2.23}$$

$$\frac{\partial u}{\partial \mathbf{n}} + \kappa|u|^\alpha u = \varphi \quad \text{on } \partial\Omega, \tag{2.24}$$

where $f \in L_q(\Omega), \varphi \in W^{1-\frac{1}{q}}(\partial\Omega)$. Theorem 1 yields the existence of a uniquely defined solution $u \in D(A)$ for every $F \in L_q(\Omega)$. $\square$

The m-accretivity of A ensures that a Rothe-sequence to the abstract initial problem (2.14), (2.15) exists. We follow the statements of [13], chapter 8.2., for the definition of a Rothe sequence.
Let us start with a discretization of the time interval $I = [0, T]$ considering a sequence of time steps:

$$\tau_\ell = \frac{T}{2^\ell}, \quad \ell = 1, 2, ...$$

Thus we get for the ℓ-th partition of $I = [0, T]$ the time instants

$$t_{\ell k} = k\tau_\ell, \; k = 0, 1, ..., 2^\ell.$$

In what follows we omit the index ℓ for simplicity and write τ instead of τ_ℓ and $t_k = k\tau$ instead of $t_{\ell k}$.
We consider the values of the right-hand side $f \in C(I, L_q(\Omega))$ from (2.14), at the nodal points $t = \kappa\tau, k = 1, ..., \frac{T}{\tau}$ and denote them by $f_\tau^k = f(k\tau)$. Then we define $u_\tau^k \in D(A)$ by the recursive implicit Euler formula:

$$\frac{u_\tau^k - u_\tau^{k-1}}{\tau} + A(u_\tau^k) = f_\tau^k, \tag{2.25}$$

$$u_\tau^0 = u_0. \tag{2.26}$$

The piecewise defined affine interpolant $u_\tau \in C(I, D(A))$ is given by

$$u_\tau(t) = (\frac{t}{\tau} - (k-1))u_\tau^k + (k - \frac{t}{\tau})u_\tau^{k-1} \text{ for } (k-1)\tau < t \leq k\tau, \; k = 1, ..., \frac{T}{\tau}. \tag{2.27}$$

The piecewise constant interpolant $\bar{u}_\tau(t) \in L^\infty(I, D(A))$ is defined as

$$\bar{u}_\tau(t) = u_\tau^k \quad \text{for } (k-1)\tau < t \leq k\tau, \; k = 1, ..., \frac{T}{\tau}. \tag{2.28}$$

The derivative $\frac{d}{dt}u_\tau$ belongs to $L^\infty(I, D(A))$; the derivative of $\bar{u}_\tau(t)$ exists in the distributional sense only. In [13], Chapter 9, p.277, Lemma 9.3 and Lemma 9.4, the following results are proved in a Banach space X:

Lemma 1 *Let $A : D(A) \subset X \to X$ be m-accretive, $f \in L_1(I, X), u_0 \in \overline{D(A)} \subset X$. Then for arbitrary τ there exists u_τ and it holds*

$$\|u_\tau\|_{C(I,X)} \leq C, \quad \|\bar{u}_\tau\|_{L^\infty(I,X)} \leq C. \tag{2.29}$$

Moreover, if $f \in W^{1,\infty,1}(I, X, X), u_0 \in D(A)$, then

$$\|\frac{du_\tau}{dt}\|_{L_\infty(I,X)} \leq \|f\|_{W^{1,\infty,1}(I,X,X)} + \|A(u_0)\|_X. \tag{2.30}$$

Remark 1 *The interpolant $u_\tau(t)$ is indeed from $D(A)$ since for $k = 1, ..., \frac{T}{\tau}$*

$$A(u_\tau^k) = f_\tau^k - \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \quad \in X.$$

2.2 Regularity in space and time

The existence of solutions of the problem (2.14), (2.15) can be shown by results of Kato [9], [10], looking for the limit of the sequence $\{u_\tau\}$ for $\tau \to 0$.
We introduce a strong solution by the following conditions, see [13], p.275: Let be $X = L^q(\Omega)$, where q is given by Theorem 1. The function $u = u(t,x)$ is a strong solution of problem (2.14), (2.15), if

1. $u \in W^{1,\infty,1}(I, X, X)$.
2. $\{A(u(t)\}_{t\geq 0}$ is bounded.
3. $u(t) \in D(A)$ for all $t \in I$.
4. u satisfies the initial problem (2.14), (2.15) for a.e. $t \in I$.

Now, we cite the corresponding theorem 9.5 from [13] p.278:

Theorem 3 *Let $A : D(A) \subset X \to X$ be m-accretive, $f \in W^{1,\infty,1}(I, X, X), u_0 \in D(A)$. Then there is an $u \in W^{1,\infty,\infty}(I, X, X)$ such that $u_\tau \to u$ in $C(I, X)$ and this u is a strong solution of the problem (2.14), (2.15).*

Taking the assertion $u(t) \in D(A)$ for all $t \in I$ into account, then we get the following corollary for our initial problem (2.14), (2.15):

Corollary 1 *Let $q \geq 1$ be given by Theorem 1. If $f = f(t,x) \in W^{1,\infty,1}(I, L_q(\Omega), L_q(\Omega)), \varphi(t,\cdot) \in W^{1-\frac{1}{q},q}(\partial\Omega)\ \forall t \in I$ and $u_0 \in D(A)$, then for the solution u of problem (2.14), (2.15) it holds: $u \in C(I, D(A)) \cap W^{1,\infty,\infty}(I, L_q(\Omega), L_q(\Omega))$ and $u(t,\cdot) \in W^{2,q}(\Omega)\ \forall t \in I$.*

3 Some advanced semilinear problems

The considered initial boundary value problem problem (2.1), (2.2) and (2.3) was quite well to handle, since the nonlinear boundary condition was explicitely given. The question occurs, whether we can transfer the used methods to more general nonlinear problems. In this section we study the following semilinear problems:

$$\frac{\partial u}{\partial t} - \triangle u + c(u) = f(t,x) \quad \text{for } (t,x) \in Q, \tag{3.31}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} + b(u) = \varphi(t,x) \quad \text{for } (t,x) \in \Sigma, \tag{3.32}$$

$$u(0,x) = u_0(x) \quad \text{for } x \in \Omega. \tag{3.33}$$

where f and φ are given functions and the properties of the expressions $c(u)$ and $b(u)$ are to specify.

3.1 The stationary boundary value problem

First of all we have to discuss the solvability and regularity of the stationary boundary value problem in a polygon Ω with the boundary $\partial\Omega$:

$$-\triangle u + c(u) = f \quad \text{in } \Omega, \tag{3.34}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} + b(u) = \varphi \quad \text{on } \partial\Omega. \tag{3.35}$$

The corresponding weak formulation in $V = H^1(\Omega)$ reads:
Find an $u \in V$ such that

$$\begin{aligned} \langle A(u), v\rangle = a(u,v) &= \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c(u) v dx + \int_{\partial\Omega} b(u) v dS \\ &= \int_\Omega f v dx + \int_{\partial\Omega} \varphi v dS = \langle L, v\rangle, \ \forall v \in V, \end{aligned} \tag{3.36}$$

where $A : V \to V^*$ and $L \in V^*$. This formulation makes sense, if we can guarantee that $c(u)$ and $b(u)$ are from V^* for every $u \in V$ and if we use the dual pairing $\langle c(u), v\rangle$ and $\langle b(u), v\rangle$ instead of the corresponding integrals. But, in order to get regularity results we want to ensure that for $u \in V$ the expressions $c(u) \in L_{q_1}(\Omega)$ and $b(u)|_{\partial\Omega} \in W^{1-\frac{1}{q_2},q_2}(\partial\Omega)$, where $q_1 \geq 1$ is arbitrary and $1 < q_2 < 2$.

Lemma 2 *Assume that the nonlinear terms $c(s)$ and $b(s)$ satisfy the following growth conditions for all $s \in \mathbb{R}$:*
G1 $|c(s)| \leq k_c(1 + |s|^{\gamma_c})$ *with a constant $\gamma_c \geq 1$ and a positive constant k_c.*
G2 $|b(s)| \leq k_b(1 + |s|^{\gamma_b})$ *with a constant $\gamma_b \geq 1$ and a positive constant k_b.*
G3 $|\frac{db(s)}{ds}| \leq \tilde{k}_b(1 + |s|^{\tilde{\gamma}_b})$ *with a constant $\tilde{\gamma}_b \geq 1$ and a positive constant $\tilde{k}_b$.*
Then for $u \in H^1(\Omega)$

$$c(u) \in L_{q_1}(\Omega) \textit{ for any } q_1 \geq 1, \tag{3.37}$$

$$b(u) \in W^{1-\frac{1}{q_2},q_2}(\partial\Omega) \ \textit{ with } 1 < q_2 < 2. \tag{3.38}$$

Later, in Lemma 13, we have to consider the stronger case, that $0 \leq \gamma_c, \gamma_b \leq 1$. Since $|s|^a < 1 + |s|^b$ for $0 \leq a \leq b$ this case is included too.

Proof. Since $u \in H^1(\Omega)$ the assertion (3.37) follows from the imbedding $H^1(\Omega) \subset L_q(\Omega)$ for any $q \geq 1$. The same arguments leads to $b(u) \in L_{q_2}(\Omega)$. Therefore it remains to show, that $\nabla b(u) \in L_{q_2}(\Omega)$. Then the trace $b(u)|_{\partial\Omega}$ is from $W^{1-\frac{1}{q_2},q_2}(\partial\Omega)$. Indeed, the assumption **G3** and the well-known inequality $(a+b)^\alpha \leq 2^{\alpha-1}(a^\alpha + b^\alpha)$, which is valid for $a, b \geq 0, \alpha \geq 1$, yield

$$\begin{aligned} \int_\Omega |\nabla(b(u))|^{q_2} dx &= \int_\Omega |\frac{db(u)}{du}|^{q_2} |\nabla u|^{q_2} dx \\ &\leq \tilde{k}_b^{q_2} \int_\Omega (1 + |u|^{\tilde{\gamma}_b})^{q_2} |\nabla u|^{q_2} dx \\ &\leq \tilde{c}_b(\||\nabla u|\|^{q_2}_{L_{q_2}(\Omega} + \int_\Omega |u|^{\tilde{\gamma}_b q_2} |\nabla u|^{q_2} dx) \\ &\leq \tilde{c}_b(\||\nabla u|\|^{q_2}_{L_{q_2}(\Omega} + \||u|^{\tilde{\gamma}_b q_2}\|_{L_\beta(\Omega)} \||\nabla u|^{q_2}\|_{L_{\beta'}}(\Omega)), \end{aligned} \tag{3.39}$$

where $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. The first term is finite, since $q_2 < 2$, and both factors in the second term are finite choosing $\beta' = \frac{2}{q_2}$. $\square$

In what follows we need a certain continuity of the operator A in the following sense: If $u_k \rightharpoonup u$ in $V = H^1(\Omega)$ (weak convergence), then for the nonlinear parts of the operator A we have $c(u_k) \to c(u)$ in $L_\alpha(\Omega), 1 \leq \alpha < \infty$ and $b(u_k) \to b(u)$ in $L_\gamma(\partial\Omega), 1 \leq \gamma < \infty$. To this aim we introduce the following Carathéodory property (**C**) of the expressions $c(u)(x)$ and $b(u)(x)$:

(**C**) If the sequence $\{u_n\}_n$ converges pointwise a.e. in Ω, or $\partial\Omega$ respectively, that means $u_n(x) \to u(x)$ a.e., then the pointwise convergence of the function values follows: $c(u_n)(x) \to c(u)(x)$ a.e. in Ω and $b(u_n)(x) \to b(u)(x)$ a.e. in $\partial\Omega$.

Later we need for the nonstationary problem a modified Carathéodory property:

($\tilde{\mathbf{C}}$) If the sequence $\{u_n\}_n$ converges pointwise a.e. in $Q = I \times \Omega$, or $\Sigma = I \times \partial\Omega$ respectively, that means $u_n(t,x) \to u(t,x)$ a.e., then the pointwise convergence of the function values follows: $c(u_n)(t,x) \to c(u)(t,x)$ a.e. in Q and $b(u_n)(t,x) \to b(u)(t,x)$ a.e. in Σ.

Lemma 3 *Assume (***G1***), (***G2***), (***C***). Let $\{u_k\}_k$ be a sequence of elements from V, which converges weakly to $u \in V$. Then $c(u_k) \to c(u)$ in $L_\alpha(\Omega), 1 \leq \alpha < \infty$ and $b(u_k) \to b(u)$ in $L_\gamma(\partial\Omega), 1 \leq \gamma < \infty$.*

Proof. In principle the proof is classical with a light modification related to our special function spaces, see e.g. [14], p.68. Therefore we restrict to the main steps.
step 1: We consider a weak convergent sequence in $V = H^1(\Omega)$, $u_k \rightharpoonup u$. Since $H^1(\Omega)$ is compactly imbedded in $L_\alpha(\Omega)$ for any α with $1 \leq \alpha < \infty$, there exists a subsequence $\mathbf{u}_{nk} \to u$ in $L_\alpha(\Omega)$. It follows [11], p.88, that there is a sub-subsequcence u_{nkl} which converges pointwise to u for a.e. $x \in \Omega$. Considering the compact imbedding of $H^1(\Omega)$ in $L_\gamma(\partial\Omega)$ see e.g. [13],p.17, 1.36b, we get the pointwise convergence of a sub-subsequence a.e. for $x \in \partial\Omega$.
step 2: The assumption (**C**) leads to the pointwise convergence a.e. of the sequences $\{c(u_{nkl}\}$ and $\{b(u_{nkl}\}$ respectively. Due to the growth conditions (**G1**) and (**G2**) and the convergence of $\|u_{nk}\| \to \|u\|$ in the norms of $L_\alpha(\Omega)$ or in $L_\gamma(\partial\Omega)$ respectively, we get majorants

$$\int_\Omega |c(u_{nkl}) - c(u)|^p dx \leq k_c \int_\Omega (2 + |u_{knl}|^{\gamma_c p} + |u|^{\gamma_c p}) dx \leq const + const\|u\|^\alpha_{L_\alpha(\Omega)}, \quad \alpha = \gamma_c p.$$

$$\int_{\partial\Omega} |b(u_{nkl}) - b(u)|^p dS \leq k_c \int_{\partial\Omega} (2 + |u_{knl}|^{\gamma_b p} + |u|^{\gamma_b p}) dS \leq const + const\|u\|^\gamma_{L_\gamma(\partial\Omega)}, \quad \gamma = \gamma_b p.$$

The classical theorem of Lebesgue on the dominated convergence, see [11] p.60, [14] p.68, yields that $\|c(u_{nkl}) - c(u)\|_{L_p(\Omega)} \to 0$ and $\|b(u_{nkl}) - b(u)\|_{L_p(\partial\Omega)} \to 0$. Since we can start in step 1 with an arbitrary subsequence of u_n we get finally that $\|c(u_n) - c(u)\|_{L_\alpha(\Omega)} \to 0$ and
$\|b(u_n) - b(u)\|_{L_\gamma(\partial\Omega)} \to 0$ for any $\alpha, \gamma \in [1, \infty)$.

$\square$

3.1.1 Existence of weak solutions

We use the theory of pseudomonotone operators in order to discuss under which conditions a weak solution of problem (3.36) in $V = H^1(\Omega)$ exists. Let us remind of the definition of the pseudomonotonicity: The operator $A : V \to V^*$ is called pseudomonotone, if
(**PM1**) A is bounded,
(**PM2**) From $u_k \rightharpoonup u$ and $\limsup_{k\to\infty}\langle A(u_k), u_k - u\rangle \le 0$ it follows that $\langle A(u), u - v\rangle \le \liminf_{k\to\infty}\langle A(u_k), u_k - v\rangle \ \forall v \in V$.

Theorem 4 *(Brézis 1968)*
If the operator $A : V \to V^*$ is pseudomonotone and coercive then A is surjective.

Now, we consider the operator A belonging to our semilinear problem.

Lemma 4 *Assume that for the operator A, given by (3.36), the conditions* **G1**, **G2** *and* **C** *are satisfied. Then A is pseudomonotone.*

Proof. In the proof we use standard arguments, see e.g. [13], p. 48. But, we restrict to our special operator and make the proof as simple as possible.
We start with the property **PM1** and show that the range $A\{u \in V, \|u\|_V \le R\}$ is bounded in V^*:

$$\begin{aligned}
&\sup_{\|u\|_V\le R}\|A(u)\|_{V^*} = \sup_{\|u\|_V\le R}\sup_{\|v\|_V\le 1}\langle A(u), v\rangle\\
&= \sup_{\|u\|_V\le R}\sup_{\|v\|_V\le 1}\Big(\int_\Omega \nabla u\cdot\nabla v dx + \int_\Omega c(u)v dx + \int_{\partial\Omega} b(u)v dS\Big)\\
&\le \sup_{\|u\|_V\le R}\sup_{\|v\|_V\le 1}\left(\|\nabla u\|_{L_2(\Omega)}\|\nabla v\|_{L_2(\Omega)} + k_c\int_\Omega(1+|u|^{\gamma_c})|v|dx + k_b\int_{\partial\Omega}(1+|u|^{\gamma_b})|v|dS\right)\\
&\le R + \text{const} + \sup_{\|u\|_V\le R}\sup_{\|v\|_V\le 1}\left(k_c\int_\Omega(|u|^{\gamma_c})|v|dx + k_b\int_{\partial\Omega}(|u|^{\gamma_b})|v|dS\right)\\
&\le \text{const} + \sup_{\|u\|_V\le R}\sup_{\|v\|_V\le 1}\left(k_c\||u|^{\gamma_c}\|_{L_{\gamma_c'}(\Omega)}\|v\|_{L_{\gamma_c}(\Omega)} + k_b\||u|^{\gamma_b}\|_{L_{\gamma_b'}(\partial\Omega)}\|v\|_{L_{\gamma_b}(\partial\Omega)}\right)\\
&\le \text{const} + \text{const}\sup_{\|u\|_V\le R}\sup_{\|v\|_V\le 1}\|u\|_V\|\|v\|_V \le \text{const}.
\end{aligned}$$

For the estimation of v we have used the imbeddings $H^1(\Omega) \subset L_\alpha(\Omega)$, and $H^1(\Omega) \subset L_\gamma(\partial\Omega)$ for any $1 \le \alpha < \infty, 1 \le \gamma < \infty$. In the same way we estimate u, remarking that $\||u|^{\gamma_c}\|_{L_{\gamma_c'}(\Omega)} = \|u\|^{\gamma_c}_{L_{\gamma_c\gamma_c'}(\Omega)}$ and $\||u|^{\gamma_b}\|_{L_{\gamma_b'}(\partial\Omega)} = \|u\|^{\gamma_b}_{L_{\gamma_b\gamma_b'}(\partial\Omega)}$.

Now, we discuss the property **PM2**: We start with a sequence $u_k \rightharpoonup u$ in V and assume $\limsup_{k\to\infty}\langle A(u_k), u_k - u\rangle \le 0$. We shall show that $\langle A(u), u - v\rangle \le \liminf_{k\to\infty}\langle A(u_k), u_k - v\rangle \ \forall v \in V$. For any $w \in V$ we have:

$$0 \le \langle\nabla(u_k - w), \nabla(u_k - w)\rangle = \langle A(u_k), u_k - w\rangle - \langle\nabla w, \nabla(u_k - w)\rangle - \langle c(u_k), u_k - w\rangle - \langle b(u_k), u_k - w\rangle.$$

We take an $\varepsilon \in [0, 1]$ and set $w = (1-\varepsilon)u + \varepsilon v$ in the second term of the dual pairs:

$$\begin{aligned}
&\langle A(u_k), u_k - ((1-\varepsilon)u + \varepsilon v)\rangle - \langle\nabla w, \nabla(u_k - ((1-\varepsilon)u + \varepsilon v))\rangle\\
&-\langle c(u_k), u_k - ((1-\varepsilon)u + \varepsilon v)\rangle - \langle b(u_k), u_k - ((1-\varepsilon)u + \varepsilon v)\rangle \ge 0.
\end{aligned}$$

Thus we get

$$\begin{aligned}\varepsilon\langle A(u_k), u-v\rangle \geq & -\langle A(u_k), u_k-u\rangle + \langle \nabla w, \nabla(u_k-u)\rangle + \varepsilon\langle \nabla w, \nabla(u-v)\rangle \\ & + \langle c(u_k), u_k-u\rangle + \varepsilon\langle c(u_k), u-v\rangle + \langle b(u_k), u_k-u\rangle + \varepsilon\langle b(u_k), u-v\rangle\end{aligned}$$

and

(3.40)
$$\begin{aligned}\varepsilon \liminf_{k\to\infty}\langle A(u_k), u-v\rangle \geq & -\limsup_{k\to\infty}\langle A(u_k), u_k-u\rangle + \lim_{k\to\infty}\langle \nabla w, \nabla(u_k-u)\rangle + \varepsilon\langle \nabla w, \nabla(u-v)\rangle \\ & + \lim_{k\to\infty}\langle c(u_k), u_k-u\rangle + \varepsilon\lim_{k\to\infty}\langle c(u_k), u-v\rangle + \lim_{k\to\infty}\langle b(u_k), u_k-u\rangle \\ & + \varepsilon\lim_{k\to\infty}\langle b(u_k), u-v\rangle.\end{aligned}$$

For the following terms of the right hand side of (3.40) it holds:

- $-\limsup_{k\to\infty}\langle A(u_k), u_k-u\rangle \geq 0$ by assumption.
- $\lim_{k\to\infty}\langle \nabla w, \nabla(u_k-u)\rangle = 0$, since $\nabla u_k \rightharpoonup \nabla u$.
- $\lim_{k\to\infty}\langle c(u_k), u_k-u\rangle = 0$ and $\lim_{k\to\infty}\langle b(u_k), u_k-u\rangle = 0$ due to Lemma 3.
- $\lim_{k\to\infty}\langle c(u_k), u-v\rangle = \langle c(u), u-v\rangle$ and $\lim_{k\to\infty}\langle b(u_k), u-v\rangle = \langle b(u), u-v\rangle$ due to Lemma 3.

Therefore, the inequality (3.40) reads:

(3.41)
$$\begin{aligned}\varepsilon\liminf_{k\to\infty}\langle A(u_k), u-v\rangle \geq & \ \varepsilon\langle \nabla w, \nabla(u-v)\rangle \\ & + \varepsilon\langle c(u), u-v\rangle + \varepsilon\langle b(u), u-v\rangle.\end{aligned}$$

Dividing by ε, inserting $w = (1-\varepsilon)u + \varepsilon v$ and taking the limit $\varepsilon \to 0$ we get the assertion

$$\liminf_{k\to\infty}\langle A(u_k), u-v\rangle \geq \langle A(u), u-v\rangle \ \forall v \in V.$$

□

It remains to investigate the coercivity, that means to discuss, under which conditions for the operator $A : V \to V^*$ it holds

(3.42)
$$\lim_{\|u\|_V\to\infty}\frac{\langle A(u), u\rangle}{\|u\|_V} = \infty.$$

This question was discussed in [13], p. 51, and we cite the result:

Lemma 5 *If there are constants $c_1 > 0, c_2 < \infty$ and functions $k_1 \in L_1(\Omega), k_2 \in L_1(\partial\Omega)$ such that*
*(**Co-c**) $c(s)s \geq c_1|s|^q - k_1$,*
*(**Co-b**) $b(s)s \geq -c_2|s|^{q_1} - k_2$*
for all $s \in \mathbb{R}$ and some $1 < q_1 < q \leq 2$, then A is coercive.

The proof in [13] is related to the application of appropiate Poincaré and Young inequalities. To cover our special boundary value problem in Section 2 we modify it:

Lemma 6 *If there are constants $c_1 > 0, c_2 < \infty$ and functions $k_1 \in L_1(\partial\Omega), k_2 \in L_1(\Omega)$ such that*
*(**co-b**) $b(s)s \geq c_1|s|^q - k_1$,*
*(**co-c**) $c(s)s \geq -c_2|s|^{q_1} - k_2$*
for all $s \in \mathbb{R}$ and some $1 < q_1 < q \leq 2$, then A is coercive.

Proof. We remind to the following Poincaré inequality, [13], p.21:

$$\|u\|_V \leq C(\|\nabla u\|_{L_2(\Omega)} + \|u\|_{L_q(\partial\Omega)}), \tag{3.43}$$

where $V = H^1(\Omega)$, $1 \leq q < \infty$ and $C > 0$ is an appropriate constant. It follows for $q \leq 2$ that there is a constant $C_1 > 0$, such that

$$\|u\|_V^q \leq C_1(\|\nabla u\|_{L_2(\Omega)}^q + \|u\|_{L_q(\partial\Omega)}^q) \leq C_1(1 + \|\nabla u\|_{L_2(\Omega)}^2 + \|u\|_{L_q(\partial\Omega)}^q).$$

Thus we get

$$\|\nabla u\|_{L_2(\Omega)}^2 + \|u\|_{L_q(\partial\Omega)}^q \geq \frac{\|u\|_V^q}{C_1} - 1. \tag{3.44}$$

Furthermore, the Young inequality for $a, b \in \mathbb{R}^+$

$$ab \leq \varepsilon a^p + C_\varepsilon b^{p'} \quad \forall \varepsilon > 0, 1 < p < \infty,\ 1/p + 1/p' = 1,$$

implies that for $1 \leq q_1 < q \leq 2$ and $p = \frac{q}{q_1} > 1$ we have

$$|u|^{q_1} 1 \leq \varepsilon |u|^{q_1 p} + C_\varepsilon = \varepsilon |u|^q + C_\varepsilon.$$

It follows, there is a constant $C_2 > 0$ with

$$\|u\|_{L_{q_1}(\Omega)}^{q_1} \leq \varepsilon \|u\|_{L_q(\Omega)}^q + C_\varepsilon \text{meas}(\Omega) \leq \varepsilon C_2 \|u\|_{L_2(\Omega)}^q + C_\varepsilon \text{meas}(\Omega). \tag{3.45}$$

Due to **co-c**, **co-b** and (3.44) we get the following estimates with some constants K_i:

$$\begin{aligned}
\langle Au, u \rangle &= \int_\Omega \nabla u \cdot \nabla u dx + \int_\Omega c(u) u dx + \int_{\partial\Omega} b(u) u dS && (3.46)\\
&\geq \|\nabla u\|_{L_2(\Omega)}^2 - c_2 \|u\|_{L_{q_1}(\Omega)}^{q_1} - K_2 + c_1 \|u\|_{L_q(\partial\Omega)}^q - K_1 \\
&\geq \min\{1, c_1\}(\|\nabla u\|_{L_2(\Omega)}^2 + \|u\|_{L_q(\partial\Omega)}^q) - c_2 \|u\|_{L_{q_1}(\Omega)}^{q_1} - K_3 \\
&\geq \min\{1, c_1\}(\frac{\|u\|_V^q}{C_1} - 1) - c_2 \|u\|_{L_{q_1}(\Omega)}^{q_1} - K_3. \\
&= \min\{1, c_1\}\frac{\|u\|_V^q}{C_1} - c_2 \|u\|_{L_{q_1}(\Omega)}^{q_1} - K_4. && (3.47)
\end{aligned}$$

Further, from the inequality (3.45) it follows that for a sufficient small $\varepsilon > 0$ there exists a constant $C_3 > 0$ such that

$$
\begin{aligned}
\langle Au, u\rangle &\geq \min\{1, c_1\}\frac{\|u\|_V^q}{C_1} - c_2 C_2 \varepsilon \|u\|_{L_2(\Omega)}^q - K_{\varepsilon,5} \\
&\geq C_3\|u\|_V^q - K_{\varepsilon,5}.
\end{aligned} \tag{3.48}
$$

Note, that $K_{\varepsilon,5}$ is a finite constant for an appropriate small fixed ε.
Finally, dividing by $\|u\|_V$ and taking into account that $q > 1$, we get the coercivity (3.46). □

Remark 2 *We have investigated in Section 2 of this paper the case that $c(u) = 0, b(u)u = \kappa|u|^{\alpha+2}$ what is covered by Lemma 6, condition* **co-b** *and* **co-c**.

Summarizing the results we get the following theorem on the existence of weak solutions of the semilinear problem (3.34), (3.35):

Theorem 5 *Assume (***G1***), (***G2***), (***C***) and (***Co-c***), (***Co-b***) or (***co-b***,(***co-c***), then the operator $A : V \to V^*$, given by (3.36), is surjective.*

3.1.2 Regularity results

We prove a regularity theorem for the semilinear problem (3.36), which is from the same quality as the regularity Theorem 1.

Theorem 6 *Let the assumptions of Theorem 5 and the growth condition (***G3***) be satisfied. If $f \in L_q(\Omega)$ and $\varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$ for the right-hand sides of (3.34) and (3.35), then a weak solution $u \in H^1(\Omega)$ of (3.36) belongs to $W^{2,q}(\Omega)$, where q is given by Theorem 1.*

Proof. Due to Theorem 5 a weak solution $u \in H^1(\Omega)$ exists. Moreover, Lemma 2 implies that $c(u) \in L_{q_1}(\Omega)$ for any $q_1 \geq 1$ and $b(u) \in W^{1-\frac{1}{q_2},q_2}(\partial\Omega)$ with $1 < q_2 < 2$. We shift the nonlinear terms to the right-hand side and consider the Neumann problem

$$
\begin{aligned}
-\triangle u = f - c(u) = F \quad &\text{in } \Omega, \\
\frac{\partial u}{\partial \boldsymbol{n}} = \varphi - b(u) = \Phi \quad &\text{on } \partial\Omega.
\end{aligned}
$$

If $q < 2$, then we set $q_2 = q$. The regularity theory for linear problems in polygons yields the assertion, see[8].
If $q \geq 2$, then it follows at first that $u \in W^{2,q_2}(\Omega),\ 1 < q_2 < 2$. Since $W^{2,q_2}(\Omega) \subset W^{1,q+1}(\Omega)$ we can

modify the estimate (3.39):

$$\begin{aligned}(3.49)\qquad \int_\Omega |\nabla(b(u))|^q dx &= \int_\Omega |\frac{db(u)}{du}|^q |\nabla u|^q dx \\ &\leq \tilde{k}_b^q \int_\Omega (1+|u|^{\tilde{\gamma}_b})^q |\nabla u|^q dx \\ &\leq \tilde{c}_b(\||\nabla u|\|^q_{L_q(\Omega} + \int_\Omega |u|^{\tilde{\gamma}_b q}|\nabla u|^q dx) \\ &\leq \tilde{c}_b(\||\nabla u|\|^q_{L_q(\Omega} + \||u|^{\tilde{\gamma}_b q}\|_{L_\beta(\Omega)}\||\nabla u|^q\|_{L_{\beta'}}(\Omega)),\end{aligned}$$

where $\frac{1}{\beta}+\frac{1}{\beta'}=1$. The first term is finite and choosing $\beta'=\frac{q+1}{q}$ both factors in the Hölder inequality are finite too. Therefore, the trace of $b(u)$ is from $W^{1-\frac{1}{q},q}(\partial\Omega)$ and the regularity theory for linear problems implies that q can be choosen as in Theorem 1. □

3.2 The nonstationary problem

We consider now the initial boundary value problem (3.31), (3.32), (3.33.) Theorem 6 makes it possible to proceed as in Section 2.
We choose the Banach space $X=L_q(\Omega)$, where q is given by theorem 1 and introduce the operator $A=-\triangle+c(\cdot)$, $A:\text{dom}A=D(A)\subset X\to X$, where for a given $\varphi=\varphi(t,\cdot)$

$$\begin{aligned}(3.50)\qquad D(A)=\{u(t,\cdot)\in L_q(\Omega):A(u(t,\cdot))=-\triangle u(t,\cdot)+c(u(t,\cdot))\in L_q(\Omega);\\ \frac{\partial u}{\partial \boldsymbol{n}}+b(u(t,\cdot))=\varphi(t,\cdot)\in W^{1-\frac{1}{q},q}(\partial\Omega)\ \forall t\in I\}.\end{aligned}$$

The formal initial problem reads:

Find a solution $u\in C(I,D(A))\cap W^{1,p_1,p_2}(I,X,X)$ for appropriate p_1,p_2 and $u_0\in\overline{D(A)}\subset X$ such that

$$(3.51)\qquad \frac{du}{dt}+A(u(t))=f(t)\quad\text{for}\quad t\in I,$$

$$(3.52)\qquad u(0)=u_0.$$

3.2.1 The m-accretivity

In section 2.1 we have proved the m-accretivity of the operator A defined by (2.13) for the explictly given nonlinear Newton conditions using the monotonicity of the nonlinear boundary part. In order to ensure the m-accretivity for our semilinear problem we have to assume additionally the monotonicity of the nonlinear terms $c(u)$ and $b(u)$.

Theorem 7 *Let the assumptions of Theorem 6 be satisfied. Moreover, suppose*
Mon-c: *For the mapping* $c : \mathbb{R} \to \mathbb{R}$ *it holds* $(c(s_1) - c(s_2))(s_1 - s_2) \geq 0$ *for any* $s_1, s_2 \in \mathbb{R}$.
Mon-b: *For the mapping* $b : \mathbb{R} \to \mathbb{R}$ *it holds* $(b(s_1) - b(s_2))(s_1 - s_2) \geq 0$ *for any* $s_1, s_2 \in \mathbb{R}$.
Then the operator A*, defined by (3.50) is m-accretive.*

Proof. As in the proof of Theorem 2 we choose to every $u \in X = L_q(\Omega)$ the following functional from $X^* = L_{\frac{q}{q-1}}(\Omega)$

$$u^* = \frac{u|u|^{q-2}}{\|u\|_{L_q(\Omega)}^{q-2}}.$$

Now we consider for arbitrary $u, v \in D(A), u \neq v$, the dual pairing

$$\langle (u-v)^*, Au - Av \rangle = \frac{1}{\|u-v\|_{L_q(\Omega)}^{q-2}} \int_\Omega (u-v)|u-v|^{q-2}(-\triangle(u-v) + c(u) - c(v))dx.$$

Due to the regularity results $\nabla|u-v| = \mathrm{sgn}(u-v)\nabla(u-v)$ exists, and we can apply partial integration and get

$$\begin{aligned}
&\int_\Omega (u-v)|u-v|^{q-2}\left(-\triangle(u-v) + c(u) - c(v)\right) dx \\
&= \int_\Omega \nabla((u-v)|u-v|^{q-2}) \cdot \nabla(u-v) + (u-v)|u-v|^{q-2}(c(u) - c(v))dx \\
&\quad - \int_{\partial\Omega} (u-v)|u-v|^{q-2}\frac{\partial(u-v)}{\partial \boldsymbol{n}} ds \\
&= I_1 + I_2.
\end{aligned}$$

We estimate the first integral on Ω. The assumption **Mon-c** implies

$$I_1 = \int_\Omega (q-1)|u-v|^{q-2}\nabla(u-v) \cdot \nabla(u-v) + |u-v|^{q-2}(c(u) - c(v))(u-v)dx \geq 0.$$

In the second boundary integral we insert the nonlinear Newton condition and use the monotonicity property **Mon-b** . It follows

$$I_2 = \int_{\partial\Omega} ((u-v)(b(u) - b(v))|u-v|^{q-2})ds \geq 0.$$

Thus we get the accretivity of the operator A, defined by (3.50).
Now, we show that $Id + A : D(A) \subset L_q(\Omega) \to L_q(\Omega)$ is a surjective mapping that means: For any $\varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$ the boundary value problem

$$\begin{aligned}
-\triangle u + c(u) + u &= F \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \mathbf{n}} + b(u) &= \varphi \quad \text{on } \partial\Omega,
\end{aligned}$$

has a solution in $L_q(\Omega)$ for every $F \in L_q(\Omega)$. Considering $c(u) + u$ instead of $c(u)$ all assumptions of theorem 5 are valid and the surjectivity follows. Thus, the operator A, defined by (3.50), is m-accretive. $\square$

3.2.2 The regularity in space and time, uniqueness

Now, we are ready to formulate the results analogously to subsection 2.2 for our more general semi-linear inital boundary value problem:

Theorem 8 *Assume, the growth conditions (***G1***), (***G2***), (***G3***), the Carathéodory condition ($\tilde{\mathbf{C}}$), the coercivity conditions (***Co-c***), (***Co-b***) or (***co-b***),(***co-c***) and the monotonicity conditions (***Mon-c***), (***Mon-b***) are satisfied. Then the operator $A : D(A) \subset L_q(\Omega) \to L_q(\Omega)$, where $D(A)$ is defined by (3.50), is m-accretive. Furthermore, let be the right hand side of (3.31) $f \in W^{1,\infty,1}(I, L_q(\Omega), L_q(\Omega))$, and $u_0 \in D(A)$. Then u is a strong solution of (3.31), (3.32), compare subsection 2.2, that means $u \in C(I, D(A)) \cap W^{1,\infty,\infty}(I, L_q(\Omega), L_q(\Omega))$. Moreover, $u(t,\cdot) \in W^{2,q}(\Omega)$ for a.e. $t \in I$.*

Proof. The main assertion of Theorem 8 follows from theorem 9.5 from [13], p.278, and theorem 7. Since u is a strong solution we have $\frac{\partial u}{\partial t}(t,\cdot) \in L_q(\Omega)$ for a.e. $t \in I$. Thus we get finally $u(t,\cdot) \in W^{2,q}(\Omega)$ for a.e. $t \in I$. □

The **uniqueness** of the weak solution of the **stationary** problem (3.36), is guaranteed if the monotonicity conditions (**Mon-c**) and (**Mon-b**) are strenghtened to strongly monotonicity conditions.

For the nonstationary problem we have the following result:

Lemma 7 *Assume that all suppositions of theorem 8 are satisfied. Then the strong solution of the nonstationary problem (3.31), (3.32) is uniquely defined.*

Proof. Let us take two strong solutions u_1 and u_2. Since the strong solutions belong to $W^{1,2,2}(I, V, V^*)$, the terms in the following equation are well defined:

$$\langle \frac{\partial(u_1(t,\cdot) - u_2(t,\cdot))}{\partial t}, u_1(t,\cdot) - u_2(t,\cdot)\rangle_{V^* \times V} + \int_\Omega \nabla(u_1(t,\cdot) - u_2(t,\cdot)) \cdot \nabla(u_1(t,\cdot) - u_2(t,\cdot))dx +$$
$$\int_\Omega (c(u_1(t,\cdot)) - c(u_2(t,\cdot))(u_1(t,\cdot)) - u_2(t,\cdot))dx + \int_{\partial\Omega} (b(u_1(t,\cdot)) - b(u_2(t,\cdot)))(u_1(t,\cdot) - u_2(t,\cdot))dS = 0$$

Integration over t yields

$$0 = \int_0^t \langle \frac{\partial(u_1 - u_2)}{\partial t}, u_1 - u_2\rangle_{V^* \times V} d\tau + \int_0^t \int_\Omega (c(u_1) - c(u_2))(u_1 - u_2)dx d\tau$$
$$+ \int_0^t \int_{\partial\Omega} (b(u_1) - b(u_2))(u_1 - u_2)dS d\tau$$

Due to the monotonicity of c and b the corresponding integral terms are nonnegative and by partial integration of the first term we get

$$0 \geq \int_0^t \langle \frac{\partial(u_1 - u_2)}{\partial t}, u_1 - u_2\rangle_{V^* \times V} d\tau = \frac{1}{2}\|u_1(t) - u_2(t)\|^2_{L_2(\Omega)} - \frac{1}{2}\|u_1(0) - u_2(0)\|^2_{L_2(\Omega)}$$
$$= \frac{1}{2}\|u_1(t) - u_2(t)\|^2_{L_2(\Omega)}.$$

It follows that $u_1(t) = u_2(t)$ for any $t \in I$. □

4 Semilinear problems with advection term

The results of section 3 can be transferred partly to semilinear problems with a nonlinear advection term:

$$\frac{\partial u}{\partial t} - \triangle u + c(u)\vec{v} \cdot \nabla u = f(t,x) \quad \text{for } (t,x) \in Q, \tag{4.53}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} + b(u) = \varphi(t,x) \quad \text{for } (t,x) \in \Sigma, \tag{4.54}$$

$$u(0,x) = u_0(x) \quad \text{for } x \in \Omega, \tag{4.55}$$

where f and φ are given functions and $\vec{v}$ is a known sufficient smooth velocity vector. We assume, that $\vec{v} = \vec{v}(t,x)$ is measurable on Q, $\vec{v}(t,\cdot)$ is defined for all $t \in I$ and $\vec{v}(t,\cdot) \in [L_\infty(\Omega)]^2$ or $\vec{v}(t,\cdot) = \vec{v} \in [C(\overline{\Omega})]^2$ for all $t \in I$.

4.1 The stationary boundary value problem

We consider the stationary boundary value problem to (4.53), (4.54) in a polygon Ω with the boundary $\partial\Omega$:

$$-\triangle u + c(u)\vec{v} \cdot \nabla u = f \quad \text{in } \Omega, \tag{4.56}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} + b(u) = \varphi \quad \text{on } \partial\Omega. \tag{4.57}$$

The corresponding weak formulation in $V = H^1(\Omega)$ reads: Find an $u \in V$ such that $\forall v \in V$

$$\begin{aligned} \langle A(u), v\rangle = a(u,v) &= \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c(u)\vec{v} \cdot \nabla u v dx + \int_{\partial\Omega} b(u) v dS \\ &= \int_\Omega f v dx + \int_{\partial\Omega} \varphi v dS = \langle L, v\rangle, \end{aligned} \tag{4.58}$$

where $A : V \to V^*$ and $L \in V^*$. The weak formulation is reasonable if the same growth conditions as in Lemma 2 are satisfied.

Lemma 8 *Assume that the nonlinear terms $c(s)$ and $b(s)$ satisfy the following growth conditions for all $s \in \mathbb{R}$:*
G1 $|c(s)| \le k_c(1+|s|^{\gamma_c})$ *with $\gamma_c \ge 1$ and a positive constant k_c.*
G2 $|b(s) \le k_b(1+|s|^{\gamma_b})$ *with $\gamma_b \ge 1$ and a positive constant k_b.*
G3 $|\frac{db(s)}{ds}| \le \tilde{k}_b(1+|s|^{\tilde{\gamma}_b})$ *with $\tilde{\gamma}_b \ge 1$ and a positive constant $\tilde{k}_b$.*
Then for $u \in H^1(\Omega)$ and $\vec{v} = (v_1, v_2) \in [L_\infty(\Omega)]^2$ it holds

$$c(u)\vec{v} \cdot \nabla u \in L_{q_2}(\Omega), \tag{4.59}$$

$$b(u) \in W^{1-\frac{1}{q_2},q_2}(\partial\Omega) \ \textit{with}\ 1 < q_2 < 2. \tag{4.60}$$

Proof. The relation (4.60) was already proved by the estimation (3.39). We show that $c(u)\vec{v} \in L_{q_1}(\Omega)$ for any $q_1 \geq 1$. We apply the following multiplication theorem in Sobolev spaces, see [17], p.26:
Let be Ω a bounded domain, which satisfies a cone condition. Let be $m, h, k \in I\!N \cup \{0\}, p, q, r \geq 1$ real numbers with

$$\frac{m+h+k}{n} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}\,. \tag{4.61}$$

Furthermore, assume $\frac{np}{n-hp} \geq r$, if $hp < n$ and $\frac{nq}{n-kq} \geq r$, if $kq < n$. Then for $u \in W^{m+h,p}(\Omega)$ und $v \in W^{m+k,q}(\Omega)$ it holds that $uv \in W^{m,r}(\Omega)$. Moreover, there is a constant C independent of u und v such that

$$\|uv\|_{m,r} \leq C\|u\|_{m+h,p}\|v\|_{m+k,q}\,.$$

Due to (3.37) it holds that $c(u) \in L_{q_1}(\Omega)$ for any $q_1 \geq 1$. We consider a fixed number $r = q_1$ and remark that we have $c(u) \in L_{2q_1+1}(\Omega)$ too and by assumption $\vec{v} \in [L_{2q_1+1}(\Omega)]^2$. We apply the multiplication theorem for the products uv_1 and uv_2 setting $p = q = 2q_1 + 1, m = h = k = 0$. The inequality (4.61) is satisfied and it holds $hp < n$ and $\frac{np}{n-kp} \geq r$. Thus we get that $c(u)\vec{v} \in [L_{q_1}(\Omega)]^2$. Now, we consider the product $c(u)\vec{v} \cdot \nabla u$. We apply again the multiplication theorem to the factors $c(u)\vec{v} \in [L_{q_1}(\Omega)]^2$, where q_1 is sufficient large,and $\nabla u \in [L_2(\Omega)]^2$. It comes out that

$$\|c(u)\vec{v} \cdot \nabla u\|_{L_{q_2}(\Omega)} \leq \tilde{C}\|c(u)\vec{v}\|_{L_{q_1}(\Omega)}\|\nabla u\|_{L_2(\Omega)}. \tag{4.62}$$

□

Remark 3 *The inequality (4.62) can be modified for a fixed $\vec{v} \in [L_\infty(\Omega)]^2$:*

$$\|c(u)\vec{v} \cdot \nabla u\|_{L_{q_2}(\Omega)} \leq C\|c(u)\|_{L_{q_1}(\Omega)}\|\nabla u\|_{L_2(\Omega)}. \tag{4.63}$$

Indeed, let us start with $q_1 > 1$ and choose $\tilde{q_1} = q_1 - \varepsilon > 1$ for a small $\varepsilon > 0$. Analogously to (4.62) we get

$$\|c(u)\vec{v} \cdot \nabla u\|_{L_{q_2}(\Omega)} \leq \hat{C}\|c(u)\vec{v}\|_{L_{\tilde{q_1}}(\Omega)}\|\nabla u\|_{L_2(\Omega)}.$$

Applying the multiplication theorem to the product $c(u)\vec{v}$ it follows:

$$\|c(u)\vec{v}\|_{L_{\tilde{q_1}}(\Omega)} \leq \overline{C}\|c(u)\|_{Lq1(\Omega)}\|\vec{v}\|_{L^\infty(\Omega)}.$$

and finally the inequality (4.63).
In what follows we assume for simplicity, that $\vec{v} \in [C(\overline{\Omega})]^2$ for all $t \in I$.

4.1.1 Existence of weak solutions for semilinear problems with advection term

In order to show the existence of weak solutions we proceed as in subsection 3.1.1. We consider the operator $A : V \to V^*$ given by (4.58) and start with the pseudomonotonicity:

Lemma 9 *Assume that for the operator A given by (4.58) the conditions* **G1**, **G2** *and* **C** *are satisfied. Then A is pseudomomotone.*

Proof. At first we show property **PM1**, that means, the range $A\{u \in V, \|u\|_V \leq R\}$ is bounded in V^*. Indeed, using that V is continuously imbedded in $L_\alpha(\Omega), 1 \leq \alpha < \infty$ or respectively that the traces of functions from V belong to $L_\gamma(\partial\Omega), 1 \leq \gamma < \infty)$ we have

$$
\begin{aligned}
&\sup_{\|u\|_V \leq R} \|A(u)\|_{V^*} = \sup_{\|u\|_V \leq R} \sup_{\|v\|_V \leq 1} \langle A(u), v \rangle \\
&= \sup_{\|u\|_V \leq R} \sup_{\|v\|_V \leq 1} (\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c(u)\vec{v} \cdot \nabla u v dx + \int_{\partial\Omega} b(u) v dS) \\
&\leq \sup_{\|u\|_V \leq R} \sup_{\|v\|_V \leq 1} \left(\|\nabla u\|_{L_2(\Omega)} \|\nabla v\|_{L_2(\Omega)} + k_c C_1 \int_\Omega (1 + |u|^{\gamma_c}) |\nabla u| |v| dx + k_b \int_{\partial\Omega} (1 + |u|^{\gamma_b}) |v| dS \right) \\
&\leq R + \sup_{\|u\|_V \leq R} \sup_{\|v\|_V \leq 1} \left(k_c C_1 \|\nabla u\|_{L_2(\Omega)} \|(1 + |u|^{\gamma_c}) v\|_{L_2(\Omega)} \right) + \sup_{\|u\|_V \leq R} \sup_{\|v\|_V \leq 1} k_b \int_{\partial\Omega} (1 + |u|^{\gamma_b}) |v| dS \\
&\leq R + k_c C_1 R \sup_{\|u\|_V \leq R} \sup_{\|v\|_V \leq 1} \|(1 + |u|^{\gamma_c}) v\|_{L_2(\Omega)} + \sup_{\|u\|_V \leq R} \sup_{\|v\|_V \leq 1} k_b \int_{\partial\Omega} (1 + |u|^{\gamma_b}) |v| dS \leq C^*,
\end{aligned}
$$

where C^* is a constant depending on $R, C_1, k_b, k_c, \gamma_b, \gamma_c$.
Here we have used the multiplication theorem for $p > 2$

$$\|(|u|^{\gamma_c}) v\|_{L_2(\Omega)} \leq \||u|^{\gamma_c}\|_{L_p(\Omega)} \|v\|_V$$

and the ideas of the proof of Lemma 4.
Now, we come to the property **PM2**. We consider a sequence $u_k \rightharpoonup u$ in V and assume $\limsup_{k\to\infty} \langle A(u_k), u_k - u \rangle \leq 0$. In the proof of Lemma 4 we replace the nonlinear term $c(u)$ by $c(u)\vec{v} \cdot \nabla u$. Nearly all considerations are the same and it remains to show

1. $\lim_{k\to\infty} \langle c(u_k)\vec{v} \cdot \nabla u_k, u_k - u \rangle = 0$,
2. $\lim_{k\to\infty} \langle c(u_k)\vec{v} \cdot \nabla u_k, u - v \rangle = \langle c(u)\vec{v} \cdot \nabla u, u - v \rangle$.

We start with the first item and use the inequality (4.63), Lemma 3 and the boundedness of $(\|\nabla u_k\|_{L_2(\Omega)})_k$ due to the weak convergence $\nabla u_k \rightharpoonup \nabla u$:

$$
\begin{aligned}
|\langle c(u_k)\vec{v} \cdot \nabla u_k, u_k - u \rangle| &\leq \|c(u_k)\vec{v} \cdot \nabla u_k\|_{L_{q_2}(\Omega)} \|u_k - u\|_{L_{q_2'}(\Omega)} \\
&\leq C_6 \|c(u_k)\|_{L_{q_1}(\Omega)} \|\nabla u_k\|_{L_2(\Omega)} \|u_k - u\|_{L_{q_2'}(\Omega)} \\
&\leq C_7 \|u_k - u\|_{L_{q_2'}(\Omega)}.
\end{aligned}
$$

The compact imbedding of $H^1(\Omega)$ into $L_{q_2'}(\Omega)$ yields

$$|\langle c(u_k)\vec{v}\cdot\nabla u_k, u_k-u\rangle| \to 0 \text{ for } k\to\infty.$$

Now, we come to the second item. Using again the estimate (4.63) we get similar to the first item

$$\begin{aligned}
&|\langle c(u)\vec{v}\cdot\nabla u - c(u_k)\vec{v}\cdot\nabla u_k, u-v\rangle| = |\langle (c(u)-c(u_k))\vec{v}\cdot\nabla u + c(u_k)\vec{v}\cdot(\nabla u-\nabla u_k), u-v\rangle| \\
&\le C_8\|c(u)-c(u_k)\|_{L_{q_1}(\Omega)}\|\nabla u\|_{L_2(\Omega)}\|u-v\|_{L_{q_2'}(\Omega)} \\
&+ |\langle (c(u_k)-c(u))\vec{v}\cdot(\nabla u-\nabla u_k) + c(u)\vec{v}\cdot(\nabla u-\nabla u_k), u-v\rangle| \\
&\le C_9\|c(u)-c(u_k)\|_{L_{q_1}(\Omega)}(\|\nabla u\|_{L_2(\Omega)} + \|\nabla u-\nabla u_k\|_{L_2(\Omega)})\|u-v\|_{L_{q_2'}(\Omega)} \\
&+ |\langle c(u)\vec{v}\cdot(\nabla u-\nabla u_k), u-v\rangle| = I_k + J_k.
\end{aligned}$$

The term $I_k = C_9\|c(u)-c(u_k)\|_{L_{q_1}(\Omega)}(\|\nabla u\|_{L_2(\Omega)} + \|\nabla u-\nabla u_k\|_{L_2(\Omega)}\|u-v\|_{L_{q_2'}(\Omega)}) \to 0$, since $\|\nabla u-\nabla u_k\|_{L_2(\Omega)}$ is bounded due to the weak convergence $\nabla u_k \rightharpoonup \nabla u$.
The term $J_k = |\langle c(u)\vec{v}\cdot(\nabla u-\nabla u_k), u-v\rangle|$ can be written as
$J_k = |\langle c(u)(u-v)\vec{v}, (\nabla u-\nabla u_k)\rangle| = |\langle \vec{w}, (\nabla u-\nabla u_k)\rangle|$, where $\vec{w}\in[L_2(\Omega)]^2$. Again the weak convergence $\nabla u_k \rightharpoonup \nabla u$ implies that $J_k\to 0$. $\square$

The crucial point is to show the coercivity. We discuss this problem. First we modify the growth condition (**co-c**) in Lemma 6:

Lemma 10 *If there are constants $c_1>0, c_2<\infty, \varepsilon_2>0$ and functions $k_1\in L_1(\partial\Omega), k_2\in L_1(\Omega)$ such that*
*(**co-b**) $b(s)s\ge c_1|s|^q - k_1$ for $s\in\mathbb{R}$,*
*(**ko-c**) $\vec{t}\cdot\vec{t} + c(s)s\vec{v}\cdot\vec{t} \ge \varepsilon_2|\vec{t}|^2 - c_2|s|^{q_1} - k_2$ for all $s\in\mathbb{R},\ \vec{t}=(t_1,t_2)\in\mathbb{R}^2$*
and for some $1<q_1<q\le 2$, then A is coercive.

The proof is very similar to the proof of Lemma 6, inserting in (3.46) instead of $c(u)u$ the expression $c(u)u\vec{v}\cdot\nabla u$ and using the Poincaré (3.44) and the Young inequality (3.45).

The condition (**ko-c**) is not satisfied generally. Therefore, we formulate another coercivity lemma:

Lemma 11 *Let be C the constant in the Poincaré inequality (3.43). Assume, there are constants $c_1>0, c_2<\infty, \varepsilon_2>0$ with*

$$c_2 < \frac{\min\{\varepsilon_2+c_2, c_1\}}{2C^2}. \tag{4.64}$$

and there are functions $k_1\in L_1(\partial\Omega), k_2\in L_1(\Omega)$ such that
*(**Ko-b**) $b(s)s\ge c_1|s|^2 - k_1$ for $s\in\mathbb{R}$,*
*(**Ko-c**) $\vec{t}\cdot\vec{t} + c(s)s\vec{v}\cdot\vec{t} \ge \varepsilon_2|\vec{t}|^2 - c_2|s|^2 - k_2$ for all $s\in\mathbb{R},\ \vec{t}=(t_1,t_2)\in\mathbb{R}^2$,*
then A is coercive.

Proof. It is

$$\langle Au, u\rangle = \int_\Omega \nabla u \cdot \nabla u dx + \int_\Omega c(u)\vec{v}\cdot\nabla u dx + \int_{\partial\Omega} b(u) u dS$$
$$\geq \int_\Omega (\varepsilon_2|\nabla u|^2 - c_2|u|^2 - K_2)dx + \int_{\partial\Omega}(c_1|u|^2 - K_1)dS.$$

Using the identity

$$\|u\|^2_{L_2(\Omega)} = \|u\|^2_V - \||\nabla u|\|^2_{L_2(\Omega)}$$

we get

$$\langle Au, u\rangle \geq (\varepsilon_2 + c_2)\int_\Omega (|\nabla u|^2 - K_2)dx - c_2\|u\|^2_V + \int_{\partial\Omega}(c_1|u|^2 - K_1)dS.$$

The Poincaré inequality (3.43) implies

$$\langle Au, u\rangle \geq \frac{\min\{\varepsilon_2 + c_2, c_1\}}{2C^2}\|u\|^2_V - \int_\Omega K_2 dx - c_2\|u\|^2_V - \int_{\partial\Omega} K_1 dS.$$
$$= (\frac{\min\{\varepsilon_2 + c_2, c_1\}}{2C^2} - c_2)\|u\|^2_V - const.$$

Since the factor before $\|u\|^2_V$ is positive by assumption, the coerciveness follows. □

We discuss whether lemma 11 is reasonable.

Lemma 12 *If $|c(s)| < K$ then the condition (**Ko-c**) is satisfied. If additionally (**Ko-b**) holds and if $K|\vec{v}|$ is sufficiently small, see (4.65) and (4.66), then A is coercive.*

Proof. It is

$$\vec{t}\cdot\vec{t} + c(s)s\vec{v}\cdot\vec{t} \geq \vec{t}\cdot\vec{t} - K|s||\vec{v}||\vec{t}|.$$

We apply Young's inequality for $a, b \in \mathbb{R}^+$ and $0 < \varepsilon < 1$

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2.$$

Thus we get:

$$\vec{t}\cdot\vec{t} + c(s)s\vec{v}\cdot\vec{t} \geq (1-\varepsilon)\vec{t}\cdot\vec{t} - \frac{1}{4\varepsilon}K^2|\vec{v}|^2|s|^2|.$$

Setting $\varepsilon_2 = 1 - \varepsilon$ and $c_2 = \frac{1}{4\varepsilon}K^2|\vec{v}|^2$ it follows condition (**Ko-c**).
Now we discuss the coerciveness estimate $c_2 < \frac{\min\{\varepsilon_2+c_2,c_1\}}{2C^2}$:

1. Let be $\min\{\varepsilon_2 + c_2, c_1\} = \varepsilon_2 + c_2$. Then the estimate $c_2 < \frac{1-\varepsilon+c_2}{2C^2}$ holds, if $c_2(2C^2 - 1) < 1 - \varepsilon$. If $2C^2 - 1 \leq 0$ then c_2 can be choosen arbitrary. If $2C^2 - 1 > 0$ then we demand

$$K^2|\vec{v}|^2 < \frac{(1-\varepsilon)4\varepsilon}{2C^2 - 1}. \tag{4.65}$$

For example setting $\varepsilon = \frac{1}{2}$ in the estimate (4.65) we get

$$K^2|\vec{v}|^2 < \frac{1}{2C^2 - 1}.$$

2. Let be $\min\{\varepsilon_2 + c_2, c_1\} = c_1$. Then the coerciveness condition reads: $c_2 < \frac{c_1}{2C^2}$ what leads to

$$K^2|\vec{v}|^2 < \frac{4\varepsilon c_1}{2C^2}. \tag{4.66}$$

Taking again $\varepsilon = \frac{1}{2}$ the estimate (4.66) implies

$$K^2|\vec{v}|^2 < \frac{c_1}{C^2}.$$

$\square$

Thus we get the following result on the existence of weak solutions of the semilinear problem with advection term (4.56),(4.57):

Theorem 9 *Assume (***G1***), (***G2***), (***C***) and (***co-b***),(***ko-c***) or (***G1***), (***G2***), (***C***) and (***Ko-b***),(***Ko-c***), (4.64). Then the operator $A : V \to V^*$ given by (4.58) is surjective.*

The coercivity conditions in theorem 9, in particular (4.64), are unimpressive. If we consider the nonstationary problem, then they can be relaxed by a so-called semi-coerciveness condition, compare [13], p.202. We formulate this condition for our special case.

Remark 4 *The operator A is called semi-coercive if there are constants $m_0 > 0$, m_1 and m_2, such that*

$$\langle Au, u\rangle \geq m_0 \||\nabla u|\|^2_{L_2(\Omega)} - m_1 \||\nabla u|\|_{L_2(\Omega)} - m_2 \|u\|^2_{L_2(\Omega)}. \tag{4.67}$$

We underline, that the estimate (4.67) holds if we cancel in Lemma 11 the condition (4.64).

4.1.2 Regularity results for the stationary semilinear problem with advection term

We prove a regularity theorem for the semilinear problem with advection term (4.56),(4.57) based on the regularity theorem 1.

Theorem 10 *Let the assumptions of Theorem 9 and the growth condition (***G3***) be satisfied. If* $f \in L_q(\Omega)$ *and* $\varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$ *from the right-hand sides of (4.56) and (4.57), then a weak solution* $u \in H^1(\Omega)$ *of (4.58) belongs to* $W^{2,q}(\Omega)$*, where* q *is given by theorem 1.*

Proof. We proceed as in the proof of Theorem 6. Due to Theorem 9 a weak solution $u \in H^1(\Omega)$ exists. Moreover, Lemma 8 yields, that

$$\begin{aligned} &c(u)\vec{v} \cdot \nabla u \in L_{q_2}(\Omega), \\ &b(u) \in W^{1-\frac{1}{q_2},q_2}(\partial\Omega) \ \text{ with } 1 < q_2 < 2. \end{aligned}$$

We shift the nonlinear terms to the right-hand side and consider the Neumann problem

$$\begin{aligned} -\triangle u &= f - c(u)\vec{v} \cdot \nabla u = F \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} &= \varphi - b(u) = \Phi \quad \text{on } \partial\Omega. \end{aligned}$$

If $q < 2$, we can set $q_2 = q$ and the regularity theory for linear problems in polygons yields the assertion, see [8].
If $q \geq 2$, then it follows that $u \in W^{2,q_2}(\Omega)$ (where $1 < q_2 < 2$). Since $W^{2,q_2}(\Omega) \subset W^{1,q+1}(\Omega)$ we can modify the estimate (4.61) taking q_1 sufficiently large:

$$\|c(u)\vec{v} \cdot \nabla u\|_{L_q(\Omega)} \leq C\|c(u)\|_{L_{q_1}(\Omega)}\|\nabla u\|_{L_{q+1}(\Omega)}. \tag{4.68}$$

Together with the estimate (3.39) we get that the right hand sides F and Φ of the above linearized Neumann problem belong to $L_q(\Omega)$ and $W^{1-\frac{1}{q},q}(\partial\Omega)$ respectively, where q is given by theorem 1. The regularity theory for linear problems implies the assertion. □

4.2 The nonstationary semilinear problem with advection term

We consider now the initial boundary value problem (4.53), (4.54), (4.55.) Since the m-accretivity can not be shown, we can not proceed as before. Therefore, we use the pseudomonotonicity directly to derive results about the time-space behaviour of solutions.
We start with the weak formulation of the initial-value problem on $V^* \times V$, where $V = H^1(\Omega)$. Find a solution $u \in W^{1,2,2}(I, V, V^*)$ such that $\forall v \in V$

$$\langle \frac{\partial u}{\partial t}, v\rangle_{V^* \times V} + \langle A(t,u), v\rangle = \int_\Omega f(t,\cdot)v dx + \int_{\partial\Omega} \varphi(t,\cdot)v dS = \langle L, v\rangle \tag{4.69}$$

$$u(0,\cdot) = u_0(\cdot) \text{ in } \Omega. \tag{4.70}$$

where $A : I \times V \to V^*$ is given by

$$\langle A(t,u), v\rangle = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c(u)\vec{v}(t,\cdot) \cdot \nabla u\, v dx + \int_{\partial\Omega} b(u) v dS \tag{4.71}$$

If we assume, that $\vec{v} \in C(I, [C(\overline{\Omega})]^2)$ and the assumptions of Lemma 9 are satisfied for every $t \in I$ then follows $A : I \times V \to V^*$ is well defined and pseudomonotone.

In order to study the behaviour of the solution also in time we introduce the operator

$$[\mathcal{A}(u)](t) := A(u(t)). \tag{4.72}$$

We discuss under which conditions $\mathcal{A} : L_2(I,V) \cap L_\infty(I, L_2(\Omega)) \to L_2(I, V^*)$ is bounded. To this end we denote $L_2(\Omega) = H$, having in mind the evolution triple $V \subset H \subset V^*$.

Lemma 13 *Assume that the nonlinear terms satisfy the growth conditions* **G1** *and* **G2** *with* $0 \le \gamma_c < 1$, $0 \le \gamma_b \le 1$ *and* $\vec{v} \in C(I, [C(\overline{\Omega})]^2)$.
Then $\mathcal{A} : L_2(I,V) \cap L_\infty(I,H) \to L_2(I,V^*)$ *is bounded.*

Proof. We have to show: for every bounded set described by

$$\|u\|_{L_2(I,V)} < R_V, \|u\|_{L_\infty(I,H)} < \mathbb{R}_H,$$

the range $\|\mathcal{A}u\|_{L_2(I,V^*)}$ is bounded too.
Since $\|\mathcal{A}u\|_{L_2(I,V^*)} = (\int_0^T \|\mathcal{A}u\|^2_{V^*} dt)^{\frac{1}{2}}$ we start with the estimate of $\|\mathcal{A}u\|_{V^*}$:

$$\begin{aligned}\|\mathcal{A}u\|_{V^*} &= \sup_{\|v\|_V \le 1} \left(\int_\Omega \nabla u(t) \cdot \nabla v dx + \int_\Omega c(u(t))\vec{v}(t) \cdot \nabla u(t) v dx + \int_{\partial\Omega} b(u(t)) v dS) \right) \\ &\le \|\nabla u(t)\|_{L_2(\Omega)} + I_1 + I_2.\end{aligned}$$

Using the boundedness of $\vec{v}$ and the multiplication theorem fom the proof of Lemma 8 we estimate $I_1 = \sup_{\|v\|_V \le 1} \int_\Omega c(u(t))\vec{v}(t) \cdot \nabla u(t) v dx$:

$$\begin{aligned} I_1 &\le C_1 \sup_{\|v\|_V \le 1} \|c(u(t))v\|_{L_2(\Omega)} \|\nabla u(t)\|_{L_2(\Omega)} \\ &\le C_2 \|1 + |u(t)|^{\gamma_c}\|_{L_p(\Omega)} \|\nabla u(t)\|_{L_2(\Omega)} \\ &\le C_2 \|\nabla u(t)\|_{L_2(\Omega)} + C_2 \|u(t)\|^{\frac{2}{p}}_{L_2(\Omega)} \|\nabla u(t)\|_{L_2(\Omega)}.\end{aligned}$$

Here is $p > 2$ and $\gamma_c = \frac{2}{p} < 1$.
For the second term $I_2 = \sup_{\|v\|_V \le 1} \int_{\partial\Omega} b(u(t)) v dS$ it holds for $q > 1$ and $\gamma_b \le 1$

$$\begin{aligned} I_2 &\le C_3 \sup_{\|v\|_V \le 1} \|b(u(t)\|_{L_q(\partial\Omega)} \|v\|_V \\ &\le C_4 \|1 + |u(t)|^{\gamma_b}\|_{L_q(\partial\Omega)} \\ &\le C_5 + C_4 \|u(t)\|_{L^{\gamma_b}_{\gamma_b q}(\partial\Omega)} \\ &\le C_6 + C_7 \|u(t)\|\|_V.\end{aligned}$$

Here we have used the imbedding $V \subset L_{\gamma_b q}(\partial\Omega)$.
Thus we get finally

$$\|\mathcal{A}u\|_{V^*} \leq C_6 + C_8\|u(t)\||_V + C_2\|u(t)\|_H^{\frac{2}{p}}\|u(t)\|_V.$$

It follows the estimate

$$\|\mathcal{A}u\|_{L_2(I,V^*)} \leq \left(\int_0^T C_9 + C_{10}\|u(t)\||_V^2 + C_{11}\|u(t)\|_H^{\frac{4}{p}}\|u(t)\|_V^2 dt\right)^{\frac{1}{2}}$$
$$\leq C_{12} + C_{10}R_V + C_{11}R_H^{\frac{2}{p}}R_V \leq C_{13}.$$

□

Now, we consider the weak formulated boundary value problem (4.69), (4.70). If we assume that the right hand side L belongs to $L_2(I, V^*)$ and that we can guarantee that a weak solution $u \in L_2(I, V)$ exists, then $u \in W^{1,2,2}(I, V, V^*)$. It follows from proposition23.23, p. 422, in [16], that $u \in L_\infty(I, H)$. Therefore the suppposition in Lemma 13 makes sense.

The existence of a weak solution follows from Theorem 8.9, p.209 in [13] based on a result of Papageorgiou [15].

Theorem 11 *Let $A : V \to V^*$ be pseudomonotone and semi-coercive and $\mathcal{A} : L_2(I, V) \cap L_\infty(I, H) \to L_2(I, V^*)$ be bounded. Assume, that the right hand side L of (4.69) belongs to $L_2(I, V^*)$ and the initial datum of (4.70) $u_0 \in H$. Then the weak formulated initial boundary value problem (4.69), (4.70) possesses a solution from $W^{1,2,2}(I, V, V^*)$.*

Lemma 9, Lemma 10, Lemma 11, Lemma 12 and Remark 4 guarantee the pseudomonotonicity and semi-coerciveness of the operator $A : V \to V^*$ and Lemma13 the boundedness of $\mathcal{A} : L_2(I, V) \cap L_\infty(I, H) \to L_2(I, V^*)$.

Let us remark that the regularity with respect to the space variables in Theorem 11 is not optimally and that further investigations are necessary.

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